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A generalization of a theorem of Lekkerkerker to Ostrowski's decomposition of natural numbers

by

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1. Introduction. In 1939, Zeckendorf observed (and later published in [10]) the fact that every natural number can be expressed uniquely as a sum of nonadjacent Fibonacci numbers. Equivalently, if we write $F_1 = 1$, $F_2 = 2$, and for $k > 2$, $F_k = F_{k-1} + F_{k-2}$, then given a natural number n, there exists a unique sequence $\{c_k\}_{k=1}^{\infty}$ of integers with the following three properties:

(i) The natural number n can be expressed as

$$n = \sum_{k=1}^{\infty} c_k F_k.$$

(ii) For each index k, $c_k \in \{0, 1\}$, and for all but finitely many k, $c_k = 0$.
(iii) For all indices k, $k > 1$, if $c_k = 1$, then $c_{k-1} = 0$.

This representation is now known as the *Zeckendorf decomposition* of natural numbers.

Twelve years later, Lekkerkerker [7] independently rediscovered Zeckendorf's result and further included an analysis of the asymptotic behavior of the average number of terms (nonzero summands) required in such a decomposition (see also the work of Daykin [3]). In particular, Lekkerkerker proved:

Theorem 1.1. *For a natural number n, let $\sigma(n)$ denote the number of terms in the Zeckendorf decomposition of n, and $\psi(k)$ denote the average number of terms in the Zeckendorf decomposition among all integers n sat-*

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isfying $F_k \le n < F_{k+1}$; that is,

$$\psi(k) = \frac{1}{F_{k+1} - F_k} \sum_{n=F_k}^{F_{k+1}-1} \sigma(n).$$

Then

$$\lim_{k\to\infty} \frac{\psi(k)}{k} = \frac{5-\sqrt{5}}{10} = 0.2763\ldots.$$

It is well-known that the ratios F_{k+1}/F_k comprise the complete list of convergents (or "best rational approximants") of $\varphi = (1+\sqrt{5}\,)/2$. More generally, given an irrational real number α, we denote its (simple) continued fraction expansion

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

by $\alpha = [a_0, a_1, a_2, \ldots]$, in which each *partial quotient* a_n is an integer and for all $n > 0$, $a_n \ge 1$. We define the nth *convergent* of α, $p_n(\alpha)/q_n(\alpha) = p_n/q_n$, to be $p_n/q_n = [a_0, a_1, \ldots, a_n]$, with $\gcd(p_n, q_n) = 1$; in particular, $p_0 = a_0$ and $q_0 = 1$. We refer to the denominator q_n as the nth *continuant* of α. With the standard declaration $q_{-1} = 0$, the continuants satisfy the following second-order linear recurrence for all $n \ge 1$:

$$q_n = a_n q_{n-1} + q_{n-2} \tag{1.1}$$

(we note that the elements of the sequence $\{p_n\}$ enjoy the same recurrence relation with $p_{-1} = 1$). There are many sources that provide the basic properties of continued fractions that we employ throughout this work; see, for example, [1] or [6].

The Zeckendorf decomposition is, in fact, a special case of a much more general theorem first found by Ostrowski [8] nearly twenty years earlier. Specifically, in 1922 Ostrowski proved the following:

THEOREM 1.2. *Let α be an irrational real number having continued fraction expansion $\alpha = [a_0, a_1, a_2, \ldots]$ and let q_k denote the kth continuant associated with α. Then given any integer $n \ge 0$, there exists a unique sequence $\{c_k\}_{k=0}^{\infty}$ of integers such that:*

(i) *The natural number n can be expressed as*

$$n = \sum_{k=0}^{\infty} c_k q_k.$$

(ii) *For each index k, $0 \le c_k \le a_{k+1}$, and for all but finitely many k, $c_k = 0$.*

(iii) *The coefficient* c_0 *satisfies* $0 \leq c_0 < a_1$, *and for all* $k > 0$, *if* $c_k = a_{k+1}$, *then* $c_{k-1} = 0$.

We call the expansion given in the previous theorem the *Ostrowski* α-*decomposition* of n and say that any sequence of allowable coefficients $\{c_k\}$ satisfies the *Ostrowski conditions*. That is, a sequence of integers $\{c_k\}$ satisfies the Ostrowski conditions with respect to $\alpha = [a_0, a_1, \ldots]$ if properties (ii) and (iii) of Theorem 1.2 hold. Given that $\varphi = [1, 1, 1, \ldots] = [\overline{1}]$ and, in this case, $p_k/q_k = F_{k+1}/F_k$, we see that the Ostrowski φ-decomposition of a natural number n coincides with the Zeckendorf decomposition of n.

Here in this paper we extend the asymptotic result of Lekkerkerker for the Zeckendorf decomposition of natural numbers to the more general Ostrowski α-decomposition of natural numbers for an arbitrary real quadratic irrational α. Toward this end, we first recall the celebrated theorem of Lagrange stating that a real number α is a quadratic irrational if and only if its continued fraction expansion is eventually periodic, that is, if and only if $\alpha = [a_0, a_1, \ldots, a_{t-1}, \overline{a_t, \ldots, a_{t+T-1}}]$, where the bar overscores the periodic string $(a_t, \ldots, a_{t+T-1})$. As an aside, we remark that in the results that follow there is no implicit assumption that the period length T is minimal. Next, we write $\overline{\alpha}$ for the (algebraic) conjugate of α, and recall that a quadratic is called *reduced* if its continued fraction expansion is *purely* periodic. And finally, two real numbers, $\alpha = [a_0, a_1, \ldots]$ and $\beta = [b_0, b_1, \ldots]$, are said to be *equivalent* if the tail of one continued fraction equals the tail of the other continued fraction; that is, if there exists integers N and m so that for all $n \geq N$, $a_n = b_{n+m}$.

We now state our main theorem, which generalizes Lekkerkerker's result to all quadratic irrational real numbers.

THEOREM 1.3. *If* α *is a quadratic irrational real number, then the asymptotic average of the number of terms in the Ostrowski* α*-decomposition exists and depends only on a reduced quadratic equivalent to* α. *Moreover, that limit can be computed explicitly in terms of the following quantities. Let* $\alpha^* = [\overline{a_0, \ldots, a_{T-1}}]$ *be a reduced quadratic irrational equivalent to* α. *Let* $q_n = q_n(\alpha)$ *and* $q_n(\alpha^*)$ *denote the* n*th continuant associated with* α *and* α^*, *respectively. Define* $M = p_{T-1}(\alpha^*) + q_{T-2}(\alpha^*)$, *and let* θ *and* $\overline{\theta}$, $\theta > \overline{\theta}$, *denote the zeros of the polynomial* $x^2 - Mx + (-1)^T$. *Finally, write* s_k *for the* k*th continuant of the auxiliary quadratic* $a_1 + (a_0 - \overline{\alpha^*})^{-1}$, *define* $\sigma(n)$ *to be the number of terms in the Ostrowski* α*-decomposition of* n, *and let* $\psi(k)$ *denote the average number of terms in the Ostrowski* α*-decomposition among all integers* n *satisfying* $q_k \leq n < q_{k+1}$; *that is,*

$$\psi(k) = \frac{1}{q_{k+1} - q_k} \sum_{n=q_k}^{q_{k+1}-1} \sigma(n).$$

Then

$$\lim_{k\to\infty}\frac{\psi(k)}{k}$$
$$=\frac{1}{T}\sum_{h=0}^{T-1}\frac{(s_{T+h}-s_{T+h-1}-(s_h-s_{h-1})\overline{\theta})(q_{2T-h}(\alpha^*)-q_{T-h}(\alpha^*)\overline{\theta})}{\theta(\theta-\overline{\theta})(q_T(\alpha^*)-q_0(\alpha^*)\overline{\theta})}.$$

If we apply Theorem 1.3 to the celebrated quadratic $\alpha=\varphi=[\overline{1}]$, then $\alpha^*=\varphi$, $T=1$, $\theta=\varphi$, $\overline{\theta}=\overline{\varphi}$, the corresponding auxiliary quadratic is $1+(1-\overline{\varphi})^{-1}=\varphi$, and hence $s_k=q_k=F_k$. Thus Theorem 1.3 implies that

$$\lim_{k\to\infty}\frac{\psi(k)}{k}=\frac{-\overline{\varphi}(2-\overline{\varphi})}{\varphi(\varphi-\overline{\varphi})(1-\overline{\varphi})}=\frac{5-\sqrt{5}}{10},$$

which reproduces Lekkerkerker's original result.

We note that in 2008, Vipismakul [9], one of the first author's Honors Thesis students, using an approach different from the one presented here, was able to prove the special case of Theorem 1.3 in which α has period length 2 ($T=2$). That is, for $\alpha=[a_0,a_1,\ldots,a_{t-1},\overline{a,b}]$, the asymptotic result in Theorem 1.3 yields

$$\lim_{k\to\infty}\frac{\psi(k)}{k}=\frac{a+b+2-2G+(2(G^2-G-1)-G(a+b))\theta}{-2G+2(G^2-2)\theta},$$

where $G=ab+2$ and $\theta=[1+ab,\overline{1,ab}]$. The natural connection between G and the continued fraction expansion of α is visible through the identity

$$G=\operatorname{Trace}\left(\begin{pmatrix}a&1\\1&0\end{pmatrix}\begin{pmatrix}b&1\\1&0\end{pmatrix}\right)=\operatorname{Trace}\begin{pmatrix}ab+1&a\\b&1\end{pmatrix}$$

(note that the quantity M in Theorem 1.3 is the product of T such matrices corresponding to the period of α). It is interesting to observe that, in this case, if the geometric mean of a and b is fixed, then the asymptotic ratio depends only on the arithmetic mean of a and b; in particular, the limit is maximized when the arithmetic mean is minimized. For example, if we consider all possible (a,b) satisfying $ab=100$, then our formula reveals:

$\alpha=[\overline{a,b}]$	$\lim_{k\to\infty}\psi(k)/k$
$[\overline{1,100}]$	$0.4950\ldots$
$[\overline{2,50}]$	$0.7353\ldots$
$[\overline{4,25}]$	$0.8481\ldots$
$[\overline{5,20}]$	$0.8677\ldots$
$[\overline{10,10}]$	$0.8922\ldots$

In fact, the previous observations offer some intuition into our main result. As a heuristic, we can think of the limit of $\psi(k)/k$ as the asymptotic probability that some randomly chosen term in an Ostrowski α-decomposition has a nonzero coefficient. To illustrate this idea, if we return to $\alpha = [\,\overline{10, 10}\,]$, then the allowable coefficients in the Ostrowski α-decomposition range from 0 to 10. If the coefficients were selected at random, then the probability of picking a nonzero coefficient equals $10/11 \approx 0.909$. However, we expect the actual asymptotic limit to be slightly lower due to the Ostrowski condition that this coefficient would be *forced* to equal 0 if the coefficient of the term after it were to be 10. Indeed, as stated above, the actual limit equals $0.8922\ldots$, which is extremely close to our 0.909 estimate.

We establish Theorem 1.3 by first considering a *weaker* asymptotic limit—instead of averaging between *consecutive* continuants (between q_k and q_{k+1}), we first jump by the period length T and average over the range from q_k to q_{k+T}. In Section 2 we prove that this weaker limit exists in the special case of reduced quadratic irrationals and then, in Section 3, indicate how to extend the result to arbitrary quadratic irrational real numbers. Given that this weaker limit exists, in Section 4 we first show that the asymptotic limit averaging between consecutive continuants exists and agrees with the weaker limit found in Section 3. We then conclude by demonstrating that two equivalent quadratic irrationals have the same asymptotic limit; in particular, the limit is unaffected by any pre-period, which will complete the proof of our main result.

2. A weakened asymptotic limit: the reduced case. Let α be a quadratic irrational real number having a continued fraction expansion with period length T. Here we introduce a new, weaker asymptotic limit that measures the average number of terms in the Ostrowski α-decompositions, in which the average is computed over intervals between continuants whose indices jump by the period length T. More precisely, if we let $\sigma(n)$ be the number of terms in the Ostrowski α-decomposition of n, and $\xi(k)$ be the average number of terms in the Ostrowski α-decomposition among all integers n satisfying $q_k \le n < q_{k+T}$, that is,

$$\xi(k) = \frac{1}{q_{k+T} - q_k} \sum_{n=q_k}^{q_{k+T}-1} \sigma(n),$$

then we define the *asymptotic period-jumping average of the number of terms in the Ostrowski α-decomposition* by

$$\lim_{k\to\infty} \frac{\xi(k)}{k}.$$

In this section we prove that this asymptotic limit exists and can be explicitly computed for all reduced quadratic irrational α. In particular, here we prove the following:

PROPOSITION 2.1. *If α is a reduced quadratic irrational having a continued fraction expansion $\alpha = [\overline{a_0, \ldots, a_{T-1}}]$, then the asymptotic period-jumping average of the number of terms in the Ostrowski α-decomposition exists. Moreover, that limit can be computed explicitly in terms of the following quantities. Define $M = p_{T-1}(\alpha) + q_{T-2}(\alpha)$, let θ and $\overline{\theta}$, $\theta > \overline{\theta}$, denote the zeros of the polynomial $x^2 - Mx + (-1)^T$, and write s_k for the kth continuant of the auxiliary quadratic $a_1 + (a_0 - \overline{\alpha})^{-1}$. Then*

$$\lim_{k\to\infty} \frac{\xi(k)}{k} = \frac{1}{T}\sum_{h=0}^{T-1} \frac{(s_{T+h} - s_{T+h-1} - (s_h - s_{h-1})\overline{\theta})(q_{2T-h} - q_{T-h}\overline{\theta})}{\theta(\theta - \overline{\theta})(q_T - q_0\overline{\theta})}.$$

To establish Proposition 2.1, we begin by defining several natural quantities and then proving a number of useful lemmas. For a fixed, reduced quadratic $\alpha = [\overline{a_0, a_1, \ldots, a_{T-1}}]$ and natural number m, we write the Ostrowski α-decomposition of m as

$$m = \sum_{j\ge 0} c_j q_j,$$

where q_j denotes the jth continuant of α and the coefficients $\{c_j\}$ satisfy the Ostrowski conditions as defined in Section 1.

Given integers n and k, we define $b_{n,k}$ to be the number of Ostrowski α-decompositions that involve no more than the first n continuants $(q_0, q_1, \ldots, q_{n-1})$ and have exactly k nonzero coefficients. That is,

$$b_{n,k} = \Big|\Big\{\sum_{j=0}^{n-1} c_j q_j : \text{exactly } k \text{ coefficients satisfy } c_j \neq 0\Big\}\Big|.$$

If we forego the parameter k and write B_n for the number of Ostrowski α-decompositions that involve no more than the first n continuants, then it follows that

$$B_n = \sum_{k=0}^{n} b_{n,k}$$

(in particular, $B_0 = 1$). We begin by establishing the following elegant albeit elementary result.

LEMMA 2.2. *For all integers $n \ge 0$, $B_n = q_n$.*

Proof. We recall that B_n equals the number of permissible Ostrowski α-decompositions of the form $\sum_{j=0}^{n-1} c_j q_j$. Alternatively, we can express this quantity by conditioning on the value of the last coefficient, c_{n-1}. If $0 \le c_{n-1} < a_n$, then there are no additional constraints on c_{n-2} beyond the condition that $0 \le c_{n-2} \le a_{n-1}$. Thus, in this case, there are B_{n-1} allowable ways to choose the first $n-1$ coefficients. As there are a_n different choices for the c_{n-1} coefficient ($0 \le c_{n-1} < a_n$), we find that, in this case, there is a total of $a_n B_{n-1}$ different permissible expansions of the form $\sum_{j=0}^{n-1} c_j q_j$.

On the other hand, if $c_{n-1} = a_n$, the Ostrowski conditions imply that c_{n-2} must be 0. Hence, in this case, there are B_{n-2} ways to choose the first $n-2$ coefficients. Therefore the total number of permissible Ostrowski α-decompositions of the form $\sum_{j=0}^{n-1} c_j q_j$ equals $a_n B_{n-1} + B_{n-2}$, that is, we conclude that for all $n \ge 2$,

$$B_n = a_n B_{n-1} + B_{n-2}.$$

The previous identity reveals that the sequences $\{B_n\}$ and $\{q_n\}$ satisfy the same second-order linear recurrence relation. Thus to prove that $B_n = q_n$, we need only show that $B_0 = q_0$ and $B_1 = q_1$. Of course we have $q_0 = 1$ and $q_1 = a_1$, and we recall that $B_0 = 1$. Finally, we note that $B_1 = a_1$ since, in this case, the only coefficient appearing in the expansion, c_0, may be any of the a_1 integers satisfying $0 \le c_0 < a_1$. Hence we conclude that the two sequences are equal. ∎

We now define the quantity D_n to be the total number of nonzero terms in all α-decompositions involving no more than the first n continuants. That is,

$$D_n = \sum_{k=0}^{q_n - 1} \sigma(k),$$

where we recall that $\sigma(k)$ equals the number of terms in the Ostrowski α-decomposition of k. Equivalently, we may write

$$D_n = \sum_{k=0}^{n} k b_{n,k}.$$

We first show that the sequence $\{D_n\}$ satisfies a somewhat exotic recurrence relation.

LEMMA 2.3. *For all integers $n \ge 2$,*

$$D_n = a_n D_{n-1} + D_{n-2} + B_n - B_{n-1}.$$

Proof. We first claim that for all integers $k \ge 1$ and $n \ge 2$, we have the recurrence identity

$$b_{n,k} = b_{n-1,k} + b_{n-2,k-1} + (a_n - 1) b_{n-1,k-1}.$$

To establish this claim, we first recall that $b_{n,k}$ equals the number of n-tuples $(c_0, c_1, \ldots, c_{n-1})$ of coefficients having k nonzero c_j, $0 \le j \le n-1$, satisfying the Ostrowski conditions. Again, another way to express this quantity is to condition on the value of the last coefficient, c_{n-1}. If $c_{n-1} = 0$, then there are $b_{n-1,k}$ ways to choose the remaining coefficients. If $c_{n-1} = a_n$, then c_{n-2} is forced to be 0, so there are now $b_{n-2,k-1}$ different ways to select the remaining coefficients. Finally, if c_{n-1} takes on one of the $a_n - 1$ other possible values, then there are no additional restrictions on c_{n-2}, and hence there are $b_{n-1,k-1}$ ways of choosing the remaining coefficients. Combining these three quantities produces the desired recurrence and verifies our claim.

In view of the definition of D_n together with the recurrence identity we just established and the recurrence relation implicit in Lemma 2.2, we have

$$\begin{aligned} D_n &= \sum_{k=0}^{n} k b_{n,k} = \sum_{k=1}^{n} k\big(b_{n-1,k} + b_{n-2,k-1} + (a_n - 1)b_{n-1,k-1}\big) \\ &= \sum_{k=0}^{n} k b_{n-1,k} + \sum_{k=0}^{n-1} (k+1) b_{n-2,k} + (a_n - 1)\sum_{k=0}^{n-1}(k+1)b_{n-1,k} \\ &= D_{n-1} + D_{n-2} + B_{n-2} + (a_n - 1)(D_{n-1} + B_{n-1}) \\ &= a_n D_{n-1} + D_{n-2} + (a_n - 1)B_{n-1} + B_{n-2} \\ &= a_n D_{n-1} + D_{n-2} + B_n - B_{n-1}. \ \blacksquare \end{aligned}$$

We now introduce a suite of auxiliary objects that will be central to our analysis. Given a reduced quadratic $\gamma = [\overline{d_0, d_1, \ldots, d_{S-1}}]$, we use periodicity to extend the indices of the sequence of partial quotients, $\{d_n\}$, to all integers in the natural way: For any integer n, we let j be the unique integer satisfying $0 \le n + jS < S$ and then define $d_n = d_{n+jS}$. Returning to our given reduced quadratic $\alpha = [\overline{a_0, a_1, \ldots, a_{T-1}}]$, for an integer n satisfying $0 \le n < T$, we define the auxiliary quadratic $\beta_n = [\overline{a_{n+1}, a_n, \ldots, a_0, a_{T-1}, \ldots, a_{n+2}}]$. For an arbitrary integer n, we let j be the unique integer satisfying $0 \le n + jT < T$ and define $\beta_n = \beta_{n+jT}$. Finally, we let $r_{n,k}/s_{n,k}$ denote the kth convergent of β_n and abbreviate r_k/s_k for $r_{0,k}/s_{0,k}$.

The continuants $s_{n,k}$ allow us to uncover some deeper structure within the sequence of D_n. To this end, we begin with an elementary lemma showing that the doubly-indexed $s_{n,k}$ enjoy a familiar recurrence relation.

Lemma 2.4. *For any integer n, we have*

$$s_{n,k} = a_n s_{n-1,k-1} + s_{n-2,k-2}.$$

Proof. We first observe that

$$\frac{r_{n-1,k-1}}{s_{n-1,k-1}} = a_n + \frac{1}{r_{n-2,k-2}/s_{n-2,k-2}} = \frac{a_n r_{n-2,k-2} + s_{n-2,k-2}}{r_{n-2,k-2}}.$$

Recalling that the integers $r_{n,k}$ and $s_{n,k}$ are relatively prime, we conclude that the numerator and denominator of the rightmost fraction are also relatively prime. Hence we have $s_{n-1,k-1} = r_{n-2,k-2}$ and $r_{n-1,k-1} = a_n r_{n-2,k-2} + s_{n-2,k-2}$, which, upon combining these identities, yields

$$s_{n,k} = a_n s_{n-1,k-1} + s_{n-2,k-2}. \ \blacksquare$$

In view of Lemma 2.4, we may now express D_n as a sum of B_j. In particular we have:

LEMMA 2.5. *For all integers $n \geq 0$,*

$$D_n = \sum_{k=1}^{n} (s_{n,k} - s_{n,k-1}) B_{n-k}.$$

Proof. We proceed by induction on n and begin by noting that the identity holds when $n = 0$, since in this case both quantities equal 0. The identity also holds for $n = 1$, because $D_1 = a_1 - 1$, $B_0 = 1$, $s_{1,1} = a_1$ and $s_{1,0} = 1$. We now assume the identity holds for indices up to some fixed n, $n \geq 1$. In view of Lemmas 2.2–2.4, together with the facts that $s_{n,-1} = 0$ and $s_{n,0} = 1$, we have

$$\begin{aligned}
D_{n+1} &= a_{n+1} D_n + D_{n-1} + B_{n+1} - B_n \\
&= a_{n+1} D_n + D_{n-1} + (a_{n+1} - 1) B_n + B_{n-1} \\
&= a_{n+1} \Big(\sum_{k=1}^{n} (s_{n,k} - s_{n,k-1}) B_{n-k} \Big) + \sum_{k=1}^{n-1} (s_{n-1,k} - s_{n-1,k-1}) B_{n-1-k} \\
&\quad + (a_{n+1} - 1) B_n + B_{n-1} \\
&= \sum_{k=1}^{n} a_{n+1} (s_{n,k} - s_{n,k-1}) B_{n-k} + \sum_{k=2}^{n} (s_{n-1,k-1} - s_{n-1,k-2}) B_{n-k} \\
&\quad + (a_{n+1} - 1) B_n + B_{n-1} \\
&= \sum_{k=2}^{n} (a_{n+1} s_{n,k} + s_{n-1,k-1} - a_{n+1} s_{n,k-1} - s_{n-1,k-2}) B_{n-k} \\
&\quad + a_{n+1} (s_{n,1} - s_{n,0}) B_{n-1} + (a_{n+1} - 1) B_n + B_{n-1} \\
&= \sum_{k=2}^{n} (s_{n+1,k+1} - s_{n+1,k}) B_{n-k} + (s_{n+1,2} - s_{n+1,1}) B_{n-1} \\
&\quad + (s_{n+1,1} - s_{n+1,0}) B_n \\
&= \sum_{k=3}^{n+1} (s_{n+1,k} - s_{n+1,k-1}) B_{n+1-k} + (s_{n+1,2} - s_{n+1,1}) B_{n-1} \\
&\quad + (s_{n+1,1} - s_{n+1,0}) B_n \\
&= \sum_{k=1}^{n+1} (s_{n+1,k} - s_{n+1,k-1}) B_{n+1-k},
\end{aligned}$$

which thus establishes the identity for all $n \geq 0$. ∎

Recall that we defined $\xi(m)$ to equal the average number of terms in the Ostrowski α-decomposition among all integers j satisfying $q_m \leq j < q_{m+T}$

and that we defined D_m by $D_m = \sum_{k=0}^{q_m-1} \sigma(k)$. Hence we have

$$\xi(m) = \frac{D_{m+T} - D_m}{q_{m+T} - q_m}.$$

Our aim is to compute

$$\lim_{m\to\infty} \frac{\xi(m)}{m} = \lim_{m\to\infty} \frac{D_{m+T} - D_m}{m(q_{m+T} - q_m)},$$

and toward this goal, for each integer k, $0 \le k < T$, we let

$$L_k = \lim_{n\to\infty} \frac{D_{k+(n+1)T} - D_{k+nT}}{(k+nT)(q_{k+(n+1)T} - q_{k+nT})}.$$

So L_k represents the limit of the subsequence formed along indices $i \equiv k \bmod T$. Our strategy is to first show that L_k exists for each k and then prove that these T limits are, in fact, equal.

In view of Lemma 2.5, we see that to study the growth rate of the difference $D_{m+T} - D_m$, it is enough to estimate the quantities B_n and $s_{n,k}$. In order to estimate the quantity B_n, we first recall from Lemma 2.2 that $B_n = q_n$. Thus it is sufficient to produce an asymptotic expression for q_n in the case $n \equiv k \bmod T$. Given that the continued fraction expansion for α is purely periodic with period length T, it follows from Lemma 3 of [2] (see also [5]) that we have, for any nonnegative integer k,

$$q_{k+nT} = Mq_{k+(n-1)T} + (-1)^{T+1} q_{k+(n-2)T}, \tag{2.1}$$

in which $M = p_{T-1} + q_{T-2}$. We now collect some basic facts about the associated characteristic polynomial; the proof is straightforward and we do not include it here.

LEMMA 2.6. *Let M be the integer defined above and $f_M(x) = x^2 - Mx + (-1)^T$ be the characteristic polynomial associated with the recurrence in* (2.1). *Then $f_M(x)$ has two distinct real roots. Moreover, if we let θ denote the larger root of $f_M(x)$ and $\overline{\theta}$ denote the smaller, then $|\theta| > 1$ and $|\overline{\theta}| < 1$.*

In view of identity (2.1) and Lemma 2.6, we see that there exists a real constant x_k such that

$$q_{k+nT} = x_k \theta^n + O(1), \tag{2.2}$$

which, in view of Lemma 2.2, immediately provides us with a useful estimate for B_{k+nT}.

We now turn our attention to estimating the quantity $s_{n,k}$. In view of the definition of β_n given after the proof of Lemma 2.3, we note that, due to periodicity, for all integers j, $s_{n,k} = s_{n+jT,k}$. For a fixed n, we can again apply Lemma 3 of [2] to deduce that

$$s_{n,k+jT} = N_n s_{n,k+(j-1)T} + (-1)^{T+1} s_{n,k+(j-2)T},$$

where $N_n = r_{n,T-1} + s_{n,T-2}$. We now show that the coefficients N_n are, in fact, all equal to the constant M appearing in identity (2.1).

LEMMA 2.7. *For all integers n, $N_n = M$.*

Proof. For an integer j, we define the matrix M_j by

$$M_j = \begin{pmatrix} a_j & 1 \\ 1 & 0 \end{pmatrix}.$$

By the well-known connection between products of such 2×2 matrices and continued fractions (see, e.g., [1]), we have $M = \mathrm{Trace}(M_0 M_1 \cdots M_{T-1})$. Similarly, we find that $N_n = \mathrm{Trace}(M_{n+1} M_n \cdots M_0 M_{T-1} M_{T-2} \cdots M_{n+2})$. Therefore

$$\begin{aligned} N_n &= \mathrm{Trace}(M_{n+1} M_n \cdots M_0 M_{T-1} \cdots M_{n+2}) \\ &= \mathrm{Trace}((M_{n+1} M_n \cdots M_0 M_{T-1} \cdots M_{n+2})^T) \\ &= \mathrm{Trace}(M_{n+2}^T M_{n+3}^T \cdots M_{T-1}^T M_0^T \cdots M_{n+1}^T) \\ &= \mathrm{Trace}((M_{n+2} M_{n+3} \cdots M_{T-1})(M_0 \cdots M_{n+1})) \\ &= \mathrm{Trace}(M_0 M_1 \cdots M_{T-1}) = M. \ \blacksquare \end{aligned}$$

Lemma 2.7 reveals that the characteristic polynomial $x^2 - N_n x + (-1)^T$, for any index n, is identical to $x^2 - Mx + (-1)^T$. Moreover, by Lemma 2.6, we conclude that for each n, $x^2 - N_n x + (-1)^T$ has zeros at θ and $\overline{\theta}$, with $|\theta| > 1$ and $|\overline{\theta}| < 1$. Hence there exists a real constant $y_{n,k}$ satisfying

$$s_{n,k+jT} = y_{n,k} \theta^j + O(1). \tag{2.3}$$

We now show that the limit L_k exists for each k. In particular, we prove:

LEMMA 2.8. *For each k, $0 \le k < T$, the limit L_k exists. Moreover, given the constants x_k and $y_{n,k}$ as defined above, we have*

$$\lim_{n\to\infty} \frac{D_{k+(n+1)T} - D_{k+nT}}{(k+nT)(q_{k+(n+1)T} - q_{k+nT})} = \frac{1}{Tx_k} \sum_{h=0}^{T-1} (y_{k,h} - y_{k,h-1}) x_{k-h}.$$

REMARK. We note that the previous limit is an average of T values that has then been scaled by the factor $1/x_k$. Recall that T is the period length of the continued fraction expansion for α.

Proof of Lemma 2.8. In view of Lemma 2.5, together with Lemma 2.2 and identities (2.2) and (2.3), we have (with the change of variables $i = mT + h$ in the second equality):

$$\begin{aligned}
D_{k+nT} = & \sum_{i=1}^{k+nT} (s_{k+nT,i} - s_{k+nT,i-1}) B_{k+nT-i} \\
= & \sum_{h=0}^{T-1} \Big(\sum_{m=0}^{n+\lfloor (k-h)/T \rfloor} (s_{k,Tm+h} - s_{k,Tm+h-1}) B_{(n-m)T+(k-h)} \Big) - B_{k+nt} \\
= & \sum_{h=0}^{k} \Big(\sum_{m=0}^{n} ((y_{k,h} - y_{k,h-1})\theta^m + O(1))(x_{k-h}\theta^{n-m} + O(1)) \Big) \\
& + \sum_{h=k+1}^{T-1} \Big(\sum_{m=0}^{n-1} ((y_{k,h} - y_{k,h-1})\theta^m + O(1))(x_{k-h}\theta^{n-m} + O(1)) \Big) \\
& - B_{k+nt} \\
= & \sum_{h=0}^{k} (((y_{k,h} - y_{k,h-1})x_{k-h})(n+1)\theta^n + O(\theta^n)) \\
& + \sum_{h=k+1}^{T-1} (((y_{k,h} - y_{k,h-1})x_{k-h})n\theta^n + O(\theta^n)) - B_{k+nt} \\
= & \, n\theta^n \sum_{h=0}^{T-1} (y_{k,h} - y_{k,h-1})x_{k-h} + O(\theta^n).
\end{aligned}$$

If we now define

$$C_k = \sum_{h=0}^{T-1} (y_{k,h} - y_{k,h-1})x_{k-h},$$

then our previous identity can be expressed as

$$D_{k+nT} = C_k n\theta^n + O(\theta^n).$$

Hence we conclude that

$$\frac{D_{k+(n+1)T} - D_{k+nT}}{(k+nT)(q_{k+(n+1)T} - q_{k+nT})} = \frac{n\theta^n(\theta-1)C_k + O(\theta^n)}{(k+nT)(x_k\theta^n(\theta-1) + O(1))},$$

which upon letting $n \to \infty$ reveals that

$$L_k = \frac{C_k}{Tx_k},$$

and thus completes our proof. ∎

In view of Lemma 2.8, we now demonstrate that our desired asymptotic limit exists by proving that the quantity C_k/x_k is independent of k. To establish this assertion, we require several lemmas and begin with a study of the growth rates of x_k and $y_{n,k}$.

Lemma 2.9. *Let θ and its algebraic conjugate $\overline{\theta}$ be as defined in Lemma 2.6. Then the constants $y_{n,k}$ and x_k can be explicitly given as*

$$y_{n,k} = \frac{s_{n,k+T} - s_{n,k}\overline{\theta}}{\theta - \overline{\theta}} \quad \textit{and} \quad x_k = \frac{q_{k+T} - q_k\overline{\theta}}{\theta - \overline{\theta}}.$$

Proof. Given Lemma 2.6 and the recurrence relation that followed the lemma, together with the definition of $y_{n,k}$ from identity (2.3), we conclude that

$$s_{n,k+jT} = y_{n,k}\theta^j + \gamma\overline{\theta}^j$$

for some constant γ. Letting $j = 0$ and $j = 1$, respectively, reveals

$$s_{n,k} = y_{n,k} + \gamma \quad \text{and} \quad s_{n,k+T} = y_{n,k}\theta + \gamma\overline{\theta}. \tag{2.4}$$

The first identity in (2.4) yields

$$\gamma = s_{n,k} - y_{n,k},$$

which, in view of the second identity of (2.4), gives

$$s_{n,k+T} = y_{n,k}\theta + (s_{n,k} - y_{n,k})\overline{\theta},$$

or equivalently

$$y_{n,k} = \frac{s_{n,k+T} - s_{n,k}\overline{\theta}}{\theta - \overline{\theta}}.$$

An analogous argument allows us to sharpen the estimate of (2.2) and derive the corresponding formula for x_k. ∎

Lemma 2.10. *For all integers n and k,*

$$y_{n,k} = a_n y_{n-1,k-1} + y_{n-2,k-2} \quad \textit{and} \quad x_k = a_k x_{k-1} + x_{k-2}.$$

Proof. The first recurrence identity follows immediately from Lemmas 2.4 and 2.9. In particular, we write

$$\begin{aligned} y_{n,k} &= \frac{a_n s_{n-1,k+T-1} + s_{n-2,k+T-2} - (a_n s_{n-1,k-1} + s_{n-2,k-2})\overline{\theta}}{\theta - \overline{\theta}} \\ &= a_n \frac{s_{n-1,k+T-1} - s_{n-1,k-1}\overline{\theta}}{\theta - \overline{\theta}} + \frac{s_{n-2,k+T-2} - s_{n-2,k-2}\overline{\theta}}{\theta - \overline{\theta}} \\ &= a_n y_{n-1,k-1} + y_{n-2,k-2}. \end{aligned}$$

The recurrence relation for x_k follows in a similar manner. ∎

Lemma 2.11. *For all integers k and n, we have*

$$x_k = \theta x_{k-T} \quad \textit{and} \quad y_{n,k} = \theta y_{n,k-T}.$$

Proof. We recall that from the estimate in (2.2) we have

$$q_{k+iT} = x_k\theta^i + O(1) \quad \text{and} \quad q_{(k-T)+jT} = x_{k-T}\theta^j + O(1).$$

Letting $i = j-1$ in the first estimate and then equating the previous two estimates allows us to conclude that

$$x_k\theta^{j-1} = x_{k-T}\theta^j + O(1),$$

after which dividing by θ^{j-1} and letting $j \to \infty$ yields $x_k = x_{k-T}\theta$. An analogous argument establishes the corresponding result for $y_{n,k}$. ∎

LEMMA 2.12. *For any integer k,*

$$C_k = \theta C_{k-T}.$$

Proof. We first note that from the definition of $y_{k,h}$ and the basic properties of $s_{k,h}$, it follows that for all k and h, we have $y_{k-T,h} = y_{k,h}$. This observation together with the definition of C_k and Lemma 2.11 implies

$$\begin{aligned} C_{k-T} &= \sum_{h=0}^{T-1}(y_{k-T,h} - y_{k-T,h-1})x_{k-T-h} \\ &= \sum_{h=0}^{T-1}(y_{k,h} - y_{k,h-1})x_{k-h}\theta^{-1} = \theta^{-1}C_k. \ \blacksquare \end{aligned}$$

LEMMA 2.13. *For any integer k,*

$$C_k = a_kC_{k-1} + C_{k-2}.$$

Proof. The definition of C_k together with Lemma 2.10 yields

$$\begin{aligned} C_k &= \sum_{h=0}^{T-1}(y_{k,h} - y_{k,h-1})x_{k-h} \\ &= \sum_{h=0}^{T-1}(a_ky_{k-1,h-1} + y_{k-2,h-2} - a_ky_{k-1,h-2} - y_{k-2,h-3})x_{k-h} \\ &= a_k\sum_{h=0}^{T-1}(y_{k-1,h-1} - y_{k-1,h-2})x_{k-h} \\ &\quad + \sum_{h=0}^{T-1}(y_{k-2,h-2} - y_{k-2,h-3})x_{k-h} \\ &= a_k\sum_{h=-1}^{T-2}(y_{k-1,h} - y_{k-1,h-1})x_{(k-1)-h} \\ &\quad + \sum_{h=-2}^{T-3}(y_{k-2,h} - y_{k-2,h-1})x_{(k-2)-h}. \end{aligned} \tag{2.5}$$

An application of Lemma 2.11 reveals

$$\begin{aligned}(y_{k,h} - y_{k,h-1})x_{k-h} &= (y_{k,h+T}\theta^{-1} - y_{k,(h+T)-1}\theta^{-1})x_{k-(h+T)}\theta \\ &= (y_{k,h+T} - y_{k,(h+T)-1})x_{k-(h+T)}.\end{aligned}$$

This identity allows us to re-index the sums in (2.5) to conclude that

$$\begin{aligned}C_k &= a_k \sum_{h=0}^{T-1} (y_{k-1,h} - y_{k-1,h-1})x_{(k-1)-h} + \sum_{h=0}^{T-1} (y_{k-2,h} - y_{k-2,h-1})x_{(k-2)-h} \\ &= a_k C_{k-1} + C_{k-2}. \blacksquare\end{aligned}$$

LEMMA 2.14. *Given C_k and x_k as previously defined, we have*

$$\frac{C_0}{x_0} = \frac{C_1}{x_1}.$$

Proof. For any real numbers δ and γ, we denote the set $(\delta, \gamma) \cup (\gamma, \delta)$ by $\langle \delta, \gamma \rangle$; that is, $\langle \delta, \gamma \rangle$ is the *open* interval having endpoints δ and γ. Recall that if a, b, c, d are positive real numbers satisfying

$$\frac{a}{b} \neq \frac{c}{d},$$

then

$$\frac{a+c}{b+d} \in \left\langle \frac{a}{b}, \frac{c}{d} \right\rangle.$$

We proceed by contradiction; that is, we assume that $C_0/x_0 \neq C_1/x_1$. We now claim that for all $k \geq 1$, $C_{k-1}/x_{k-1} \neq C_k/x_k$. To establish this claim we induct on k, noting that by our previous assumption, the assertion holds for $k = 1$.

We now assume that $C_{k-1}/x_{k-1} \neq C_k/x_k$ for some $k \geq 1$. By Lemmas 2.10 and 2.13, we have

$$\frac{C_{k+1}}{x_{k+1}} = \frac{a_{k+1}C_k + C_{k-1}}{a_{k+1}x_k + x_{k-1}},$$

and thus

$$\frac{C_{k+1}}{x_{k+1}} \in \left\langle \frac{a_{k+1}C_k}{a_{k+1}x_k}, \frac{C_{k-1}}{x_{k-1}} \right\rangle = \left\langle \frac{C_k}{x_k}, \frac{C_{k-1}}{x_{k-1}} \right\rangle,$$

and in particular, recalling that this interval is open, we deduce that $C_k/x_k \neq C_{k+1}/x_{k+1}$. Therefore we conclude that for all natural numbers k, $C_{k-1}/x_{k-1} \neq C_k/x_k$. We note that this argument actually proves more, namely that for all $k \geq 2$,

$$\frac{C_k}{x_k} \in \left\langle \frac{C_{k-1}}{x_{k-1}}, \frac{C_{k-2}}{x_{k-2}} \right\rangle.$$

Hence for all $k \geq 2$ we have

$$\left\langle \frac{C_k}{x_k}, \frac{C_{k-1}}{x_{k-1}} \right\rangle \subseteq \left\langle \frac{C_{k-1}}{x_{k-1}}, \frac{C_{k-2}}{x_{k-2}} \right\rangle,$$

which, after repeated applications, implies that

$$\frac{C_T}{x_T} \in \left\langle \frac{C_{T-1}}{x_{T-1}}, \frac{C_{T-2}}{x_{T-2}} \right\rangle \subseteq \left\langle \frac{C_0}{x_0}, \frac{C_1}{x_1} \right\rangle;$$

in particular, $C_T/x_T \neq C_0/x_0$. However, Lemmas 2.11 and 2.12 yield

$$\frac{C_T}{x_T} = \frac{\theta C_0}{\theta x_0} = \frac{C_0}{x_0},$$

which contradicts our established claim. Therefore we conclude that our original assumption is false, that is, $C_0/x_0 = C_1/x_1$, as desired. ∎

LEMMA 2.15. *For all integers $k \geq 1$,*

$$\frac{C_k}{x_k} = \frac{C_0}{x_0}.$$

Proof. We proceed by induction on k. By Lemma 2.14, we see that the identity is valid in the case $k = 1$. We now assume the identity holds for all indices k up to some fixed index $j \geq 1$. By Lemmas 2.10 and 2.13, we have

$$\frac{C_{j+1}}{x_{j+1}} = \frac{a_{j+1}C_j + C_{j-1}}{a_{j+1}x_j + x_{j-1}}. \tag{2.6}$$

On the other hand, our induction hypothesis yields

$$\frac{a_{j+1}C_j}{a_{j+1}x_j} = \frac{C_0}{x_0} = \frac{C_{j-1}}{x_{j-1}},$$

which, in view of (2.6), implies that $C_{j+1}/x_{j+1} = C_0/x_0$, and thus completes our proof. ∎

Putting all our observations together, we now show that for any reduced quadratic irrational α, the asymptotic period-jumping average of the number of terms in the Ostrowski α-decomposition exists and is given by the quantity stated in Proposition 2.1.

Proof of Proposition 2.1. For any integer k, $0 \leq k < T$, by an application of Lemma 2.8 we have

$$\lim_{n\to\infty} \frac{\xi(k+nT)}{k+nT} = \frac{C_k}{Tx_k},$$

which, in view of Lemma 2.15, implies that

$$\lim_{n\to\infty} \frac{\xi(k+nT)}{k+nT} = \frac{C_0}{Tx_0}.$$

Given that these limits are constant as k ranges over a complete residue class modulo T, we conclude that

$$\lim_{k\to\infty} \frac{\xi(k)}{k} = \frac{C_0}{Tx_0}.$$

Applying the definition of C_0 and Lemma 2.9 to evaluate x_0 yields

$$\lim_{k\to\infty} \frac{\xi(k)}{k} = \frac{1}{T}\,\frac{\theta-\overline{\theta}}{q_T-q_0\overline{\theta}}\sum_{h=0}^{T-1}(y_{0,h}-y_{0,h-1})x_{-h}.$$

By the T-periodicity of the sequence $\{x_n\}$, we can replace the terms x_{-h} in the previous sum with x_{T-h}. Moreover, we recall our abbreviated notation $s_h = s_{0,h}$ for the hth continuant of $\beta_0 = [\overline{a_1, a_0, a_{T-1}, a_{T-2}, \ldots, a_2}]$, which, by a result of Galois [4] (see [1]), can be expressed formally as

$$\beta_0 = [a_1, a_0, -1/\overline{\alpha}] = a_1 + (a_0 - \overline{\alpha})^{-1}.$$

These observations together with another application of Lemma 2.9 to evaluate $y_{0,h}$ allow us to deduce that

$$\lim_{k\to\infty} \frac{\xi(k)}{k} = \frac{1}{T}\sum_{h=0}^{T-1}\frac{(s_{T+h}-s_{T+h-1}-(s_h-s_{h-1})\overline{\theta})(q_{2T-h}-q_{T-h}\overline{\theta})}{\theta(\theta-\overline{\theta})(q_T-q_0\overline{\theta})}. \ \blacksquare$$

3. A weakened asymptotic limit: extended to arbitrary quadratic irrationals. With mostly minor changes, our proof of Proposition 2.1 can be extended to all real quadratic irrationals α. In particular, here we outline an argument showing that for any quadratic irrational real number α, the asymptotic period-jumping average of the number of terms in the Ostrowski α-decomposition exists and is explicitly given in the following result.

PROPOSITION 3.1. *If α is a real quadratic irrational having a continued fraction expansion $\alpha = [a_0, a_1, \ldots, a_{t-1}, \overline{a_t, \ldots, a_{t+T-1}}]$, with pre-period length t and period length T, then the asymptotic period-jumping average of the number of terms in the Ostrowski α-decomposition exists. Moreover, that limit can be computed explicitly in terms of the following quantities. Let $\alpha^* = [\overline{a_0, \ldots, a_{T-1}}]$ be a reduced quadratic irrational equivalent to α. Let $q_n = q_n(\alpha)$ and $q_n(\alpha^*)$ denote the nth continuant associated with α and α^*, respectively. Define $M = p_{T-1}(\alpha^*) + q_{T-2}(\alpha^*)$, and let θ and $\overline{\theta}$, $\theta > \overline{\theta}$, denote the zeros of the polynomial $x^2 - Mx + (-1)^T$. Finally, write s_k^* for the kth continuant of the auxiliary quadratic $a_{t+1} + \left(a_t - \overline{\alpha^*}\right)^{-1}$, define $\sigma(n)$ to be the number of terms in the Ostrowski α-decomposition of n, and let $\xi(k)$ denote the average number of terms in the Ostrowski α-decomposition among all integers n satisfying $q_k \le n < q_{k+T}$; that is,*

$$\xi(k) = \frac{1}{q_{k+T}-q_k}\sum_{n=q_k}^{q_{k+T}-1}\sigma(n).$$

Then

$$\lim_{k\to\infty}\frac{\xi(k)}{k}=\frac{1}{T}\sum_{h=0}^{T-1}\frac{(s^*_{T+h}-s^*_{T+h-1}-(s^*_h-s^*_{h-1})\overline{\theta})(q_{2T+t-h}-q_{T+t-h}\overline{\theta})}{\theta(\theta-\overline{\theta})(q_{T+t}-q_t\overline{\theta})}.$$

Proof. Here we require auxiliary objects analogous to the β_n from our previous argument. For any integer $n\ge 0$, we write $\beta_n=[a_{n+1},a_n,\dots,a_1,a_0]$, and for integers n satisfying $t\le n<t+T$, we define

$$\beta^*_n=[\overline{a_{n+1},a_n,\dots,a_t,a_{t+T-1},\dots,a_{n+2}}].$$

For an index $n\notin[t,t+T)$, we extend our definition of β^*_n by periodicity; that is, we declare $\beta^*_n=\beta^*_{n+jT}$ for all integers j. Again we write $r_{n,k}/s_{n,k}$ for the kth convergent of β_n and now let $r^*_{n,k}/s^*_{n,k}$ denote the kth convergent of β^*_n and write $s^*_k=s^*_{t,k}$. We note that if $n+1\ge t$ (recall that t is the length of the pre-period of α), then the first $n-t+2$ partial quotients of β_n agree with the first $n-t+2$ partial quotients of β^*_n, so for all integers k, $0\le k\le n-t+1$, $r_{n,k}=r^*_{n,k}$ and $s_{n,k}=s^*_{n,k}$.

We also extend our definition of $s^*_{n,k}$ to negative indices k by applying the usual recurrence that defines the continuants, where we define negatively indexed partial quotients by periodicity. For example, if $\beta^*_n=[\overline{b_0,b_1}]$, then $s^*_{n,0}=1$ and $s^*_{n,1}=b_1$, and the recurrence $s^*_{n,1}=b_1s^*_{n,0}+s^*_{n,-1}$ allows us to compute $s^*_{n,-1}=0$, and hence $s^*_{n,0}=b_0s^*_{n,-1}+s^*_{n,-2}$ gives $s^*_{n,-2}=1$, and $s^*_{n,-1}=b_{-1}s^*_{n,-2}+s^*_{n,-3}$ yields $s^*_{n,-3}=-b_1$.

With these new definitions, we see that Lemmas 2.2 through 2.5 can be proven exactly as before. Furthermore, the argument in Lemma 2.4 also applies to show that whenever $n\ge t$,

$$s^*_{n,k}=a_ns^*_{n-1,k-1}+s^*_{n-2,k-2}.$$

Because of the initial run of t nonperiodic partial quotients, we alter our definition of the limit L_k and declare

$$L_k=\lim_{n\to\infty}\frac{D_{k+(n+1)T+t}-D_{k+nT+t}}{(k+nT+t)(q_{k+(n+1)T+t}-q_{k+nT+t})},$$

where again D_n denotes the total number of nonzero terms in all Ostrowski α-decompositions involving no more than the first n continuants of α.

We now let $M=p_{T-1}(\alpha^*)+q_{T-2}(\alpha^*)$ and denote the larger and smaller zeros of the polynomial $x^2-Mx+(-1)^T$ by θ and $\overline{\theta}$, respectively. The methods used in Section 2 to deduce closed formulas for q_n and $s_{n,k}$ can be applied analogously in this new context. In particular, we have the following results.

Lemma 3.2. *Given an integer k, there exists a constant x_{k+t} such that for all integers n satisfying $k+nT\ge 0$, we have*

$$q_{k+t+nT}=x_{k+t}\theta^n+O(1).$$

LEMMA 3.3. *Given integers k and n, with $k \geq 0$, there exists a constant $y_{n,k}$ such that for all nonnegative integers j, we have*

$$s^*_{n,k+jT} = y_{n,k}\theta^j + O(1).$$

We note that since $s_{n,k} = s^*_{n,k}$ for integers k satisfying $0 \leq k \leq n-t+1$, Lemma 3.3 also implies:

LEMMA 3.4. *Given integers k and n, with $k \geq 0$, the $y_{n,k}$ from Lemma 3.3 can be chosen such that whenever $i \geq 0$ and $k + jT \leq n - t + 1$, we have*

$$s_{n+iT,k+jT} = y_{n,k}\theta^j + O(1).$$

Given that the sequence $\{\beta^*_n\}$ is periodic with period length T, we again note that $y_{n,k} = y_{n+jT,k}$ for all natural numbers j.

We now estimate the quantity D_{k+jT+t}. In the purely periodic case, finding such an estimate was straightforward since we had closed formulas for $s_{n,k}$ and q_n. In the generalized case, we still have closed forms, but they are slightly more delicate due to the presence of a pre-period. Thus we must handle such estimates with a bit more care. In particular, we have

$$\begin{aligned} D_{k+jT+t} &= \sum_{i=1}^{k+jT+t} (s_{k+jT+t,i} - s_{k+jT+t,i-1})B_{k+jT+t-i} \\ &= \sum_{h=0}^{T-1}\Big(\sum_{m=0}^{j+\lfloor\frac{k-h+t}{T}\rfloor} (s_{k+jT+t,Tm+h} - s_{k+jT+t,Tm+h-1})B_{(j-m)T+(k-h)+t}\Big) \\ &\quad - B_{k+jT+t}. \end{aligned}$$

We know that $k - h + t \geq -h > -T$, so $\lfloor (k-h+t)/T \rfloor \geq -1$. Thus we write

$$D_{k+jT+t} =$$

$$\sum_{h=0}^{T-1}\Big(\sum_{m=j}^{j+\lfloor\frac{k-h+t}{T}\rfloor} (s_{k+jT+t,Tm+h} - s_{k+jT+t,Tm+h-1})B_{(j-m)T+(k-h)+t}\Big) \tag{3.1}$$

$$- B_{k+jT+t}$$

$$+\sum_{h=0}^{T-1}\Big(\sum_{m=0}^{j-1}(s_{k+jT+t,Tm+h} - s_{k+jT+t,Tm+h-1})B_{(j-m)T+(k-h)+t}\Big). \tag{3.2}$$

To examine the double sum (3.1), we first consider the innermost sum. As α is a real quadratic irrational, it is badly approximable, that is, there exists a constant K such that $a_i \leq K$ for all $i \geq 0$. Thus for any natural number w, we have

$$B_w = q_w = a_w q_{w-1} + q_{w-2} \leq K q_{w-1} + q_{w-1} = (K+1)q_{w-1}.$$

Repeated application of this inequality reveals that $B_w \le (K+1)^w q_0 = (K+1)^w$. Now in the sum (3.1), we have $m \ge j$, so $j - m \le 0$ and thus $B_{(j-m)T+(k-h)+t} \le B_{k-h+t} \le (K+1)^{k-h+t}$. We know that $0 \le k, h < T$, so $k - h + t \le T + t$ and thus we have

$$B_{(j-m)T+(k-h)+t} \le (K+1)^{T+t},$$

so all the terms B_i in the sum (3.1) are bounded by a constant independent of m and j. Hence we conclude that the inner sum of (3.1) is

$$O\Big(\sum_{m=j}^{j+\lfloor (k-h+t)/T\rfloor} s_{k+jT+t,Tm+h} - s_{k+jT+t,Tm+h-1}\Big) = O\Big(\sum_{m=j}^{j+\lfloor (k-h+t)/T\rfloor} s_{k+jT+t,Tm+h}\Big),$$

where the second estimate follows from the fact that

$$s_{k+jT+t,Tm+h} \ge s_{k+jT+t,Tm+h-1}.$$

If $h + mT \le k + jT$, we have $s_{k+jT+t,h+mT} \le s_{k+jT+t,k+jT}$, and if $h + mT > k + jT$, then we argue as above to conclude

$$s_{k+jT+t,h+mT} \le (K+1)^{(h-k)+(m-j)T} s_{k+jT+t,k+jT}.$$

We note that $h-k \le T-0 = T$ and $m-j \le \lfloor (k-h+t)/T\rfloor \le \lfloor (T+t)/T\rfloor$, so the exponent on $K+1$ is bounded by the constant $T + 1 + \lfloor t/T\rfloor$. Hence we see that

$$s_{k+jT+t,h+mT} = O(s_{k+jT+t,k+jT}).$$

The indices of $s_{k+jT+t,k+jT}$ satisfy the hypotheses of Lemma 3.4 and thus we have

$$s_{k+jT+t,k+jT} = y_{k+jT+t,k}\theta^j + O(1).$$

By periodicity, $y_{k+jT+t,k} = y_{k+(j+1)T+t,k}$, so $y_{k+jT+t,k}$ as a function of j is $O(1)$. Hence we see that

$$s_{k+jT+t,k+jT} = O(\theta^j),$$

and therefore

$$s_{k+jT+t,h+mT} = O(\theta^j)$$

as well. Therefore the inner sum of (3.1) is

$$O\Big(\sum_{m=j}^{j+\lfloor (k-h+t)/T\rfloor} s_{k+jT+t,Tm+h}\Big) = O\Big(\sum_{m=j}^{j+\lfloor (k-h+t)/T\rfloor} \theta^j\Big).$$

The total number of terms in this last summation is bounded by

$$j + \left\lfloor \frac{k-h+t}{T} \right\rfloor - j + 1 \le \left\lfloor \frac{T-0+t}{T} \right\rfloor + 1 \le 2 + \left\lfloor \frac{t}{T} \right\rfloor = O(1),$$

and so the entire innermost summation of (3.1) is equal to $O(\theta^j)$, and thus the entire double sum of (3.1) can be estimated by

$$\sum_{h=0}^{T-1} O(\theta^j) = O(\theta^j).$$

As for the last B_{k+jT+t} term in (3.1), we recall that $B_{k+jT+t} = q_{k+jT+t} = x_{k+t}\theta^j + O(1)$. Here x_{k+t} is bounded by a constant, since there are only finitely many values of x_{k+t} for $0 \leq k < T$. Hence we also discover that $B_{k+jT+t} = O(\theta^j)$, and so we conclude that the entirety of (3.1) is $O(\theta^j)$.

We now turn to the double sum (3.2). To apply Lemma 3.4 to produce a closed form for $s_{n,k}$ in this sum, we require that $mT + h \leq k + jT + 1$. However, we have $m \leq j-1$, and thus $mT+h \leq jT+h-T < jT \leq k+jT+1$; hence we can apply the estimate from Lemma 3.4.

To apply the estimate implicitly given in Lemma 3.2 for $B_{(j-m)T+(k-h)+t} = q_{(j-m)T+(k-h)+t}$, we require that the indices satisfy $(j-m)T + (k-h) \geq 0$. This inequality clearly holds, since, in the double sum (3.2), we have $(j-m)T + (k-h) \geq T + (k-h) > 0$.

Applying the estimates of Lemmas 3.2 and 3.4 to (3.2) reveals

$$\begin{aligned}
&\sum_{h=0}^{T-1}\Big(\sum_{m=0}^{j-1}((y_{k+jT+t,h} - y_{k+jT+t,h-1})\theta^m + O(1))(x_{k-h+T+t}\theta^{j-m-1} + O(1))\Big)\\
&\qquad = \sum_{h=0}^{T-1}(((y_{k+jT+t,h} - y_{k+jT+t,h-1})x_{k-h+T+t})j\theta^{j-1} + O(\theta^j))\\
&\qquad = j\theta^{j-1}\Big(\sum_{h=0}^{T-1}(y_{k+jT+t,h} - y_{k+jT+t,h-1})x_{k-h+T+t}\Big) + O(\theta^j)\\
&\qquad = j\theta^{j-1}\Big(\sum_{h=0}^{T-1}(y_{k+T+t,h} - y_{k+T+t,h-1})x_{k-h+T+t}\Big) + O(\theta^j).
\end{aligned}$$

Combining these two estimates for (3.1) and (3.2) yields

$$D_{k+jT+t} = C_{k+t}j\theta^{j-1} + O(\theta^j),$$

where again we define

$$C_k = \sum_{h=0}^{T-1}(y_{k+T,h} - y_{k+T,h-1})x_{k-h+T}.$$

Hence our average becomes

$$\frac{D_{k+(j+1)T+t} - D_{k+jT+t}}{(k+jT+t)(q_{k+(j+1)T+t} - q_{k+jT+t})} = \frac{j\theta^{j-1}(\theta-1)C_{k+t} + O(\theta^j)}{(k+jT+t)(x_{k+t}\theta^j(\theta-1) + O(1))},$$

and thus the limit we desire along subsequences modulo T is given by

$$L_k = \lim_{j\to\infty} \frac{D_{k+(j+1)T+t} - D_{k+jT+t}}{(k+jT+t)(q_{k+(j+1)T+t} - q_{k+jT+t})} = \frac{C_{k+t}}{T\theta x_{k+t}},$$

where the T in the denominator arises because $j/(k+jT+t) \to 1/T$ as $j \to \infty$.

From this point, the remainder of the argument is virtually identical to the purely periodic case. In particular, we show that the quantities C_{k+t}/x_{k+t} do not depend upon the value of k. Our proof for the purely periodic case required closed formulas for x_k and $y_{n,k}$ and several recurrence relations. The analogous lemmas hold in this more general setting and the proofs follow in a similar fashion. In particular, we can establish the following results.

Lemma 3.5. *For all integers n and k, $k \ge 0$, we have the following formulas for $y_{n,k}$ and x_k:*

$$y_{n,k} = \frac{s^*_{n,k+T} - \overline{\theta} s^*_{n,k}}{\theta - \overline{\theta}} \quad \textit{and} \quad x_{k+t} = \frac{q_{k+T+t} - \overline{\theta} q_{k+t}}{\theta - \overline{\theta}}.$$

Lemma 3.6. *For all integers n, $n \ge t$, we have*

$$y_{n,k} = a_n y_{n-1,k-1} + y_{n-2,k-2} \quad \textit{and} \quad x_n = a_n x_{n-1} + x_{n-2}.$$

Lemma 3.7. *For all integers n and k, $k \ge 0$, we have*

$$y_{n,k} = \theta^{-1} y_{n,k+T} \quad \textit{and} \quad x_{k+t} = \theta^{-1} x_{k+t+T}.$$

Lemma 3.8. *For all integers k, $k \ge t$, we have*

$$C_k = \theta^{-1} C_{k+T} \quad \textit{and} \quad C_k = a_k C_{k-1} + C_{k-2}.$$

Lemma 3.9. *Given a real quadratic irrational α, having a continued fraction expansion with pre-period length t, and the notation above,*

$$\frac{C_t}{x_t} = \frac{C_{t+1}}{x_{t+1}}.$$

Lemma 3.10. *For any integer $k \ge 0$,*

$$\frac{C_{t+k}}{x_{t+k}} = \frac{C_0}{x_0}.$$

With these lemmas now at our disposal, except for the shift of t in the indices of the continuants, the proof of Proposition 3.1 exactly parallels our argument in Section 2 that established Proposition 2.1. ∎

4. The asymptotic limit for Ostrowski α-decompositions. In the previous section we showed for a quadratic irrational real number α that the asymptotic period-jumping average of the number of terms in the Ostrowski

α-decomposition exists; that is, we established that $\lim_{k\to\infty} \xi(k)/k$ exists, in which

$$\xi(k) = \frac{1}{q_{k+T} - q_k} \sum_{n=q_k}^{q_{k+T}-1} \sigma(n),$$

and $\sigma(n)$ equals the number of terms in the Ostrowski α-decomposition of n.

Here in this final section, we return to the stronger asymptotic limit introduced at the beginning of this paper, which is the ultimate generalization of Lekkerkerker's work on φ. In particular, we wish to replace the weak average $\xi(k)$ with the complete average given by

$$\psi(k) = \frac{1}{q_{k+1} - q_k} \sum_{n=q_k}^{q_{k+1}-1} \sigma(n).$$

We first apply Proposition 3.1 to prove that the asymptotic limit we seek, $\lim_{k\to\infty} \psi(k)/k$, exists and equals the asymptotics computed in Section 3. We state this result as:

Proposition 4.1. *Given a quadratic irrational real number α and the notation above, the limit*

$$\lim_{k\to\infty} \frac{\psi(k)}{k}$$

exists, and is equal to the limit given in Proposition 3.1, *that is,*

$$\lim_{k\to\infty} \frac{\psi(k)}{k} = \lim_{k\to\infty} \frac{\xi(k)}{k}.$$

Before establishing Proposition 4.1, we first introduce several new quantities. To that end, we recall two objects defined at the beginning of Section 2: B_n, which denotes the number of Ostrowski α-decompositions that involve no more than the first n continuants; and D_n, which equals the total number of nonzero terms in all α-decompositions involving no more than the first n continuants.

We now let ∂_n be the number of α-decompositions whose largest continuant is q_n, and let Δ_n denote the total number of nonzero terms in all such α-decompositions. That is,

$$\partial_n = B_{n+1} - B_n = q_{n+1} - q_n \quad \text{and} \quad \Delta_n = D_{n+1} - D_n.$$

Given this new notation, we can rewrite the limit we seek as

$$\lim_{k\to\infty} \frac{\psi(k)}{k} = \lim_{k\to\infty} \frac{\Delta_k}{k\partial_k}.$$

To establish the existence of the desired limit, it will be useful to study the various sublimits taken along indices of various residue classes modulo the period length of the continued fraction expansion of α (which we recall is T). More precisely, for a fixed integer $m \geq 0$, we define ℓ_m by

$$\ell_m = \lim_{\substack{k\to\infty \\ k\equiv m\,(T)}} \frac{\psi(k)}{k} = \lim_{\substack{k\to\infty \\ k\equiv m\,(T)}} \frac{\Delta_k}{k\partial_k} = \lim_{k\to\infty} \frac{\psi(m+kT)}{m+kT},$$

and write ℓ for the value of the limit in Proposition 3.1, that is,

$$\ell = \lim_{k\to\infty} \frac{\xi(k)}{k}.$$

Thus to prove Proposition 4.1, we need to show that for each m, the limit ℓ_m exists and, furthermore, that $\ell_m = \ell$. Toward this end, we require two additional pairs of auxiliary quantities: We define

$$u_n = \sum_{i=n}^{n+T-1} \Delta_i \quad \text{and} \quad v_n = \sum_{i=n}^{n+T-1} i\partial_i,$$

and

$$\mathcal{X}_n = \Delta_{n+T} - \Delta_n \quad \text{and} \quad \mathcal{Y}_n = (n+T)\partial_{n+T} - n\partial_n.$$

In view of the definition of Δ_n and ∂_n, the quantities $u_n, v_n, \mathcal{X}_n, \mathcal{Y}_n$ are all positive integers and satisfy the identities

$$u_{n+1} = u_n + \mathcal{X}_n \quad \text{and} \quad v_{n+1} = v_n + \mathcal{Y}_n. \tag{4.1}$$

Lemma 4.2. *If the period length T satisfies $T \geq 2$, then the sequence of ratios $\{v_k/\mathcal{Y}_k\}$ is a bounded sequence.*

Proof. Given that α is a quadratic irrational real number, its partial quotients are bounded; that is, if $\alpha = [a_0, a_1, \ldots]$, then there exists an integer A satisfying $a_k \leq A$ for all $k \geq 0$. We apply this observation to deduce two basic inequalities. The first inequality bounds the growth of the continuants; in particular,

$$q_{k+T} < (A+1)^T q_k \tag{4.2}$$

for all $k \geq 0$, which is an immediate consequence of repeated applications of the recurrence relation for the continuants given in (1.1) together with our bound on the partial quotients. The second inequality bounds the growth of the ∂_k sequence; specifically, we claim that for all $k \geq 0$,

$$\partial_k < (A+1)\partial_{k+1}. \tag{4.3}$$

To establish this inequality, we first recall the following well-known identity from the theory of continued fractions (see, for example, [1] or [6]):

$$\frac{q_{k+1}}{q_k} = [a_{k+1}, a_k, \ldots, a_1].$$

This identity, together with our bound on the partial quotients of α, reveals that

$$\begin{aligned}\frac{\partial_k}{\partial_{k+1}} &= \frac{q_{k+1}-q_k}{q_{k+2}-q_{k+1}} = \frac{1-(q_k/q_{k+1})}{(q_{k+2}/q_{k+1})-1} \\ &< \frac{1}{[a_{k+2},a_{k+1},\dots,a_1]-1} \le \frac{1}{[0,a_{k+1},a_k,\dots,a_1]} \\ &= [a_{k+1},a_k,\dots,a_1] \le A+1,\end{aligned}$$

which establishes the inequality claimed in (4.3).

Repeated application of (4.3) within the definition of v_k implies

$$v_k = \sum_{i=k}^{k+T-1} i\partial_i < (k+T-1)\sum_{i=k}^{k+T-1}\partial_i < (k+T-1)T(A+1)^T\partial_{k+T-1}.$$

Hence

$$\frac{v_k}{\mathcal{Y}_k} < T(A+1)^T\left(\frac{(k+T-1)\partial_{k+T-1}}{\mathcal{Y}_k}\right), \tag{4.4}$$

and thus to establish the lemma we need only bound the ratio within the parentheses.

Toward this end, we first note that by the hypothesis that $T \ge 2$, it easily follows that

$$(k+T)q_{k+T-1} - kq_{k+1} > 0. \tag{4.5}$$

Next, we define the constant $C = 2(A+1)^T$, and select the integer K so large that for all $k \ge K$, $(k+T)/k < 2$. This inequality, together with (4.2) and (4.5), reveals

$$C > \frac{(k+T)q_{k+T}}{kq_k} > \frac{(k+T)q_{k+T}}{kq_k + (k+T)q_{k+T-1} - kq_{k+1}},$$

which, upon taking reciprocals, then adding 1, and recalling that $a_k \ge 1$ for all $k > 0$ as well as the recurrence (1.1), yields

$$\begin{aligned}1+\frac{1}{C} &< \frac{kq_k + (k+T)q_{k+T} + (k+T)q_{k+T-1} - kq_{k+1}}{(k+T)q_{k+T}} \\ &< \frac{kq_k + (k+T)a_{k+T+1}q_{k+T} + (k+T)q_{k+T-1} - kq_{k+1}}{(k+T)q_{k+T}} \\ &= \frac{kq_k + (k+T)q_{k+T+1} - kq_{k+1}}{(k+T)q_{k+T}}.\end{aligned}$$

The previous inequality implies the weaker inequality

$$\begin{aligned}&((C+1)(k+T)-1)q_{k+T} + Ckq_{k+1} \\ &\qquad < C(k+T)q_{k+T+1} + (k+T-1)q_{k+T-1} + Ckq_k,\end{aligned}$$

which is equivalent to

$$(k+T-1)(q_{k+T}-q_{k+T-1}) < C((k+T)(q_{k+T+1}-q_{k+T}) - k(q_{k+1}-q_k)).$$

This last inequality, in view of the definition of ∂_k, can be rewritten as

$$(k+T-1)\partial_{k+T-1} < C((k+T)\partial_{k+T} - k\partial_k),$$

which, given the definition of $\mathcal{Y}_k$, yields

$$\frac{(k+T-1)\partial_{k+T-1}}{\mathcal{Y}_k} < C.$$

In view of inequality (4.4), we conclude that for all $k \geq K$, we have

$$\frac{v_k}{\mathcal{Y}_k} < T(A+1)^T C,$$

which implies that this sequence of ratios is bounded for all k, as desired. ■

Proof of Proposition 4.1. In the case $T = 1$, $\psi(k) = \xi(k)$ for all $k \geq 1$, and hence the proposition is equivalent to Proposition 3.1. Thus we now consider the remaining case $T \geq 2$. We begin by observing that

$$(4.6) \qquad \frac{\xi(k)}{k} = \frac{D_{k+T} - D_k}{k(q_{k+T} - q_k)} = \frac{\sum_{i=k}^{k+T-1} \Delta_i}{k(\sum_{i=k}^{k+T-1} \partial_i)} = \frac{u_k}{k(\sum_{i=k}^{k+T-1} \partial_i)}.$$

We now claim that the difference between v_k and the denominator of the rightmost fraction above is $o(u_n)$ as $n \to \infty$, where we are adopting the *little-o* notation. To establish this claim, we apply Proposition 3.1, which in this context implies that the limit as $k \to \infty$ of (4.6) exists and is nonzero (what we named earlier as ℓ), and deduce that

$$v_k - k\sum_{i=k}^{k+T-1} \partial_i = \sum_{i=k}^{k+T-1} (i-k)\partial_i \leq \sum_{i=k}^{k+T-1} (T-1)\partial_i = o\Big(k\sum_{i=k}^{k+T-1} \partial_i\Big) = o(u_k).$$

Hence we conclude that

$$\lim_{k\to\infty} \frac{u_k}{v_k} = \lim_{k\to\infty} \frac{u_k}{k(\sum_{i=k}^{k+T-1} \partial_i) - o(u_k)} = \lim_{k\to\infty} \frac{\xi(k)}{k} = \ell.$$

In particular, this limit implies that for any fixed integer m, $0 \leq m < T$, we have

$$\lim_{\substack{k\to\infty \\ k\equiv m\,(T)}} \left(\frac{u_{k+1}}{v_{k+1}} - \frac{u_k}{v_k}\right) = 0.$$

Applying the identities from (4.1) yields

$$0 = \lim_{\substack{k\to\infty \\ k\equiv m\,(T)}} \left(\frac{u_k + \mathcal{X}_k}{v_k + \mathcal{Y}_k} - \frac{u_k}{v_k}\right) = \lim_{\substack{k\to\infty \\ k\equiv m\,(T)}} \frac{\mathcal{X}_k v_k - \mathcal{Y}_k u_k}{\mathcal{Y}_k v_k + v_k^2} = \lim_{\substack{k\to\infty \\ k\equiv m\,(T)}} \frac{\frac{\mathcal{X}_k}{\mathcal{Y}_k} - \frac{u_k}{v_k}}{1 + \frac{v_k}{\mathcal{Y}_k}}.$$

In view of Lemma 4.2, the only way for the previous limit to equal 0 is if

$$\lim_{\substack{k\to\infty \\ k\equiv m\,(T)}} \left(\frac{\mathcal{X}_k}{\mathcal{Y}_k} - \frac{u_k}{v_k}\right) = 0,$$

and thus we see that

$$\lim_{\substack{k\to\infty \\ k\equiv m\,(T)}} \frac{\mathcal{X}_k}{\mathcal{Y}_k} = \ell.$$

Given that this limit holds for every choice of m modulo T, we conclude that

$$\lim_{k\to\infty} \frac{\mathcal{X}_k}{\mathcal{Y}_k} = \ell. \tag{4.7}$$

Finally, we claim that

$$\lim_{k\to\infty} \frac{\mathcal{X}_k}{\mathcal{Y}_k} = \lim_{k\to\infty} \frac{\Delta_k}{k\partial_k},$$

which would complete our proof.

To establish this final claim, we fix $\varepsilon > 0$. Given the limit of (4.7), there exists an integer K so that for all $k \geq K$, $|\mathcal{X}_k - \mathcal{Y}_k\ell| < \mathcal{Y}_k\varepsilon$, that is,

$$\mathcal{Y}_k(\ell - \varepsilon) < \mathcal{X}_k < \mathcal{Y}_k(\ell + \varepsilon). \tag{4.8}$$

Now for a fixed integer k, $k > KT$, we can write it as $k = m + jT$ for some integers j and m satisfying $j \geq K$ and $0 \leq m < T$. Hence, we can express the ratio we wish to study as a ratio of telescoping sums and rewrite it as

$$\begin{aligned}
\frac{\Delta_k}{k\partial_k} &= \frac{\Delta_m + \mathcal{X}_m + \mathcal{X}_{m+T} + \cdots + \mathcal{X}_{k-T}}{m\partial_m + \mathcal{Y}_m + \mathcal{Y}_{m+T} + \cdots + \mathcal{Y}_{k-T}} \\
&= \frac{(\Delta_m + \sum_{i=m}^{m+(K-1)T} \mathcal{X}_i) + \sum_{i=m+KT}^{k-T} \mathcal{X}_i}{(m\partial_m + \sum_{i=m}^{m+(K-1)T} \mathcal{Y}_i) + \sum_{i=m+KT}^{k-T} \mathcal{Y}_i} \\
&= \frac{O(1) + \mathcal{X}_{m+KT} + \mathcal{X}_{m+(K+1)T} + \cdots + \mathcal{X}_{k-T}}{O(1) + \mathcal{Y}_{m+KT} + \mathcal{Y}_{m+(K+1)T} + \cdots + \mathcal{Y}_{k-T}},
\end{aligned}$$

where the *big-O* notation represents functions of α, K, and m, and thus constant functions of k. In view of the fact that the sequences $\{\mathcal{X}_k\}$ and $\{\mathcal{Y}_k\}$ both tend to infinity as $k \to \infty$, we see that for all sufficiently large k, the $O(1)$ terms will change the last ratio above by a quantity less than ε, that is, there exists K', $K' \geq KT$, so that for all $k \geq K'$ satisfying $k \equiv m \mod T$,

$$\left| \frac{\Delta_k}{k\partial_k} - \frac{\mathcal{X}_{m+KT} + \mathcal{X}_{m+(K+1)T} + \cdots + \mathcal{X}_{k-T}}{\mathcal{Y}_{m+KT} + \mathcal{Y}_{m+(K+1)T} + \cdots + \mathcal{Y}_{k-T}} \right| < \varepsilon.$$

This inequality together with (4.8) implies that

$$\begin{aligned}
\frac{\Delta_k}{k\partial_k} &< \frac{\mathcal{X}_{m+KT} + \mathcal{X}_{m+(K+1)T} + \cdots + \mathcal{X}_{k-T}}{\mathcal{Y}_{m+KT} + \mathcal{Y}_{m+(K+1)T} + \cdots + \mathcal{Y}_{k-T}} + \varepsilon \\
&< \frac{(\ell+\varepsilon)\mathcal{Y}_{m+KT} + (\ell+\varepsilon)\mathcal{Y}_{m+(K+1)T} + \cdots + (\ell+\varepsilon)\mathcal{Y}_{k-T}}{\mathcal{Y}_{m+KT} + \mathcal{Y}_{m+(K+1)T} + \cdots + \mathcal{Y}_{k-T}} + \varepsilon \\
&= \ell + 2\varepsilon.
\end{aligned}$$

A symmetric argument, now subtracting ε and using the other inequality of (4.8), reveals that

$$\ell - 2\varepsilon < \frac{\Delta_k}{k\partial_k},$$

and thus for a fixed integer m, $0 \le m < T$, we have

$$\lim_{\substack{k\to\infty \\ k\equiv m\,(T)}} \frac{\psi(k)}{k} = \lim_{\substack{k\to\infty \\ k\equiv m\,(T)}} \frac{\Delta_k}{k\partial_k} = \ell.$$

Given that this limit holds for all residue classes modulo T, we conclude that

$$\lim_{k\to\infty} \frac{\psi(k)}{k} = \ell,$$

which completes our proof of Proposition 4.1. ∎

Finally, we study equivalent quadratic irrationals, α and β, and demonstrate that the asymptotic average of the number of terms in the Ostrowski α-decomposition equals the asymptotic average associated to the Ostrowski β-decomposition. To that end, we now write $\Delta_k(\alpha)$, $\partial_k(\alpha)$, and $\ell(\alpha)$ to explicitly highlight the dependence of these quantities on α.

Proposition 4.3. *If α and β are two equivalent quadratic irrational numbers, then*

$$\ell(\alpha) = \ell(\beta).$$

To establish this result, it is enough to show that the asymptotic average for a quadratic irrational α is equal to the asymptotic average for an equivalent reduced quadratic. That is, if $\alpha = [a_0, a_1, \ldots, a_{t-1}, \overline{a_t, \ldots, a_{t+T-1}}]$ and we write α_t for the equivalent reduced quadratic $\alpha_t = [\overline{a_t, \ldots, a_{t+T-1}}]$, then to prove the proposition it is enough to show that $\ell(\alpha) = \ell(\alpha_t)$.

Given an irrational real number $\alpha = [a_0, a_1, a_2, \ldots]$, we define the nth *complete quotient*, α_n, by $\alpha_n = [a_n, a_{n+1}, a_{n+2}, \ldots]$. We now prove two important lemmas regarding the complete quotients of α. The first lemma offers two new combinatorial identities.

Lemma 4.4. *If $\alpha = [a_0, a_1, \ldots]$ is an irrational real number, then for all integers $k \ge 2$ and $n \ge 0$,*

$$\begin{aligned}\Delta_k(\alpha_n) = a_{n+1}\Delta_{k-1}(\alpha_{n+1}) &+ (a_{n+1} - 1)\partial_{k-1}(\alpha_{n+1}) \\ &+ \Delta_{k-2}(\alpha_{n+2}) + \partial_{k-2}(\alpha_{n+2})\end{aligned}$$

and

$$\partial_k(\alpha_n) = a_{n+1}\partial_{k-1}(\alpha_{n+1}) + \partial_{k-2}(\alpha_{n+2}).$$

Proof. We begin by establishing the first identity. Recall that $\Delta_k(\alpha_n)$ represents the total number of nonzero allowable coefficients, $(c_0, c_1, \ldots, c_k)$, in any Ostrowski α_n-decomposition having q_k as the largest continuant appearing in the sum. The Ostrowski conditions require that $0 \le c_1 \le a_{n+2}$. We now consider two cases.

In the case $0 \le c_1 < a_{n+2}$, the sequences of allowable coefficients are just sequences of the form $(c_0, c_1, \dots, c_k)$, in which the subsequence $(c_1, \dots, c_k)$ is any allowable sequence of coefficients for Ostrowski α_{n+1}-decompositions. The allowable values of c_0 range from 0 to $a_{n+1} - 1$. If $c_0 = 0$, then we have a contribution of $\Delta_{k-1}(\alpha_{n+1})$ distinct, allowable sequences. If c_0 is any value in the range $0 < c_0 \le a_{n+1} - 1$, then we have a contribution of $\Delta_{k-1}(\alpha_{n+1})$ for the number of nonzero, allowable coefficients from $(c_1, c_2, \dots, c_k)$ plus we must count the nonzero term c_0 for each of these expansions (hence we add $\partial_{k-1}(\alpha_{n+1})$). Thus, in this case, we have a contribution of $(a_{n+1} - 1)(\Delta_{k-1}(\alpha_{n+1}) + \partial_{k-1}(\alpha_{n+1}))$ distinct, allowable sequences. Adding these two contributions produces the total number of nonzero terms for all allowable sequences in which $0 \le c_1 < a_{n+2}$, namely $a_{n+1}\Delta_{k-1}(\alpha_{n+1}) + (a_{n+1} - 1)\partial_{k-1}(\alpha_{n+1})$.

We now consider the remaining case $c_1 = a_{n+2}$. By the Ostrowski conditions, if $c_1 = a_{n+2}$, then $c_0 = 0$; thus our sequence must be of the form $(0, a_{n+2}, c_2, \dots, c_k)$, in which $(c_2, \dots, c_k)$ is any allowable sequence of coefficients from the Ostrowski α_{n+2}-decompositions. These sequences contribute an additional $\Delta_{k-2}(\alpha_{n+2}) + \partial_{k-2}(\alpha_{n+2})$ nonzero terms (as in the previous case, we must add $\partial_{k-2}(\alpha_{n+2})$ to count each occurrence of the nonzero a_{n+2} term).

Adding the counts found in each of the two cases above produces the desired identity. The proof of the second identity, involving $\partial_k(\alpha_n)$, follows a similar method of conditioning on the value of c_1, and thus we suppress the details here. ■

The following lemma contains the essential ingredient required to prove Proposition 4.3. Namely, if $\ell(\alpha_n)$ is not equal to $\ell(\alpha_{n+1})$ or $\ell(\alpha_{n+2})$, then $\ell(\alpha_{n+1})$ and $\ell(\alpha_{n+2})$ are also not equal, and furthermore, $\ell(\alpha_n)$ is a value strictly in between $\ell(\alpha_{n+1})$ and $\ell(\alpha_{n+2})$.

Lemma 4.5. *Given a quadratic irrational real number α and the notation above, for all integers $n \ge 0$, if $\ell(\alpha_n) < \ell(\alpha_{n+1})$, then $\ell(\alpha_{n+2}) < \ell(\alpha_n)$; and if $\ell(\alpha_{n+1}) < \ell(\alpha_n)$, then $\ell(\alpha_n) < \ell(\alpha_{n+2})$.*

Proof. Applying the first identity of Lemma 4.4, we see that for any integers $k \ge 2$ and $n \ge 0$,

$$\frac{\Delta_k(\alpha_n)}{k\partial_k(\alpha_n)} = \frac{a_{n+1}\Delta_{k-1}(\alpha_{n+1})}{k\partial_k(\alpha_n)} + \frac{\Delta_{k-2}(\alpha_{n+2})}{k\partial_k(\alpha_n)} + \frac{(a_{n+1} - 1)\partial_{k-1}(\alpha_{n+1}) + \partial_{k-2}(\alpha_{n+2})}{k\partial_k(\alpha_n)}.$$

By the second identity of Lemma 4.4, we see that the rightmost ratio above is less than $1/n$ and thus is $o(1)$ as $n \to \infty$. Hence, after multiplying by 1, we have

$$\frac{\Delta_k(\alpha_n)}{k\partial_k(\alpha_n)} = \left(\frac{\Delta_{k-1}(\alpha_{n+1})}{(k-1)\partial_{k-1}(\alpha_{n+1})}\right)\left(\frac{k-1}{k}\right)\left(\frac{a_{n+1}\partial_{k-1}(\alpha_{n+1})}{\partial_k(\alpha_n)}\right) + \left(\frac{\Delta_{k-2}(\alpha_{n+2})}{(k-2)\partial_{k-2}(\alpha_{n+2})}\right)\left(\frac{k-2}{k}\right)\left(\frac{\partial_{k-2}(\alpha_{n+2})}{\partial_k(\alpha_n)}\right) + o(1).$$

Given that from Proposition 4.1, for all n, the limit

$$\ell(\alpha_n) = \lim_{k\to\infty} \frac{\Delta_k(\alpha_n)}{k\partial_k(\alpha_n)}$$

exists and, by the second identity in Lemma 4.4, both $\partial_{k-1}(\alpha_{n+1})/\partial_k(\alpha_n)$ and $\partial_{k-2}(\alpha_{n+2})/\partial_k(\alpha_n)$ are bounded functions of k, we can collect all the terms that are of the order $1/k$, as $k \to \infty$, into the $o(1)$ term to conclude

$$\frac{\Delta_k(\alpha_n)}{k\partial_k(\alpha_n)} = \left(\frac{\Delta_{k-1}(\alpha_{n+1})}{(k-1)\partial_{k-1}(\alpha_{n+1})}\right)\left(\frac{a_{n+1}\partial_{k-1}(\alpha_{n+1})}{\partial_k(\alpha_n)}\right) + \left(\frac{\Delta_{k-2}(\alpha_{n+2})}{(k-2)\partial_{k-2}(\alpha_{n+2})}\right)\left(\frac{\partial_{k-2}(\alpha_{n+2})}{\partial_k(\alpha_n)}\right) + o(1). \tag{4.9}$$

We now assume that $\ell(\alpha_n) < \ell(\alpha_{n+1})$. Again by Proposition 4.1, for a fixed $\varepsilon > 0$, there exists an integer K so that for all $k \geq K$, the following three inequalities hold:

$$\left|\frac{\Delta_k(\alpha_n)}{k\partial_k(\alpha_n)} - \ell(\alpha_n)\right| < \varepsilon,$$
$$\left|\frac{\Delta_{k-1}(\alpha_{n+1})}{(k-1)\partial_{k-1}(\alpha_{n+1})} - \ell(\alpha_{n+1})\right| < \varepsilon,$$
$$\left|\frac{\Delta_{k-2}(\alpha_{n+2})}{(k-2)\partial_{k-2}(\alpha_{n+2})} - \ell(\alpha_{n+2})\right| < \varepsilon.$$

We now require that the integer K be chosen large enough so that for all $k \geq K$, the $o(1)$ term in identity (4.9) is less than ε in absolute value.

Next, we define

$$\mathcal{P}_{k,n} = \frac{a_{n+1}\partial_{k-1}(\alpha_{n+1})}{\partial_k(\alpha_n)},$$

and note that by Lemma 4.4, it follows that

$$1 - \mathcal{P}_{k,n} = \frac{\partial_{k-2}(\alpha_{n+2})}{\partial_k(\alpha_n)}.$$

Since $\mathcal{P}_{k,n}$ and $1 - \mathcal{P}_{k,n}$ are clearly positive, we have $0 < \mathcal{P}_{k,n} < 1$ for all k

and n. Given this new notation, we can rewrite identity (4.9) as

$$\frac{\Delta_k(\alpha_n)}{k\partial_k(\alpha_n)} = \left(\frac{\Delta_{k-1}(\alpha_{n+1})}{(k-1)\partial_{k-1}(\alpha_{n+1})}\right)\mathcal{P}_{k,n} + \left(\frac{\Delta_{k-2}(\alpha_{n+2})}{(k-2)\partial_{k-2}(\alpha_{n+2})}\right)(1-\mathcal{P}_{k,n}) + o(1),$$

which is equivalent to

$$\left(\frac{\Delta_{k-2}(\alpha_{n+2})}{(k-2)\partial_{k-2}(\alpha_{n+2})}\right)(1-\mathcal{P}_{k,n}) = \frac{\Delta_k(\alpha_n)}{k\partial_k(\alpha_n)} - \left(\frac{\Delta_{k-1}(\alpha_{n+1})}{(k-1)\partial_{k-1}(\alpha_{n+1})}\right)\mathcal{P}_{k,n} + o(1).$$

Thus for any $k \geq K$, we can apply the previous three inequalities guaranteed by our original choice of K to the above identity to conclude that

$$(\ell(\alpha_{n+2}) - \varepsilon)(1-\mathcal{P}_{k,n}) < \ell(\alpha_n) + \varepsilon - (\ell(\alpha_{n+1}) - \varepsilon)\mathcal{P}_{k,n} + \varepsilon,$$

which, as $0 < \mathcal{P}_{k,n} < 1$, implies that

$$\ell(\alpha_{n+2})(1-\mathcal{P}_{k,n}) < \ell(\alpha_n)(1-\mathcal{P}_{k,n}) - (\ell(\alpha_{n+1}) - \ell(\alpha_n))\mathcal{P}_{k,n} + 4\varepsilon,$$

or equivalently,

$$\ell(\alpha_{n+2}) < \ell(\alpha_n) - (\ell(\alpha_{n+1}) - \ell(\alpha_n))\frac{\mathcal{P}_{k,n}}{1-\mathcal{P}_{k,n}} + \frac{4\varepsilon}{1-\mathcal{P}_{k,n}}.$$

We now claim that $\mathcal{P}_{k,n}$ is bounded away from both 0 and 1 by a fixed, positive constant. Assuming the validity of this claim for the moment, we observe that as $\varepsilon \to 0$, the previous inequality reveals that

$$\ell(\alpha_{n+2}) < \ell(\alpha_n) - (\ell(\alpha_{n+1}) - \ell(\alpha_n))C$$

for some positive constant C. Therefore our hypothesis that $\ell(\alpha_n) < \ell(\alpha_{n+1})$ implies that

$$\ell(\alpha_{n+2}) < \ell(\alpha_n),$$

which is the desired inequality asserted in the lemma.

Therefore, to complete the proof, we need only establish our claim that $\mathcal{P}_{k,n}$ is bounded away from both 0 and 1 by a fixed, positive constant. We begin by recalling that because the partial quotients of α are bounded, there exists an integer A so that $a_n \leq A$ for all n. By the second identity of Lemma 4.4, we have

$$\partial_{k-1}(\alpha_{n+1}) = a_{n+2}\partial_{k-2}(\alpha_{n+2}) + \partial_{k-3}(\alpha_{n+3}) > \partial_{k-2}(\alpha_{n+2}),$$

and hence another application of Lemma 4.4 reveals

$$\begin{aligned}\partial_n(\alpha_k) &= a_{n+1}\partial_{k-1}(\alpha_{n+1}) + \partial_{k-2}(\alpha_{n+2}) \\ &< (a_{n+1}+1)\partial_{k-1}(\alpha_{n+1}) \leq (A+1)\partial_{k-1}(\alpha_{n+1}).\end{aligned}$$

Consequently, we are able to bound $\mathcal{P}_{k,n}$ away from 0 by observing that

$$\mathcal{P}_{k,n} = \frac{a_{n+1}\partial_{k-1}(\alpha_{n+1})}{\partial_k(\alpha_n)} \geq \frac{a_{n+1}(A+1)^{-1}\partial_k(\alpha_n)}{\partial_k(\alpha_n)} \geq \frac{1}{A+1} > 0,$$

and similarly, we are able to bound $\mathcal{P}_{k,n}$ away from 1 by noting that

$$1 - \mathcal{P}_{k,n} = \frac{\partial_{k-2}(\alpha_{n+2})}{\partial_k(\alpha_n)} \geq \frac{(A+1)^{-2}\partial_k(\alpha_n)}{\partial_k(\alpha_n)} = \frac{1}{(A+1)^2} > 0.$$

These two inequalities establish our claim and therefore prove the first assertion of the lemma, namely, if $\ell(\alpha_n) < \ell(\alpha_{n+1})$, then $\ell(\alpha_{n+2}) < \ell(\alpha_n)$. The proof of the second assertion, stating that if $\ell(\alpha_{n+1}) < \ell(\alpha_n)$, then $\ell(\alpha_n) < \ell(\alpha_{n+2})$, follows from a symmetric argument and hence we suppress the details. ∎

Proof of Proposition 4.3. We proceed by contradiction and assume that there exists an integer n such that $\ell(\alpha_n) \neq \ell(\alpha_{n+1})$; and without loss of generality we now assume that $\ell(\alpha_n) < \ell(\alpha_{n+1})$. By Lemma 4.5 we have $\ell(\alpha_n) > \ell(\alpha_{n+2})$, which implies $\ell(\alpha_{n+1}) > \ell(\alpha_{n+2})$. Thus we are able to apply Lemma 4.5 again. Continuing inductively, we find that

$$\cdots < \ell(\alpha_{n+4}) < \ell(\alpha_{n+2}) < \ell(\alpha_n) < \ell(\alpha_{n+1}) < \ell(\alpha_{n+3}) < \ell(\alpha_{n+5}) < \cdots,$$

in particular, we see that $\ell(\alpha_{n+i}) \neq \ell(\alpha_{n+j})$ for all positive integers i and j, $i \neq j$. However, given that the length of the pre-period of the continued fraction expansion of α is t and its period length is T, it follows that $\alpha_{n+t} = \alpha_{n+t+T}$ and thus plainly we have $\ell(\alpha_{n+t}) = \ell(\alpha_{n+t+T})$, which is a contradiction. Therefore $\ell(\alpha_n) = \ell(\alpha_{n+1})$ for all integers $n \geq 0$, and thus, in particular, $\ell(\alpha) = \ell(\alpha_t)$, as desired. ∎

Assembling all our work allows us to prove our main result in just a few lines.

Proof of Theorem 1.3. By Proposition 4.1, we know that the asymptotic average of the number of terms in the Ostrowski α-decomposition exists and equals the limit given in Proposition 3.1. Proposition 4.3 asserts that this limit is equal to the corresponding limit associated with α^*, a reduced quadratic equivalent to α. Another application of Proposition 4.1 implies that this limit is equal to the one given in Proposition 2.1, which completes our proof. ∎

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