ACTA ARITHMETICA
153.3 (2012)

The prime number theorem for Beurling's generalized numbers. New cases

by

Jan-Christoph Schlage-Puchta and Jasson Vindas (Ghent)

1. Introduction. Let $1 < p_1 \le p_2 \le \cdots$ be a non-decreasing sequence of real numbers tending to infinity. Following Beurling [3], we shall call such a sequence $P = \{p_k\}_{k=1}^{\infty}$ a set of *generalized prime numbers*. We arrange the set of all possible products of generalized primes in a non-decreasing sequence $1 < n_1 \le n_2 \le \cdots$, where every n_k is repeated as many times as it can be represented by $p_{\nu_1}^{\alpha_1} \dots p_{\nu_m}^{\alpha_m}$ with $\nu_j < \nu_{j+1}$. The sequence $\{n_k\}_{k=1}^{\infty}$ is called the set of *generalized integers*.

Let π denote the distribution of the generalized prime numbers,

$$\pi(x) = \pi_P(x) = \sum_{p_k < x} 1, \tag{1.1}$$

and let N denote the distribution of the generalized integers,

$$N(x) = N_P(x) = \sum_{n_k < x} 1. \tag{1.2}$$

Beurling's problem is then to find conditions on the function N which ensure the validity of the prime number theorem (PNT), i.e.,

$$\pi(x) \sim \frac{x}{\log x}, \qquad x \to \infty. \tag{1.3}$$

In his seminal work [3], Beurling proved that the condition

$$N(x) = ax + O\left(\frac{x}{\log^{\gamma} x}\right), \qquad x \to \infty, \tag{1.4}$$

where $a > 0$ and $\gamma > 3/2$, suffices for the PNT to hold. If $\gamma = 3/2$, then the PNT need not hold, as shown first by Beurling by a continuous analog of a

2010 *Mathematics Subject Classification*: Primary 11N05, 11N80, 11M45; Secondary 40E05, 41A60, 46F10.
Key words and phrases: Beurling generalized numbers, the prime number theorem, Tauberian theorems, asymptotic behavior of generalized functions, Dirichlet series, non-vanishing of the zeta function.

DOI: 10.4064/aa153-3-5

generalized prime number system, and then by Diamond [6] who exhibited an explicit example of generalized primes not satisfying the PNT. For further studies about Beurling's generalized numbers we refer the reader to [1, 5–7, 13, 22, 23, 38, 39, 40].

The present article studies new cases of the prime number theorem for generalized primes. Our main goal is to show the following theorem.

THEOREM 1.1. *Suppose there exist constants $a > 0$ and $\gamma > 3/2$ such that*

$$N(x) = ax + O\left(\frac{x}{\log^\gamma x}\right) \quad (\mathrm{C}), \qquad x \to \infty. \tag{1.5}$$

Then the prime number theorem (1.3) *holds.*

In (1.5) the symbol (C) stands for asymptotics *in Cesàro sense* [10]. Explicitly, this means that there exists some (possibly large) $m \in \mathbb{N}$ such that the following average estimate is satisfied:

$$\int_1^x \frac{N(t) - at}{t}\left(1 - \frac{t}{x}\right)^m dt = O\left(\frac{x}{\log^\gamma x}\right), \qquad x \to \infty. \tag{1.6}$$

We might have written (C, m) in (1.5) if (1.6) holds for a specific m; however, the value of m will be totally unimportant for our arguments and we therefore choose to omit it from the notation.

Naturally, if Beurling's condition (1.4) holds, then (1.6) is automatically satisfied *for all* $m \in \mathbb{N}$. Thus, Theorem 1.1 is a natural extension of Beurling's theorem. Observe that our theorem is sharp, namely, the PNT does not necessarily hold if $\gamma = 3/2$ in (1.5), as shown by Diamond's counterexample itself.

To demonstrate that Theorem 1.1 really generalizes Beurling's result, we construct a number system for which the counting function becomes regular only after smoothing.

PROPOSITION 1.2. *There exists a system of generalized integers such that $|N(x) - x| > c(x/\log^{4/3} x)$ infinitely often, for some constant $c > 0$, but $N(x) = x + O(x/\log^{5/3} x)$ in Cesàro sense. Furthermore, for this number system we have $\pi(x) = x/\log x + O(x/\log^{4/3-\varepsilon} x)$ for any $\varepsilon > 0$.*

Kahane [13] has provided yet another extension of Beurling's theorem by showing that the L^2 hypothesis

$$\int_1^\infty \left|\frac{(N(t) - at)\log t}{t}\right|^2 \frac{dt}{t} < \infty, \tag{1.7}$$

for some $a > 0$, also implies the PNT. The sufficiency of (1.7) was conjectured by Bateman and Diamond [1]. Observe that (1.7) includes Beurling's condition (1.4). On the other hand, (1.5) and (1.7) have completely different nature. We therefore believe that it is very unlikely that either of them implies the other. It would be interesting to find concrete examples of generalized number systems supporting this claim. It would also be of interest to find a general condition that would include both (1.5) and (1.7).

In addition to the PNT, we will show that the Möbius function of a generalized number system has mean value zero, provided that the condition (1.5) is satisfied for $\gamma > 3/2$. We actually conjecture that $\gamma > 1$ is enough for the Möbius function to have zero mean value (cf. Section 5); such a conjecture is suggested by the results of Zhang [38].

Our approach to the proof of Theorem 1.1 is through Tauberian theorems, and it is close to Landau–Ikehara's way to the PNT [12, 20] (see also [1]). Therefore, we shall first study the zeta function associated to a generalized prime number system satisfying the hypothesis of Theorem 1.1. In Section 3, we show the non-vanishing property of the zeta function on the line $\Re e\, s = 1$; it will be the main ingredient for the application of Tauberian arguments. The use of methods from distribution theory and generalized asymptotics (asymptotic analysis on distribution spaces [10, 26, 36]) is crucial for our arguments in this part of the article; it provides a very convenient language of translation between the condition (1.5) and the key properties of the zeta function. Therefore, we have chosen to include a preliminary section, Section 2.2, explaining the notation of generalized asymptotics. In Section 4 we show several Tauberian theorems for Laplace transforms and Dirichlet series. The results from Section 4 are inspired by a recent distributional proof of the PNT (for ordinary primes) obtained by the second author and R. Estrada [34]; some of such Tauberian results were implicitly obtained by the second author in his dissertation [32, Chap. II]. These Tauberian theorems involve local pseudo-function boundary behavior, which relaxes the boundary requirements to a minimum. We point out that J. Korevaar has recently made extensive use of local pseudo-function boundary behavior in complex Tauberian theory [14, 15, 17] and in number theory [15, 18]. We deduce Theorem 1.1 in Section 5, where we also provide other related results. Finally, in Section 6 we construct a number system proving Proposition 1.2.

2. Preliminaries and notation. Throughout this article, $P = \{p_k\}_{k=1}^{\infty}$ stands for a fixed set of generalized prime numbers with generalized integers $\{n_k\}_{k=1}^{\infty}$. The functions N and π are given by (1.2) and (1.1), respectively. We shall always assume that the distribution of the generalized integers satisfies (1.5) for some $\gamma > 0$. The letter s stands for complex numbers $s = \sigma + it$.

2.1. Functions related to generalized primes. We denote by $\Lambda = \Lambda_P$ the *von Mangoldt function* of P, defined on the set of generalized integers as

$$\Lambda(n_k) = \begin{cases} \log p_j & \text{if } n_k = p_j^m, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

The *Chebyshev function* of P is defined as usual by

$$\psi(x) = \psi_P(x) = \sum_{p_k^m < x} \log p_k = \sum_{n_k < x} \Lambda(n_k). \tag{2.2}$$

Observe that (1.5) (even for $\gamma > 0$) implies ([9, Lem. 3], [14], [37]) the ordinary asymptotic behavior

$$N(x) \sim ax, \qquad x \to \infty, \tag{2.3}$$

hence the Dirichlet series $\sum n_k^{-s}$ is easily seen to have abscissa of convergence less than or equal to 1. The *zeta function* of P is then the analytic function

$$\zeta(s) = \zeta_P(s) = \sum_{k=1}^{\infty} \frac{1}{n_k^s}, \qquad \Re e\, s > 1. \tag{2.4}$$

By a well known result [1, Lem. 2E], the relation (1.3) is equivalent to

$$\psi(x) \sim x. \tag{2.5}$$

Our approach to the PNT (Theorem 1.1) will be to show (2.5).

The *Möbius function* of P is defined on the generalized integers by

$$\mu(n_k) = \mu_P(n_k) = \begin{cases} (-1)^m & \text{if } n_k = p_{\nu_1} \dots p_{\nu_m} \text{ with } \nu_j < \nu_{j+1}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

Finally, note [1, Lem. 2D] that

$$\sum_{k=1}^{\infty} \frac{\Lambda(n_k)}{n_k^s} = -\frac{\zeta'(s)}{\zeta(s)}, \qquad \Re e\, s > 1. \tag{2.7}$$

Likewise, one readily verifies the identity

$$\sum_{k=1}^{\infty} \frac{\mu(n_k)}{n_k^s} = \frac{1}{\zeta(s)}, \qquad \Re e\, s > 1. \tag{2.8}$$

2.2. Distributions and generalized asymptotics. We shall make extensive use of the theory of Schwartz distributions and some elements from asymptotic analysis on distribution spaces.

We denote by $\mathcal{D}(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$ the Schwartz spaces of test functions consisting of smooth compactly supported functions and smooth rapidly decreasing functions, respectively, with their usual topologies. Their dual spaces, the spaces of distributions and tempered distributions, are denoted by $\mathcal{D}'(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$, respectively. We let $\mathcal{D}(0,\infty)$ denote the space of smooth functions supported on the interval $(0, \infty)$, and $\mathcal{D}'(0, \infty)$ is its dual. Furthermore, $\mathcal{D}_{L^2}(\mathbb{R})$ denote the space of smooth functions with all derivatives belonging to $L^2(\mathbb{R})$, $\mathcal{D}'_{L^2}(\mathbb{R})$ is its dual. The space $\mathcal{D}_{L^2}(\mathbb{R})$ is the intersection of all Sobolev spaces while $\mathcal{D}'_{L^2}(\mathbb{R})$ is their union. Moreover, $\mathcal{D}'_{L^1}(\mathbb{R})$ is the dual of $\dot{\mathcal{B}}(\mathbb{R})$, the space of smooth functions with all derivatives tending to 0 at $\pm\infty$. We refer to [27] for the well known properties of all these spaces (see also [4, 10, 26, 36, 37]).

We use the Fourier transform

$$\hat{\phi}(t) = \int_{-\infty}^{\infty} e^{-itx}\phi(x)\,dx \quad \text{ for } \phi \in \mathcal{S}(\mathbb{R}); \tag{2.9}$$

it is defined by duality on $\mathcal{S}'(\mathbb{R})$, that is, if $f \in \mathcal{S}'(\mathbb{R})$ its Fourier transform is the tempered distribution given by

$$\langle \hat{f}(t), \phi(t)\rangle = \langle f(x), \hat{\phi}(x)\rangle. \tag{2.10}$$

Let $f \in \mathcal{S}'(\mathbb{R})$ be supported in $[0, \infty)$. Its Laplace transform is the analytic function

$$\mathcal{L}\{f; s\} = \mathcal{L}\{f(x); s\} = \langle f(x), e^{-sx}\rangle, \quad \Re e\, s > 0. \tag{2.11}$$

The relation between the Laplace and Fourier transforms [4, 36] is given by $\hat{f}(t) = \lim_{\sigma\to 0^+} \mathcal{L}\{f; \sigma + it\}$, where the limit is taken in the weak topology of $\mathcal{S}'(\mathbb{R})$.

We shall employ various standard tempered distributions, following the notation of [10]. The Heaviside function is denoted by H; it is simply the characteristic function of $(0, \infty)$. The Dirac delta "function" δ is defined by $\langle \delta(x), \phi(x)\rangle = \phi(0)$; note that $H'(x) = \delta(x)$ (the derivative is understood in the distributional sense, of course). The Fourier transform of H is $\hat{H}(t) = -i/(t - i0)$, where the latter is defined as the distributional boundary value, on $\Re e\, s = 0$, of the analytic function $1/s$, $\Re e\, s > 0$, i.e.,

$$\left\langle \frac{-i}{t - i\sigma}, \phi(t) \right\rangle = \lim_{\sigma\to 0^+} \int_{-\infty}^{\infty} \frac{\phi(t)}{\sigma + it}\,dt, \quad \phi \in \mathcal{S}(\mathbb{R}).$$

We now turn to asymptotic analysis of distributions [10, 25, 26, 37], the so called *generalized asymptotics*.

Let $f \in \mathcal{D}'(\mathbb{R})$. A relation of the form

$$\lim_{h\to\infty} f(x + h) = \beta \quad \text{in } \mathcal{D}'(\mathbb{R}) \tag{2.12}$$

means that the limit is taken in the weak topology of $\mathcal{D}'(\mathbb{R})$, that is, for each $\phi \in \mathcal{D}(\mathbb{R})$,

$$\lim_{h\to\infty} \langle f(x + h), \phi(x)\rangle = \lim_{h\to\infty} (f * \check{\phi})(h) = \beta \int_{-\infty}^{\infty} \phi(x)\,dx, \tag{2.13}$$

where $\check{\phi}(\cdot) = \phi(-\,\cdot)$ and $*$ denotes convolution. Relation (2.12) is an example of the so-called *S-asymptotics* of generalized functions. We can also study error terms by introducing S-asymptotic boundedness. Let ρ be a positive function. Then we write

$$f(x+h) = O(\rho(h)) \quad \text{as } h \to \infty \text{ in } \mathcal{D}'(\mathbb{R}) \tag{2.14}$$

if for each $\phi \in \mathcal{D}(\mathbb{R})$ we have $\langle f(x+h), \phi(x)\rangle = O(\rho(h))$ for large values of h. The little o symbol and S-asymptotics as $h \to -\infty$ are defined in a similar way. With this notation we might write (2.12) as $f(x+h) = \beta + o(1)$ as $h \to \infty$ in $\mathcal{D}'(\mathbb{R})$. We may also talk about S-asymptotics in other spaces of distributions with a clear meaning. For example, if we write in (2.12) the space $\mathcal{S}'(\mathbb{R})$ instead of $\mathcal{D}'(\mathbb{R})$, it means that $f \in \mathcal{S}'(\mathbb{R})$ and (2.13) holds for $\phi \in \mathcal{S}(\mathbb{R})$. We refer the reader to [24, 26] for further properties of S-asymptotics of distributions.

On the other hand, we may attempt to study the asymptotic behavior of a distribution by looking at the behavior at large scale of the dilates $f(\lambda x)$ as $\lambda \to \infty$. In this case, we encounter the concept of *quasiasymptotic behavior* of distributions [10, 26, 31–33, 35, 37]. We will study, in connection to the PNT, two particular cases of this type of behavior: a limit of the form

$$\lim_{\lambda \to \infty} f(\lambda x) = \beta \quad \text{in } \mathcal{D}'(0,\infty), \tag{2.15}$$

and quasiasymptotic estimates

$$f(\lambda x) = O\left(\frac{\lambda^{\nu}}{\log^{\alpha} \lambda}\right) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{D}'(\mathbb{R}). \tag{2.16}$$

Needless to say, (2.15) and (2.16) should always be interpreted in the weak topology of the corresponding space, that is, after evaluation on test functions.

2.3. Pseudo-functions. A tempered distribution $f \in \mathcal{S}'(\mathbb{R})$ is called a *pseudo-function* if $\hat{f} \in C_0(\mathbb{R})$, that is, $\hat{f}$ is a continuous function which vanishes at $\pm\infty$.

The distribution $f \in \mathcal{D}'(\mathbb{R})$ is said to be *locally a pseudo-function* if it coincides with a pseudo-function on each finite open interval. The property of being locally a pseudo-function admits a characterization [15] in terms of a generalized "Riemann–Lebesgue lemma". Indeed, f is locally a pseudo-function if and only if $e^{iht} f(t) = o(1)$ as $|h| \to \infty$ in the weak topology of $\mathcal{D}'(\mathbb{R})$, i.e., for each $\phi \in \mathcal{D}(\mathbb{R})$,

$$\lim_{|h| \to \infty} \langle f(t), e^{iht}\phi(t)\rangle = 0. \tag{2.17}$$

We may call (2.17) the *generalized Riemann–Lebesgue lemma* for local pseudo-functions. It is clear that if $f \in L^1_{\mathrm{loc}}(\mathbb{R})$. Then it is locally a pseudo-function, due to the classical Riemann–Lebesgue lemma.

Two cases of local pseudo-functions will be of vital importance below. Let f be the Fourier transform of an element from $\mathcal{D}'_{L^1}(\mathbb{R})$. Then f is locally a pseudo-function. This follows directly from the fact that Fourier transforms of elements from $\mathcal{D}'_{L^1}(\mathbb{R})$ are continuous functions [27, p. 256]. Let now f be the Fourier transform of a distribution from $\mathcal{D}'_{L^2}(\mathbb{R})$ and let $g \in L^2_{\mathrm{loc}}(\mathbb{R})$; by a remark in [27, p. 256], the product $g \cdot f \in \mathcal{D}'(\mathbb{R})$ is a well defined distribution and it is locally a pseudo-function.

Let $G(s)$ be analytic on $\Re e\, s > \alpha$. We shall say that G *has local pseudo-function boundary behavior* on the line $\Re e\, s = \alpha$ if it has distributional boundary values [4] on that line,

$$\lim_{\sigma \to \alpha^+} \int_{-\infty}^{\infty} G(\sigma + it)\phi(t)\, dt = \langle f(t), \phi(t) \rangle, \qquad \phi \in \mathcal{D}(\mathbb{R}),$$

and the boundary distribution $f \in \mathcal{D}'(\mathbb{R})$ is locally a pseudo-function.

3. Properties of the zeta function. Our arguments for the proof of Theorem 1.1 rely on the properties of the zeta function. We shall derive such properties from those of the following special distribution. Define

$$v(x) = v_P(x) = \sum_{k=1}^{\infty} \frac{1}{n_k} \delta(x - \log n_k). \tag{3.1}$$

Let us verify that v is a tempered distribution. We see that $g(x) = e^{-x} N(e^x)$ is a bounded function, hence $g \in \mathcal{S}'(\mathbb{R})$; therefore, $v = g' + g \in \mathcal{S}'(\mathbb{R})$.

The distribution v is intimately related to the zeta function. In fact, its Laplace transform is, for $\Re e\, s > 0$,

$$\mathcal{L}\{v; s\} = \langle v(x), e^{-sx} \rangle = \sum_{k=1}^{\infty} \frac{1}{n_k^{s+1}} = \zeta(s+1). \tag{3.2}$$

Taking the boundary values of (3.2) on $\Re e\, s = 0$, in the distributional sense, we obtain the Fourier transform of v,

$$\hat{v}(t) = \zeta(1 + it). \tag{3.3}$$

Observe that we are interpreting (3.3) in the distributional sense and not as equality of functions, i.e., for each $\phi \in \mathcal{S}(\mathbb{R})$,

$$\langle \hat{v}(t), \phi(t) \rangle = \lim_{\sigma \to 1^+} \int_{-\infty}^{\infty} \zeta(\sigma + it)\phi(t)\, dt. \tag{3.4}$$

Next, we provide a lemma which establishes the main connection between (1.5) and the S-asymptotic properties of v.

Lemma 3.1. *The following assertions are equivalent:*

(i) *In the sense of* (1.6),

$$N(x) = ax + O\left(\frac{x}{\log^\gamma x}\right) \quad (\mathrm{C}), \qquad x \to \infty. \tag{3.5}$$

(ii) *There exists $j \in \mathbb{N}$ such that*

$$\sum_{n_k < x} \left(1 - \frac{n_k}{x}\right)^j = \frac{ax}{j+1} + O\left(\frac{x}{\log^\gamma x}\right), \qquad x \to \infty. \tag{3.6}$$

(iii) *In the sense of quasiasymptotic behavior,*

$$N'(\lambda x) = \sum_{k=1}^{\infty} \delta(\lambda x - n_k) = aH(x) + O\left(\frac{1}{\log^\gamma \lambda}\right) \tag{3.7}$$

as $\lambda \to \infty$ in $\mathcal{D}'(\mathbb{R})$.

(iv) *In the sense of S-asymptotic behavior,*

$$v(x+h) = \sum_{k=1}^{\infty} \frac{1}{n_k} \delta(x + h - \log n_k) = a + O\left(\frac{1}{h^\gamma}\right) \tag{3.8}$$

as $h \to \infty$ in $\mathcal{S}'(\mathbb{R})$.

Proof. (i)⇔(iii). The equivalence between (3.6) and (3.7) is a direct consequence of the structural theorem for quasiasymptotic boundedness [33, Thm. 5.8] (see also [32, p. 311]).

(ii)⇔(iii). By the structural theorem for quasiasymptotic boundedness [33, Thm. 5.8], (3.5) holds if and only if

$$\frac{N(\lambda x) - a\lambda x H(x)}{\lambda x} = O\left(\frac{1}{\log^\gamma \lambda}\right) \qquad \text{as } \lambda \to \infty \text{ in } \mathcal{D}'(\mathbb{R}).$$

The analog to [31, Thm. 4.1] for the big O symbol ([33, Thm. 10.71, p. 322]) implies that the above relation is equivalent to

$$N(\lambda x) = a\lambda x H(x) + O\left(\frac{\lambda}{\log^\gamma \lambda}\right) \qquad \text{as } \lambda \to \infty \text{ in } \mathcal{D}'(\mathbb{R}),$$

which turns out to be equivalent to (3.7) ([33, Thm. 5.8] again).

(iii)⇔(iv). By [24, Thm. 3] (see also [25, Prop. 10.2]), the S-asymptotics (3.8) is equivalent to the same S-asymptotics in $\mathcal{D}'(\mathbb{R})$. Multiplication by e^{x+h} shows that this holds if and only if

$$\sum_{k=1}^{\infty} \delta(x + h - \log n_k) = ae^{x+h} + O\left(\frac{e^h}{h^\gamma}\right),$$

or in terms of test functions,

$$e^{-h} \sum_{k=1}^{\infty} \varphi(\log n_k - h) = a \int_{-\infty}^{\infty} e^t \varphi(t)\, dt + O\left(\frac{1}{h^\gamma}\right), \qquad h \to \infty,$$

for each $\varphi \in \mathcal{D}(\mathbb{R})$. By writing $\lambda = e^h$ and $\phi(x) = \varphi(\log x)$, we find that (iv) is equivalent to

$$\frac{1}{\lambda}\sum_{k=1}^{\infty}\phi\biggl(\frac{n_k}{\lambda}\biggr) = a\int_0^{\infty}\phi(x)\,dx + O\biggl(\frac{1}{\log^{\gamma}\lambda}\biggr), \qquad \lambda \to \infty,$$

for each $\phi \in \mathcal{D}(0,\infty)$, i.e., the quasiasymptotic behavior (3.7) but in the space $\mathcal{D}'(0,\infty)$. Now, using the big O analog to [31, Thm. 4.1] once again, we obtain the equivalence between (iv) and (iii). ∎

Define the remainder distribution $E_1 = v - aH$. By Lemma 3.1, E_1 has the S-asymptotic bound

$$E_1(x+h) = O\biggl(\frac{1}{|h|^{\gamma}}\biggr) \quad \text{as } |h| \to \infty \text{ in } \mathcal{S}'(\mathbb{R}). \tag{3.9}$$

Indeed, $E_1(x+h) = v(x+h) - aH(x+h) = a(1-H(x+h)) + v(x+h) - a = O(1/h^{\gamma})$ as $h \to \infty$, in $\mathcal{S}'(\mathbb{R})$. On the other hand, the estimate as $h \to -\infty$ follows easily from the fact that E_1 has support in $[0,\infty)$.

We now obtain the first properties of the zeta function. From the properties of E_1, we can show the continuity of $\zeta(s)$ on $\Re e\, s = 1$, $s \neq 1$.

Proposition 3.2. *Let N satisfy* (3.5) *with $\gamma > 1$. Then $\zeta(s) - a/(s-1)$ extends to a continuous function on $\Re e\, s \geq 1$, that is,*

$$\zeta(1+it) + \frac{ia}{t - i0} \in C(\mathbb{R}). \tag{3.10}$$

Consequently, $t\zeta(1+it)$ is continuous over the whole real line and so $\zeta(1+it)$ is continuous in $\mathbb{R}\setminus\{0\}$.

Proof. Observe that $\hat{E}_1(t) = \zeta(1+it) + ia/(t-i0)$. By (3.9), $(E_1 * \phi)(h) = O(|h|^{-\gamma})$ as $h \to \infty$, and so $E_1 * \phi \in L^1(\mathbb{R})$, for each $\phi \in \mathcal{S}(\mathbb{R})$. This is precisely Schwartz's characterization [27, p. 201] of the space $\mathcal{D}'_{L^1}(\mathbb{R})$, and so $E_1 \in \mathcal{D}'_{L^1}(\mathbb{R})$. Therefore [27, p. 256], $\hat{E}_1$ is continuous. ∎

The ensuing lemma is the first step toward the non-vanishing property of ζ on $\Re e\, s = 1$, in the case $\gamma > 3/2$.

Lemma 3.3. *Let N satisfy* (3.5) *with $1 < \gamma < 2$. For each $t_0 \neq 0$ there exists $C = C_{t_0} > 0$ such that for $1 < \sigma < 2$,*

$$|\zeta(\sigma + it_0) - \zeta(1+it_0)| < C(\sigma - 1)^{\gamma - 1}. \tag{3.11}$$

Proof. Find $\varphi \in \mathcal{D}(\mathbb{R})$ such that $0 \notin \operatorname{supp}\hat{\varphi}$ and $\hat{\varphi}(t) = 1$ for t in a small neighborhood of t_0. Set $f = v * \varphi$. Then $\hat{f}(t) = \hat{\varphi}(t)\hat{v}(t) = \hat{\varphi}(t)\zeta(1+it)$. The function f is smooth and satisfies $f(x) = O(|x|^{-\gamma})$. Indeed, $\widehat{(H * \varphi)}(t) = -it^{-1}\hat{\varphi}(t) \in \mathcal{D}(\mathbb{R})$, thus $H * \varphi \in \mathcal{S}(\mathbb{R})$, and the estimate for f follows from (3.9). In particular $f \in L^1(\mathbb{R})$ and so $\hat{f}$ is continuous.

Define the harmonic function

$$U(\sigma+it)=\langle f(x)H(x),e^{-itx}e^{-\sigma x}\rangle+\langle f(x)H(-x),e^{-itx}e^{\sigma x}\rangle;$$

then U is a harmonic representation of $\hat{f}$ on $\Re e\, s>0$, in the sense that $\lim_{\sigma\to 0^+}U(\sigma+it)=\hat{f}(t)$, uniformly over $\mathbb{R}$ [4]. We claim that $\zeta(\sigma+it_0)=U(\sigma-1+it_0)+O(\sigma-1)$, $\sigma\to 1^+$. Consider $V(s)=\zeta(s+1)-U(s)$, harmonic on $\Re e\, s>0$. Because $\zeta(1+it)-\hat{f}(t)=0$ in a neighborhood of t_0, it follows that $V(s)$ converges uniformly to 0 in a neighborhood of t_0 as $\Re e\, s\to 0^+$. Then, by applying the reflection principle [29, Sect. 3.4] to the real and imaginary parts of V, we find that V admits a harmonic extension to a (complex) neighborhood of t_0. Therefore, $U(s)-\zeta(s+1)=V(s)=O(|s-t_0|)$ for $\Re e\, s>0$ sufficiently close to t_0. This shows the claim.

We now show $|U(\sigma-1+it_0)-\zeta(1+t_0)|=O((\sigma-1)^{\gamma-1})$ as $\sigma\to 1^+$; the estimate (3.11) follows immediately from this assertion. The estimate $f(x)=O(|x|^{-\gamma})$ and [10, Lem. 3.9.4, p. 153] imply the quasiasymptotics

$$e^{-i\lambda t_0 x}f(\lambda x)H(x)=\mu_+\frac{\delta(x)}{\lambda}+O\bigg(\frac{1}{\lambda^{\gamma}}\bigg)\quad \text{as } \lambda\to\infty \text{ in } \mathcal{S}'(\mathbb{R}),$$
$$e^{-i\lambda t_0 x}f(\lambda x)H(-x)=\mu_-\frac{\delta(x)}{\lambda}+O\bigg(\frac{1}{\lambda^{\gamma}}\bigg)\quad \text{as } \lambda\to\infty \text{ in } \mathcal{S}'(\mathbb{R}),$$

where $\mu_\pm=\int_0^\infty f(\pm x)e^{\mp it_0x}\,dx$, and so $\mu_-+\mu_+=\int_{-\infty}^\infty f(x)e^{-it_0x}\,dx=\hat{f}(t_0)=\zeta(1+t_0)$. The two quasiasymptotics imply

$$\begin{aligned}U\bigg(\frac{1}{\lambda}+it_0\bigg)&=\langle e^{-it_0x}f(x)H(x),e^{-x/\lambda}\rangle+\langle e^{-it_0x}f(x)H(-x),e^{x/\lambda}\rangle\\&=\lambda\langle e^{-\lambda it_0x}f(\lambda x)H(x),e^{-x}\rangle+\lambda\langle e^{-\lambda it_0x}f(\lambda x)H(-x),e^{x}\rangle\\&=\mu_+\langle\delta(x),e^{-x}\rangle+\mu_-\langle\delta(x),e^{x}\rangle+O\bigg(\frac{1}{\lambda^{\gamma-1}}\bigg)\\&=\mu_++\mu_-+O\bigg(\frac{1}{\lambda^{\gamma-1}}\bigg)=\zeta(1+t_0)+O\bigg(\frac{1}{\lambda^{\gamma-1}}\bigg),\qquad \lambda\to\infty.\end{aligned}$$

Writing $\sigma-1=1/\lambda$ proves the assertion. ∎

We are now in a position to show the non-vanishing of $\zeta(s)$ on $\Re e\, s=1$, $s\neq 1$, for $\gamma>3/2$. Actually, the proof is the same as the one of [1, Thm. 8E], but we sketch it for the sake of completeness.

THEOREM 3.4. *Let N satisfy* (3.5) *with $\gamma>3/2$. Then $t\zeta(1+it)\neq 0$ for all $t\in\mathbb{R}$. Consequently, $1/((s-1)\zeta(s))$ converges locally and uniformly to a continuous function as $\Re e\, s\to 1^+$.*

Proof. The proof is in essence the classical argument of Hadamard [14, p. 63]. Without loss of generality we assume that $3/2<\gamma<2$. We use the representation [1, Lem. 2C], which is also valid under our hypothesis,

$$\zeta(s) = \exp\bigg(\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{j}p_k^{-js}\bigg),$$

to conclude [1, Lem. 8B] that for any $m \in \mathbb{N}$ and $t_0 \in \mathbb{R}$,

$$|\zeta(\sigma)|^{m+1}|\zeta(\sigma+it_0)|^{2m}\prod_{j=1}^{m}|\zeta(\sigma+i(j+1)t_0)|^{2m-2j} \geq 1.$$

If we now fix $t_0 \neq 0$ and m, Proposition 3.2 and the above inequality imply the existence of $A = A_{m,t_0} > 0$ such that for $1 < \sigma < 2$,

$$1 \leq \frac{A|\zeta(\sigma+it_0)|^{2m}}{(\sigma-1)^{m+1}},$$

or, what is the same,

$$D(\sigma-1)^{1/2+1/(2m)} \leq |\zeta(\sigma+it_0)|,$$

with $D = A^{1/(2m)}$.

Suppose we had $\zeta(1+it_0) = 0$. Choose m such that $1/2+1/(2m) < \gamma-1$. By the inequality (3.11) of Lemma 3.3, we would have

$$D(\sigma-1)^{1/2+1/(2m)} \leq |\zeta(\sigma+it_0)| < C(\sigma-1)^{\gamma-1},$$

which is certainly absurd. Therefore, $\zeta(1+it) \neq 0$ for all $t \in \mathbb{R}\setminus\{0\}$. ∎

We now obtain the boundary behavior of $-\zeta'(s)/\zeta(s) - 1/(s-1)$.

LEMMA 3.5. *Let N satisfy* (3.5) *with $\gamma > 3/2$. Then*

$$-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}$$

has local pseudo-function boundary behavior on the line $\Re e\, s = 1$.

Proof. We work with $s+1$ instead of s and analyze the boundary behavior on $\Re e\, s = 0$. Recall E_1 was defined before (3.9). As observed in the proof of Proposition 3.2, $\zeta(s+1) - a/s = \mathcal{L}\{E_1; s\}$ converges uniformly over compact sets to the continuous function $\hat{E}_1$, as $\Re e\, s \to 1^+$, so, by Theorem 3.4,

$$G_1(s) = \frac{1}{s\zeta(s+1)}\bigg(\zeta(s+1) - \frac{a}{s}\bigg), \qquad \Re e\, s > 0,$$

converges uniformly over finite intervals to a continuous function, and thus its boundary value is a pseudo-function. Define $E_2(x) = (xE_1(x))'$. A quick computation shows that

$$\mathcal{L}\{E_2; s\} = -s\zeta'(s+1) - \frac{a}{s}, \qquad \Re e\, s > 0.$$

Since $-\zeta'(s+1)/\zeta(s+1) - 1/s = G_2(s) - G_1(s)$, where

$$G_2(s) = \frac{1}{s\zeta(s+1)}\mathcal{L}\{E_2; s\},$$

it is enough to see that $G_2(s)$ has local pseudo-function boundary behavior. Now, the S-asymptotic bound (3.9) implies that $E_2(x+h) = O(|h|^{-\gamma+1})$, and, because of the hypothesis $\gamma > 3/2$, we have $E_2 * \phi \in L^2(\mathbb{R})$ for all $\phi \in \mathcal{S}(\mathbb{R})$. But this is precisely Schwartz's characterization [27, p. 201] of the distribution space $\mathcal{D}'_{L^2}(\mathbb{R})$; thus $E_2 \in \mathcal{D}'_{L^2}(\mathbb{R})$. As remarked in Section 2.3, multiplying the Fourier transform of an element of $\mathcal{D}'_{L^2}(\mathbb{R})$ by an element of $L^2_{\mathrm{loc}}(\mathbb{R})$ always gives rise to a distribution which is locally a pseudo-function. It remains to observe that $G_2(s)$ tends in $\mathcal{D}'(\mathbb{R})$ to $\hat{E}_2(t)/(t\zeta(1+it))$, which in view of the previous argument and the continuity of $1/(t\zeta(1+it))$ is locally a pseudo-function. ∎

For future applications, we need a Chebyshev type upper estimate:

Lemma 3.6. *Let N satisfy* (3.5) *with $\gamma > 3/2$. Then $\psi(x) = O(x)$ as $x \to \infty$.*

Proof. Set $\tau(x) = e^{-x}\psi(e^x)$. The crude estimate $\tau(x) \le xe^{-x}N(e^x) = O(x)$ shows that $\tau \in \mathcal{S}'(\mathbb{R})$. Integration by parts in (2.7) gives

$$\mathcal{L}\{\tau; s\} - \frac{1}{s} = \frac{1}{s+1}\left(-\frac{\zeta'(s+1)}{\zeta(s+1)} - \frac{1}{s} - 1\right), \qquad \Re e\, s > 0,$$

which, by Lemma 3.5, has local pseudo-function boundary behavior on $\Re e\, s = 0$. Thus, we can write $\tau = H + g$, where $\hat{g}$ is locally a pseudo-function. Pick $\phi \in \mathcal{S}(\mathbb{R})$ non-negative and with $\hat{\phi} \in \mathcal{D}(\mathbb{R})$. Write φ for the inverse Fourier transform of ϕ. Then

$$\int_{-h}^{\infty} \tau(x+h)\phi(x)\,dx = \int_{-h}^{\infty} \phi(x)\,dx + \langle \hat{g}(t), e^{iht}\varphi(t)\rangle = O(1) + o(1) = O(1).$$

Notice that for x and h positive, $e^{-x}\tau(h) \le \tau(x+h)$, which follows from the non-decreasing property of ψ. Finally, setting $C = \int_0^\infty e^{-x}\phi(x)\,dx > 0$,

$$\tau(h) = C^{-1}\int_0^\infty e^{-x}\tau(h)\phi(x)\,dx \le C^{-1}\int_0^\infty \tau(x+h)\phi(x)\,dx = O(1). \ ∎$$

4. Tauberian theorems. In this section we show Tauberian theorems from which we shall derive later the PNT and prove in Section 5 that the Möbius function has mean value zero.

The ensuing theorem is from the second author's dissertation [32, Chap. II]; for the method used in the proof, see also [34].

THEOREM 4.1. *Let S be a non-decreasing function supported on $[0,\infty)$ and satisfying the growth condition $S(x)=O(e^x)$. Hence, the function*

$$\mathcal{L}\{dS;s\}=\int_0^\infty e^{-sx}\,dS(x) \tag{4.1}$$

is analytic on $\Re e\, s>1$. If there exists a constant β such that the function

$$G(s)=\mathcal{L}\{dS;s\}-\frac{\beta}{s-1} \tag{4.2}$$

has local pseudo-function boundary behavior on the line $\Re e\, s=1$, then

$$S(x)\sim\beta e^x,\qquad x\to\infty. \tag{4.3}$$

Proof. By subtracting $S(0)H(x)$, we may assume that $S(0)=0$, so the derivative of S is given by the Stieltjes integral $\langle S'(x),\phi(x)\rangle=\int_0^\infty\phi(x)\,dS(x)$. Let $M>0$ be such that $S(x)<Me^x$. Define $V(x)=e^{-x}S'(x)$.

Since $e^{-x}S(x)$ is a bounded function, it is a tempered distribution and its set of translates is, in particular, weakly bounded; because differentiation is a continuous operator, the set of translates of $(e^{-x}S(x))'$ is weakly bounded as well. Since $(e^{-x}S(x))'=-e^{-x}S(x)+V(x)$, we conclude that $V\in\mathcal{S}'(\mathbb{R})$ and $V(x+h)=O(1)$ as $h\to\infty$ in $\mathcal{S}'(\mathbb{R})$.

The Laplace transform of V on $\Re e\, s>0$ is given by

$$\mathcal{L}\{V;s\}=\langle V(x),e^{-sx}\rangle=\int_0^\infty e^{-(s+1)x}\,dS(x)=\mathcal{L}\{dS;s+1\}.$$

Observe that

$$\begin{aligned}\hat{V}(t)+\frac{\beta i}{t-i0}&=\lim_{\sigma\to0^+}\mathcal{L}\{V(x)-\beta H(x);\sigma+it\}\\&=\lim_{\sigma\to0^+}G(1+\sigma+it)\quad\text{ in }\mathcal{D}'(\mathbb{R}).\end{aligned}$$

Hence, by hypothesis, $\hat{V}(t)+i\beta/(t-i0)$ is locally a pseudo-function, therefore $e^{iht}(\hat{V}(t)+i\beta/(t-i0))=o(1)$ as $h\to\infty$ in $\mathcal{D}'(\mathbb{R})$ (Riemann–Lebesgue lemma (2.17)). Taking the inverse Fourier transform, we conclude that $V(x+h)=\beta H(x+h)+o(1)=\beta+o(1)$ as $h\to\infty$ in $\mathcal{F}(\mathcal{D}'(\mathbb{R}))$, the Fourier image of $\mathcal{D}'(\mathbb{R})$. Using the density of $\mathcal{F}(\mathcal{D}(\mathbb{R}))$ in $\mathcal{S}(\mathbb{R})$ and the boundedness of $V(x+h)$, we conclude, by applying the Banach–Steinhaus theorem [30], that $V(x+h)=\beta+o(1)$ actually in $\mathcal{S}'(\mathbb{R})$. Multiplying by e^{x+h}, we obtain $S'(x+h)\sim e^{x+h}$ in $\mathcal{D}'(\mathbb{R})$.

Let $g(u)=S(\log u)$. Then $\lim_{\lambda\to\infty}g'(\lambda u)=\beta$ in $\mathcal{D}'(0,\infty)$: indeed, let $\phi\in\mathcal{D}(0,\infty)$; then

$$\begin{aligned}\langle g'(\lambda u),\phi(u)\rangle &= -\frac{1}{\lambda^2}\int_0^\infty S(\log u)\phi'\left(\frac{u}{\lambda}\right)du\\ &= -\frac{1}{\lambda}\int_{-\infty}^\infty S(x+\log\lambda)e^x\phi'(e^x)\,dx = \frac{1}{\lambda}\langle S'(x+\log\lambda),\phi(e^x)\rangle\\ &= \int_{-\infty}^\infty e^x\phi(e^x)\,dx + o(1) = \int_0^\infty \phi(u)\,du + o(1),\qquad \lambda\to\infty.\end{aligned}$$

At this stage of the proof, we could apply first [31, Thm. 4.1] and then [9, Lem. 3] (see also [37]) to g' and automatically conclude that $S(\log u)\sim\beta u$, which is equivalent to (4.3). Alternatively, we can proceed rather directly as follows. Let $\varepsilon>0$ be an arbitrary small number; find $\phi_1,\phi_2\in\mathcal{D}(0,\infty)$ with the following properties: $0\le\phi_i\le 1$, $\operatorname{supp}\phi_1\subseteq(0,1]$, $\phi_1(u)=1$ on $[\varepsilon,1-\varepsilon]$, $\operatorname{supp}\phi_2\subseteq(0,1+\varepsilon]$, and finally, $\phi_2(u)=1$ on $[\varepsilon,1]$. Evaluating the quasiasymptotic limit of g' at ϕ_2, we obtain

$$\begin{aligned}\limsup_{\lambda\to\infty}\frac{g(\lambda)}{\lambda} &= \limsup_{\lambda\to\infty}\frac{1}{\lambda}\int_0^\lambda dg(u) \le \limsup_{\lambda\to\infty}\frac{g(\varepsilon\lambda)}{\lambda} + \frac{1}{\lambda}\int_0^\infty \phi_2\left(\frac{u}{\lambda}\right)dg(u)\\ &\le M\varepsilon + \lim_{\lambda\to\infty}\langle g'(\lambda u),\phi_2(u)\rangle = M\varepsilon + \beta\int_0^\infty \phi_2(u)\,du \le \beta+\varepsilon(M+\beta).\end{aligned}$$

Likewise, using now ϕ_1, we easily obtain

$$\beta - 2\varepsilon\beta \le \liminf_{\lambda\to\infty}\frac{g(\lambda)}{\lambda}.$$

Since ε was arbitrary, (4.3) follows. ∎

The hypothesis $S(x)=O(e^x)$ can be dropped from Theorem 4.1, as follows from Korevaar's distributional version of the Wiener–Ikehara theorem [15]; however, Theorem 4.1 will be enough for our future purposes.

Theorem 4.1 implies the following Tauberian result for Dirichlet series.

THEOREM 4.2. *Let $\{\lambda_k\}_{k=1}^\infty$ be a non-decreasing sequence of positive real numbers such that $\sum_{\lambda_k<x}1\sim ax$ for some $a\ge 0$. Let $\{c_k\}_{k=1}^\infty$ be a sequence bounded from below, i.e., there exist $k_0, M>0$ such that $c_k>-M$ for all $k\ge k_0$. Suppose that $\sum_{\lambda_k<x}c_k=O(x)$. If there exists a constant β such that*

$$G(s)=\sum_{k=1}^\infty\frac{c_k}{\lambda_k^s}-\frac{\beta}{s-1},\qquad \Re e\, s>1, \tag{4.4}$$

has local pseudo-function boundary behavior on $\Re e\, s=1$, then

$$\sum_{\lambda_k<x}c_k\sim\beta x,\qquad x\to\infty. \tag{4.5}$$

Proof. Observe first that

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k^s} - \frac{a}{s-1}$$

has local pseudo-function boundary behavior on $\Re e\, s = 1$. Indeed, it tends to $\hat{g}$, where

$$g(x) = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \delta(x - \log \lambda_k) - aH(x) = \left(1 + \frac{d}{dx}\right)\left(e^{-x} \sum_{\lambda_k < e^x} 1\right) - aH(x),$$

and in view of the assumption on $\{\lambda_k\}_{k=1}^{\infty}$ we have $g(x+h) = o(1)$. So $\hat{g}$ satisfies the Riemann–Lebesgue lemma (2.17), and hence is locally a pseudo-function. Set now $S(x) = \sum_{\lambda_k < e^x} (c_k + M)$. Then $S(x) = O(e^x)$, and

$$\int_0^{\infty} e^{-st}\, dS(t) = M \sum_{k=1}^{\infty} \frac{1}{\lambda_k^s} + \sum_{k=1}^{\infty} \frac{c_k}{\lambda_k^s}.$$

Thus, S satisfies the hypotheses of Theorem 4.1, and so

$$S(x) \sim (\beta + aM)e^x,$$

from which (4.5) follows. ∎

REMARK 4.3. In Theorem 4.2, if $c_k \geq 0$, then it is not necessary to impose any condition on the non-decreasing sequence $\{\lambda_k\}_{k=1}^{\infty}$, that is, we can remove the assumption on the asymptotics of $\sum_{\lambda_k < x} 1$. For this assertion, considering $M = 0$, the second part of the previous proof works as well.

REMARK 4.4. We emphasize that for $\lambda_k = n_k$ in Theorem 4.2, generalized integers with N satisfying (3.5) with $\gamma > 0$, the hypothesis $N(x) = \sum_{n_k < x} 1 \sim ax$, $x \to \infty$, is always satisfied (see (2.3) in Section 2.1).

Theorem 4.2 generalizes Korevaar's (apparently) weaker version of the classical Wiener–Ikehara theorem (see [16, Thm. 1.1]). It is then interesting to mention that, as shown in [16], the Wiener–Ikehara theorem is itself a consequence of such a result (and thus of Theorem 4.2). We also remark that the results from [16] were obtained via purely complex variable methods; here we use purely distributional methods!

We need a variant of Theorem 4.2 with Tauberian hypothesis of *slow oscillation*. Recall that a *complex-valued* function τ is called *slowly oscillating* [14] if

$$\lim_{h \to 0^+} \limsup_{x \to \infty} |\tau(x+h) - \tau(x)| = 0. \tag{4.6}$$

If (4.6) holds, then [14, p. 33] there exist $x_0, C > 0$ such that for $x \geq x_0$ and $h \geq 0$,

$$|\tau(x+h) - \tau(x)| \leq C(h+1). \tag{4.7}$$

Consequently, it is easy to see that $\tau(x) = O(x)$ as $x \to \infty$.

THEOREM 4.5. *Let* $T \in L^1_{\mathrm{loc}}(\mathbb{R})$ *be such that* $\operatorname{supp} T \subseteq [0,\infty)$ *and* $\tau(x) = e^{-x}T(x)$ *is slowly oscillating. Suppose there exists* $\beta \in \mathbb{R}$ *such that*

$$G(s) = \mathcal{L}\{T; s\} - \frac{\beta}{s-1} \tag{4.8}$$

has local pseudo-function boundary behavior on the line $\Re e\, s = 1$. *Then*

$$T(x) \sim \beta e^x, \qquad x \to \infty. \tag{4.9}$$

Proof. We can assume that T is real-valued, otherwise consider its real and imaginary parts separately. Observe that since $\tau(x) = O(x)$, it is a tempered distribution and the Laplace transform of T is automatically well defined for $\Re e\, s > 1$. Since the Fourier transform of a compactly supported distribution is an entire function, we can assume that (4.7) holds in fact *for all* $x \geq 0$ and $h \geq 0$.

We first need to show that $e^{-x}T(x) = \tau(x)$ is bounded. For this, pick $\eta \in \mathcal{D}(\mathbb{R})$ such that $\eta(0) = 1/(2\pi)$ and set $\varphi = \hat{\eta}$. Next, for h large enough,

$$\begin{aligned}\langle \tau(x+h), \varphi(x)\rangle &= O(1) + \langle \tau(x) - H(x), \hat{\eta}(x-h)\rangle \\ &= O(1) + \langle G(1+it), \eta(t)e^{iht}\rangle = O(1) + o(1) = O(1),\end{aligned}$$

because $G(1+it)$ is locally a pseudo-function. Observe $\int_{-\infty}^{\infty} \varphi(x)\,dx = 2\pi\eta(0) = 1$. Now, in view of (4.7),

$$\begin{aligned}|\tau(h)| &\leq O(1) + \Big|\int_{-h}^{\infty} (\tau(x+h) - \tau(h))\varphi(x)\,dx\Big| + |\tau(h)| \int_{-\infty}^{-h} |\varphi(x)|\,dx \\ &\leq O(1) + O(1)\int_{-\infty}^{\infty} (|x|+1)|\varphi(x)|\,dx + O(h)\int_{-\infty}^{-h} |\varphi(x)|\,dx = O(1).\end{aligned}$$

By adding a term of the form $Ke^xH(x)$, we may now assume $T \geq 0$. Define $S(x) = \int_0^x T(t)\,dt$. The function S is increasing and, as $T(x) = O(e^x)$, has growth $S(x) = O(e^x)$. Furthermore,

$$\mathcal{L}\{S'; s\} - \frac{\beta}{s-1} = \mathcal{L}\{T; s\} - \frac{\beta}{s-1};$$

hence, by Theorem 4.1,

$$\int_0^x T(t)\,dt \sim \beta e^x. \tag{4.10}$$

In particular, the ordinary asymptotic behavior (4.10) implies the S-asymptotic behavior $\int_0^{x+h} T(t)\,dt \sim \beta e^{x+h}$ as $h \to \infty$ in $\mathcal{D}'(\mathbb{R})$. Differentiating the latter and then dividing by e^{x+h}, we obtain

$$\tau(x+h) = \beta + o(1) \quad \text{as } h \to \infty \text{ in } \mathcal{D}'(\mathbb{R}). \tag{4.11}$$

The final step is to evaluate the S-asymptotic (4.11) at a suitable test function. Let $\varepsilon > 0$. Choose $\phi \in \mathcal{D}(\mathbb{R})$ non-negative and supported in $[0,\varepsilon]$ such that $\int_0^\varepsilon \phi(x)\,dx = 1$. Then

$$\begin{aligned}\limsup_{h\to\infty}|\tau(h)-\beta| &\le \limsup_{h\to\infty}\Big|\beta-\int_0^\infty \tau(t)\phi(t-h)\,dt\Big| \\ &\quad+\limsup_{h\to\infty}\Big|\int_0^\infty (\tau(t)-\tau(h))\phi(t-h)\,dt\Big| \\ &= \limsup_{h\to\infty}\Big|\int_h^{h+\varepsilon} (\tau(t)-\tau(h))\phi(t-h)\,dt\Big| \\ &\le \limsup_{h\to\infty}\sup_{t\in[h,h+\varepsilon]}|\tau(t)-\tau(h)|.\end{aligned}$$

Since ε was arbitrary, the slow oscillation (4.6) implies $\lim_{h\to\infty}\tau(h)=\beta$, which in turn is the same as (4.9). ∎

5. The prime number theorem and related results. The prime number theorem, Theorem 1.1, now follows directly from our previous work. Indeed, it is enough to combine Lemmas 3.5 and 3.6 with Theorem 4.2.

We give a second application of the Tauberian theorems from Section 4. We now turn our attention to the Möbius function. We show its mean value is zero and $\sum \mu(n_k)/n_k = 0$, whenever $\gamma > 3/2$. Remarkably, it is well known [8, 19] that for ordinary prime numbers either of these conditions is itself equivalent to the PNT; however, that is no longer the case for generalized number systems [38].

THEOREM 5.1. *Let N satisfy* (3.5) *with $\gamma > 3/2$. Then*

$$\lim_{x\to\infty}\frac{1}{x}\sum_{n_k<x}\mu(n_k)=0. \tag{5.1}$$

Proof. By using formula (2.8), Proposition 3.2, and Theorem 3.4, we see that

$$\sum_{k=1}^\infty \frac{\mu(n_k)}{n_k^s} = (s-1)\cdot\frac{1}{(s-1)\zeta(s)}$$

extends continuously to $\Re e\, s \ge 1$, and so this Dirichlet series has local pseudo-function boundary behavior on $\Re e\, s = 1$. Applying Theorem 4.2, we obtain (5.1) at once. ∎

COROLLARY 5.2. *Let N satisfy* (3.5) *with $\gamma > 3/2$. Then*

$$\sum_{k=1}^\infty \frac{\mu(n_k)}{n_k} = 0. \tag{5.2}$$

Proof. Exactly as in [8, p. 565], the relation (5.2) may be deduced from (5.1) by completely elementary manipulations. ∎

It is worth pointing out a possible connection between Theorem 5.1 and the results from [38]. Zhang has shown [38, Cor. 2.5] that if N satisfies Beurling's classical condition (1.4) for $\gamma > 1$, then the Möbius function has zero mean value. Observe that this result of Zhang, in combination with Diamond's example of a generalized number system satisfying (1.4) with $\gamma = 3/2$ but violating the PNT, yields a curious fact that we have already mentioned: The PNT is not equivalent to (5.1) in the context of Beurling's generalized primes. Comparison of Zhang's result and Theorem 5.1 strongly suggests the following conjecture:

Conjecture 5.3. If N satisfies the Cesàro condition (3.5) for some $\gamma > 1$, then (5.1) still holds true.

We end this section by proving an analog of Newman's Tauberian theorem [14, 21] for generalized integers. Newman used his Tauberian theorem to derive the prime number theorem for ordinary prime numbers. Indeed, Newman's way [21] to the prime number theorem was to show first Corollary 5.2 above (for ordinary integers) from his Tauberian theorem, whence the prime number theorem follows, as shown first by Landau [19]. For applications of Newman's method in the theory of generalized numbers (subject to (1.4) with $\gamma > 2$) we refer to Bekehermes's dissertation [2]. We give a version of Newman's Tauberian theorem for local pseudo-function boundary behavior rather than under the much stronger assumption of analytic continuation. Remarkably, the result holds even for $\gamma > 0$. Observe that Theorem 5.1 and Corollary 5.2 follow from Theorem 5.4.

Theorem 5.4. *Let N satisfy* (3.5) *with $\gamma > 0$. Suppose that $\{c_k\}_{k=1}^{\infty}$ is a bounded sequence of complex numbers. Let*

$$F(s) = \sum_{k=1}^{\infty} \frac{c_k}{n_k^s}, \qquad \Re e\, s > 1. \tag{5.3}$$

If there exists β such that

$$G(s) = \frac{F(s) - \beta}{s - 1}, \qquad \Re e\, s > 1, \tag{5.4}$$

has local pseudo-function boundary behavior on $\Re e\, s = 1$, then

$$\sum_{k=1}^{\infty} \frac{c_k}{n_k} = \beta \tag{5.5}$$

and

$$\lim_{x \to \infty} \frac{1}{x} \sum_{n_k < x} c_k = 0. \tag{5.6}$$

Proof. Observe that $F(s) = (s-1)G(s) + \beta$ has local pseudo-function boundary behavior on $\Re e\, s = 1$. Thus, Theorem 4.2 and Remark 4.4 imply $M(u) := \sum_{n_k<u} c_k = o(u)$, so (5.6) has been established. Set $T(x) = e^x \int_0^\infty e^{-t} M(e^t)\,dt$, so that $e^{-x}T(x)$ is slowly oscillating. Notice that T is the convolution of $M(e^x)$ and $e^x H(x)$. Hence

$$\begin{aligned}\mathcal{L}\{T; s\} &= \mathcal{L}\{M(e^x); s\}\mathcal{L}\{e^x H(x); s\} \\ &= \frac{1}{s}\mathcal{L}\Big\{\sum_{k=1}^{\infty} c_k \delta(x - \log n_k); s\Big\} \int_0^\infty e^{-(s-1)x}\,dx \\ &= \frac{1}{s(s-1)} \sum_{k=1}^{\infty} \frac{c_k}{n_k^s} = \frac{F(s)}{s(s-1)}.\end{aligned}$$

We then have

$$\mathcal{L}\{T; s\} - \frac{\beta}{s-1} = \frac{1}{s}(G(s) - \beta),$$

which has local pseudo-function boundary behavior on $\Re e\, s = 1$. Theorem 4.5 yields $\lim_{x\to\infty} \int_0^x e^{-t} M(e^t)\,dt = \beta$; thus, a change of variables shows

$$\lim_{x\to\infty} \int_0^x \frac{M(u)}{u^2}\,du = \beta. \tag{5.7}$$

We now derive (5.5) from (5.6) and (5.7):

$$\sum_{n_k<x} \frac{c_k}{n_k} = \int_0^x u^{-1}\,dM(u) = \frac{M(x)}{x} + \int_0^x \frac{M(u)}{u^2}\,du = \beta + o(1). \ \blacksquare$$

Remark 5.5. The analytic function (5.4) has local pseudo-function boundary behavior on $\Re e\, s = 1$ if and only if it has pseudo-function boundary behavior in a neighborhood of $s = 1$ on that line and F, given by (5.3), has local pseudo-function boundary behavior on $\{s : \Re e\, s = 1,\ s \neq 1\}$. So Theorem 5.4 might be reformulated in these terms.

6. Necessity of the Cesàro means. In this section we construct a number system proving that Theorem 1.1 is a proper generalization of Beurling's result. We shall do so by removing and doubling suitable blocks of ordinary primes.

Let x_i be a sequence of integers, where x_1 is chosen so large that for all $x > x_1$ the interval $[x, x + x/\log^{1/3} x]$ contains more than $x/(2\log^{4/3} x)$ prime numbers, and $x_{i+1} = \lfloor 2^{\sqrt[4]{x_i}} \rfloor$. Clearly, $i = O(\log\log x_i)$, and we may assume that $i \le \log^{1/6} x_i$.

Close to each x_i, we define four disjoint intervals $I_{i,1}, \dots, I_{i,4}$, where $I_{i,2} = [x_i, x_i + x_i/\log^{1/3} x_i]$ and $I_{i,3}$ is the contiguous interval starting at $x_i + x_i/\log^{1/3} x_i$ which contains as many prime numbers as $I_{i,2}$. Observe that each of the intervals $I_{i,2}$ and $I_{i,3}$ has at least $x_i/(2\log^{4/3} x_i)$ prime numbers. Consequently, the length of $I_{i,3}$ is at most $O(x_i/\log^{1/3} x_i)$, as follows from the classical prime number theorem. We further choose $I_{i,1}$ and $I_{i,4}$ in such a way that $I_{i,1}$ has upper bound x_i, $I_{i,4}$ has lower bound equal to the upper bound of $I_{i,3}$, and they have the properties of the following claim:

Claim 6.1. *There are intervals $I_{i,1}$ and $I_{i,4}$ as above such that $I_{i,1}$ and $I_{i,4}$ contain the same number of primes, and*

$$\prod_{\nu=1}^{i} \prod_{p\in I_{\nu,1}\cup I_{\nu,3}} \left(1-\frac{1}{p}\right)^{(-1)^{\nu+1}} \prod_{p\in I_{\nu,2}\cup I_{\nu,4}} \left(1-\frac{1}{p}\right)^{(-1)^{\nu}} = 1+O\left(\frac{1}{x_i}\right).$$

In addition, the lengths of $I_{i,1}$ and $I_{i,4}$ are $O(ix_i/\log^{1/3} x_i)$ and each of them contains $O(ix_i/\log^{4/3} x_i)$ primes.

Proof. We proceed recursively. Assume that we have defined $I_{i,1}$ and $I_{i,4}$ for $i \le k-1$ so that they satisfy the claim and additionally $A_i^{(-1)^i} > 1$, where

$$A_i = \prod_{\nu=1}^{i} \prod_{p\in I_{\nu,1}\cup I_{\nu,3}} \left(1-\frac{1}{p}\right)^{(-1)^{\nu+1}} \prod_{p\in I_{\nu,2}\cup I_{\nu,4}} \left(1-\frac{1}{p}\right)^{(-1)^{\nu}}.$$

Assume first that k is even. Then $A_{k-1} < 1$. We examine the product

$$A_k = A_{k-1} \prod_{p\in I_{k,1}\cup I_{k,3}} \left(1-\frac{1}{p}\right)^{-1} \prod_{p\in I_{k,2}\cup I_{k,4}} \left(1-\frac{1}{p}\right),$$

where $I_{k,1}$ and $I_{k,4}$ are to be defined. We pair primes $p_1 \in I_{k,2}$ and $p_2 \in I_{k,3}$. The contribution of such a pair to this product is

$$1 > \left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)^{-1} = 1-\frac{p_2-p_1}{p_1p_2}+O\left(\frac{1}{p_2p_1}\right).$$

Hence, the product over $I_{k,2}$ and $I_{k,3}$ is

$$\exp\left(O\left(\frac{x_k}{\log^{4/3} x_k}\cdot\frac{|I_{k,2}\cup I_{k,3}|}{x_k^2}\right)\right) = \exp\left(O\left(\frac{1}{\log^{5/3} x_k}\right)\right)$$

and thus

$$\begin{aligned} A_{k-1}\prod_{p\in I_{k,3}} \left(1-\frac{1}{p}\right)^{-1} \prod_{p\in I_{k,2}} \left(1-\frac{1}{p}\right) &= \exp\left(O\left(\frac{1}{\log^4 x_k}+\frac{1}{\log^{5/3} x_k}\right)\right) \\ &= 1+O\left(\frac{k}{\log^{5/3} x_k}\right). \end{aligned}$$

We now increase $I_{k,1}$ and $I_{k,4}$ in such a way that at each stage both intervals contain the same number of primes. Whenever we add one prime to both intervals, we add a factor > 1 to the product, and we stop doing so as soon as the product becomes greater than 1. Since the initial product is in fact $1 + O(k/\log^{5/3} x_k)$, we need $O(kx_k/\log^{4/3} x_k)$ primes to do so. It follows from the classical prime number theorem that $I_{k,1}$ and $I_{k,4}$ both have length $O(kx_k/\log^{1/3} x_k)$. Thus, whenever we add one more prime, the value of the product changes by $\exp(O(2k(x_k \log^{1/3} x_k)^{-1})) = \exp(o(x_k^{-1}))$. The first time the product supersedes 1, it does so by $O(x_k^{-1})$, and it follows that $A_k = 1+O(x_k^{-1})$ and $A_k > 1$. When k is odd, we proceed in a similar fashion. We now keep adding primes to $I_{k,1}$ and $I_{k,4}$ until the product becomes less than one; at that stage, we obtain $A_k < 1$ and again $A_k = 1 + O(x_k^{-1})$.

Therefore, the intervals $I_{k,1}$ and $I_{k,4}$ so constructed satisfy the requirements of the claim and in addition $A_k^{(-1)^k} > 1$. ∎

We define x_k^- to be the least integer in $I_{k,1}$, and x_k^+ the largest integer in $I_{k,4}$. Note that $x_k/\log^{1/3} x_k \le x_k^+ - x_k^- = O(kx_k/\log^{1/3} x_k)$. Since $k < \log^{1/6} x_k$, we therefore have $x_k^+ < 2x_k$ and $x_k^- > 2^{-1}x_k$, for all sufficiently large k.

Now define a sequence $P = \{p_\nu\}_{\nu=1}^{\infty}$ of generalized primes as follows. We choose one prime element for each prime number which is not in any of the intervals $I_{i,j}$; if i is even, no prime elements in $I_{i,2} \cup I_{i,4}$ and two prime elements for all prime numbers which are in one of the intervals $I_{i,1}, I_{i,3}$; if i is odd, no prime elements in $I_{i,1} \cup I_{i,3}$ and two prime elements for all prime numbers which belong to one of the intervals $I_{i,2}, I_{i,4}$. We denote by $\mathcal{G}$ the generalized number system associated to P. Let P_k be the generalized prime set which is constructed in the same way, but taking only the intervals $I_{i,j}$ with $i \le k$ into account; $\mathcal{G}_k$ then denotes its associated set of generalized integers. Let $\pi(x) = \pi_P(x)$ and $N(x) = N_P(x)$ be respectively the counting functions of P and $\mathcal{G}$. Furthermore, we denote by $N_k(x) = N_{P_k}(x)$ the one corresponding to $\mathcal{G}_k$.

We claim that N and π have the properties stated in Proposition 1.2; more precisely:

Proposition 6.2. *We have $|N(x) - x| > c(x/\log^{4/3} x)$ infinitely often, for some constant $c > 0$; however,*

$$N(x) = x + O\left(\frac{x}{\log^{5/3} x}\right) \quad (\mathrm{C}, 1),$$

i.e., its first order Cesàro mean $\overline{N}$ has asymptotics

$$\overline{N}(x) := \int_1^x \frac{N(t)}{t}\,dt = x + O\left(\frac{x}{\log^{5/3} x}\right). \tag{6.1}$$

For this system,

$$\pi(x) = \frac{x}{\log x} + O\left(\frac{x \log\log x}{\log^{4/3} x}\right).$$

Since we defined $\mathcal{G}$ via its prime elements, we control $\pi(x)$ completely. In fact, the bound for the number of prime elements follows immediately from the definition of P and the classical prime number theorem. Our task is then to compute the behavior of $N(x)$. We first estimate $N_k(x)$ for x much larger than x_k.

LEMMA 6.3. *If* $\exp(8 \log^2 x_k) \leq x < \exp(x_k^{3/5})$, *then*

$$N_k(x) = x + O\left(\frac{x}{\log^{5/3} x}\right) \tag{6.2}$$

and

$$\overline{N}_k(x) := \int_1^x \frac{N_k(t)}{t}\, dt = x + O\left(\frac{x}{\log^{5/3} x}\right). \tag{6.3}$$

Proof. Denote by $f(n)$ the number of elements of norm n in $\mathcal{G}_k$. Then $f(n)$ is multiplicative and satisfies

$$f(p^\alpha) = \begin{cases} \alpha + 1 & \text{if } \exists\ 2i \leq k\colon p \in I_{2i,1} \cup I_{2i,3}, \\ 0 & \text{if } \exists\ 2i \leq k\colon p \in I_{2i,2} \cup I_{2i,4}, \\ 0 & \text{if } \exists\ 2i+1 \leq k\colon p \in I_{2i+1,1} \cup I_{2i+1,3}, \\ \alpha + 1 & \text{if } \exists\ 2i+1 \leq k\colon p \in I_{2i+1,2} \cup I_{2i+1,4}, \\ 1 & \text{otherwise.} \end{cases}$$

Define the function $g(n)$ by the relation $f(n) = \sum_{d|n} g(d)$. Then g is multiplicative, and we have

$$g(p^\alpha) = \begin{cases} 1 & \text{if } f(p) = 2, \\ -1 & \text{if } f(p) = 0 \text{ and } \alpha = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Denote by $\mathcal{H}_k$ the set of all integers which have only prime divisors in $\bigcup_{i \leq k} I_{i,j}$, and for each integer n, let $n_{\mathcal{H}_k}$ be the largest divisor of n belonging to $\mathcal{H}_k$. Then

$$\begin{aligned} N_k(x) &= \sum_{m \in \mathcal{H}_k} \sum_{\substack{n \leq x \\ n_{\mathcal{H}_k} = m}} f(m) = \sum_{m \in \mathcal{H}_k} \sum_{\substack{n \leq x \\ m|n}} g(m) = \sum_{m \in \mathcal{H}_k} g(m) \left[\frac{x}{m}\right] \\ &= x \sum_{m \in \mathcal{H}_k} \frac{g(m)}{m} + O(|\mathcal{H}_k \cap [1, x]|) \end{aligned}$$

$$= x\prod_{i=1}^{k}\prod_{p\in I_{i,1}\cup I_{i,3}}\left(1-\frac{1}{p}\right)^{(-1)^{i+1}}\prod_{p\in I_{i,2}\cup I_{i,4}}\left(1-\frac{1}{p}\right)^{(-1)^{i}}+O(|\mathcal{H}_k\cap[1,x]|)$$
$$= x+O\left(\frac{x}{x_k}\right)+O(|\mathcal{H}_k\cap[1,x]|).$$

The first error term is negligible because $x<\exp(x_k^{3/5})$. Any element of $\mathcal{H}_k$ has only prime divisors below $2x_k$. The number of integers without large prime factors is well studied (cf., e.g., [11, 28]). Indeed, by using only the simplest estimate [11, eqn. (1.4)], we find that

$$\begin{aligned}|\mathcal{H}_k\cap[1,x]| &\le Cx^{1-\frac{1}{2\log(2x_k)}}\log x_k\\ &\le C\frac{x}{\log^{5/3}x}\left(\frac{(x_k\log x_k)^{2\log(2x_k)}}{x}\right)^{\frac{1}{2\log(2x_k)}}\\ &\le C\frac{x}{\log^{5/3}x}\qquad\text{for } \exp(8\log^2 x_k)\le x<\exp(x_k^{3/5}),\end{aligned}$$

and (6.2) follows. For (6.3), we use again the elementary estimate [11, eqn. (1.4)], so, for x in the given range,

$$\begin{aligned}\overline{N}_k(x)-x &= O\left(\frac{x}{\log^{5/3}x}\right)+O\left(\int_1^x\frac{|\mathcal{H}_k\cap[1,t]|}{t}\,dt\right)\\ &= O\left(\frac{x}{\log^{5/3}x}\right)+O\left(\log x_k\int_1^x t^{-\frac{1}{2\log(2x_k)}}\,dt\right)\\ &= O\left(\frac{x}{\log^{5/3}x}\right)+O\left(x^{1-\frac{1}{2\log(2x_k)}}\log x_k\right)=O\left(\frac{x}{\log^{5/3}x}\right). \quad\blacksquare\end{aligned}$$

Now the fact that $N(x)$ deviates far from x follows immediately. For $x<x_{k+1}^{+}$, we have $N(x)=N_k(x)$ with the exception of the missing and doubled primes from $[x_{k+1}^{-},x_{k+1}^{+}]$. Observe that, because $x_{k+1}=\lfloor\exp(x_k^{1/4}\log 2)\rfloor$,

$$[x_{k+1}^{-},x_{k+1}^{+}]\subset[\exp(8\log^2 x_k),\exp(x_k^{3/5})).$$

Since we changed more than $x/(4\log^{4/3}x)$ primes when x is the upper bound of either the interval $I_{k+1,1}$ or $I_{k+1,2}$, we deduce from Lemma 6.3 that $|N(x)-x|$ becomes as large as $cx/\log^{4/3}x$, for a fixed constant $c>0$.

To show (6.1), we bound the Cesàro means of N in the range $x_k^{-}\le x<x_{k+1}^{-}$. We start by observing that $N(x)=N_k(x)$ within this range, so (6.3) gives (6.1) for $\exp(8\log^2 x_k)\le x<x_{k+1}^{-}$. Assume now that

$$x_k^{-}\le x<\exp(8\log^2 x_k).$$

Lemma 6.3 implies that

$$\overline{N}_{k-1}(x) = \int_1^x \frac{N_{k-1}(t)}{t}\,dt = x + O\biggl(\frac{x}{\log^{5/3} x}\biggr),$$

because, by construction of the sequence, the interval $[x_k^-, \exp(8\log^2 x_k)]$ is contained in $[\exp(8\log^2 x_{k-1}), \exp(x_{k-1}^{3/5}))$. Therefore, it suffices to prove that

$$\overline{N}_k(x) - \overline{N}_{k-1}(x) = \int_{x_k^-}^x \frac{N_k(t) - N_{k-1}(t)}{t}\,dt \tag{6.4}$$

has growth order $O(x/\log^{5/3} x)$. Note that only the intervals $\nu\cdot(I_{k,1}\cup\cdots\cup I_{k,4})$ contribute to the integral (6.4). If $x > mx_k^+$, then, by the classical prime number theorem, the contribution of each of the intervals $\nu\cdot(I_{k,1}\cup\cdots\cup I_{k,4})$ is

$$O\biggl(\frac{kx_k}{\log^{4/3} x_k}\biggr)\cdot\int_{\nu x_k^-}^{\nu x_k^+}\frac{dt}{t} = O\biggl(\frac{k}{\log^{4/3} x_k}\biggr) = O\biggl(\frac{kx}{mx_k\log^{4/3} x_k}\biggr)$$
$$= O\biggl(\frac{x}{m\log^{5/3} x}\biggr),$$

because obviously $x < \exp(x_k^{3/5})$ and $k < \log x_k$. Since there are m such intervals, the contribution of the intervals which are completely below x is of the right order.

It remains to bound the contribution of the intervals $\nu\cdot(I_{k,1}\cup\cdots\cup I_{k,4})$ such that $\nu x_k^- < x < \nu x_k^+$. Since the logarithmic length of the four intervals is $O(k/\log^{1/3} x_k)$, the number of such indices ν is $O(kx/(x_k\log^{1/3} x_k))$. As above, the contribution of each of these intervals to (6.4) is at most $O(k/\log^{4/3} x_k)$; thus, their total contribution has order of growth below $O(k^2x/(x_k\log^{5/3} x_k)) = O(x/\log^{5/3} x)$. That $\overline{N}$ has asymptotics (6.1) now follows at once; consequently, Proposition 6.2 has been fully established.

Acknowledgements. J. Vindas gratefully acknowledges support by a Postdoctoral Fellowship of the Research Foundation–Flanders (FWO, Belgium).

References

[1] P. T. Bateman and H. G. Diamond, *Asymptotic distribution of Beurling's generalized prime numbers*, in: Studies in Number Theory, Math. Assoc. Amer. and Prentice-Hall, Englewood Cliffs, NJ, 1969, 152–210.

[2] T. Bekehermes, *Allgemeine Dirichletreihen und Primzahlverteilung in arithmetischen Halbgruppen*, Dissertation, Technische Universität Clausthal, 2003.

[3] A. Beurling, *Analyse de la loi asymptotique de la distribution des nombres premiers généralisés*, Acta Math. 68 (1937), 255–291.

[4] H. Bremermann, *Distributions, Complex Variables and Fourier Transforms*, Addison-Wesley, Reading, MA, 1965.

[5] H. G. Diamond, *Asymptotic distribution of Beurling's generalized integers*, Illinois J. Math. 14 (1970), 12–28.

[6] —, *A set of generalized numbers showing Beurling's theorem to be sharp*, ibid. 14 (1970), 29–34.

[7] —, *Chebyshev estimates for Beurling generalized prime numbers*, Proc. Amer. Math. Soc. 39 (1973), 503–508.

[8] —, *Elementary methods in the study of the distribution of prime numbers*, Bull. Amer. Math. Soc. (N.S.) 7 (1982), 553–589.

[9] Ju. N. Drožžinov and B. I. Zav'jalov, *Quasiasymptotic behavior of generalized functions, and Tauberian theorems in the complex domain*, Mat. Sb. (N.S.) 102 (144) (1977), 372–390.

[10] R. Estrada and R. P. Kanwal, *A Distributional Approach to Asymptotics. Theory and Applications*, 2nd ed., Birkhäuser, Boston, 2002.

[11] A. Hildebrand and G. Tenenbaum, *Integers without large prime factors*, J. Théor. Nombres Bordeaux 5 (1993), 411–484.

[12] S. Ikehara, *An extension of Landau's theorem in the analytic theory of numbers*, J. Math. Phys. M.I.T. 10 (1931), 1–12.

[13] J.-P. Kahane, *Sur les nombres premiers généralisés de Beurling. Preuve d'une conjecture de Bateman et Diamond*, J. Théor. Nombres Bordeaux 9 (1997), 251–266.

[14] J. Korevaar, *Tauberian Theory. A Century of Developments*, Grundlehren Math. Wiss. 329, Springer, Berlin, 2004.

[15] —, *Distributional Wiener–Ikehara theorem and twin primes*, Indag. Math. (N.S.) 16 (2005), 37–49.

[16] —, *The Wiener–Ikehara theorem by complex analysis*, Proc. Amer. Math. Soc. 134 (2006), 1107–1116.

[17] —, *A Tauberian theorem for Laplace transforms with pseudofunction boundary behavior*, in: Complex Analysis and Dynamical Systems II, Contemp. Math. 382, Amer. Math. Soc., Providence, RI, 2005, 233–242.

[18] —, *Prime pairs and the zeta function*, J. Approx. Theory 158 (2009), 69–96.

[19] E. Landau, *Über einige neuere Grenzwertsätze*, Rend. Circ. Mat. Palermo 34 (1912), 121–131.

[20] —, *Über den Wienerschen neuen Weg zum Primzahlsatz*, Sitz.-Ber. Preuss. Akad. Wiss. 1932, 514–521.

[21] D. J. Newman, *Simple analytic proof of the prime number theorem*, Amer. Math. Monthly 87 (1980), 693–696.

[22] B. Nyman, *A general prime number theorem*, Acta Math. 81 (1949), 299–307.

[23] R. Olofsson, *Properties of the Beurling generalized primes*, J. Number Theory 131 (2011), 45–58.

[24] S. Pilipović, *S-asymptotic of tempered and $\mathcal{K}_1'$-distributions I*, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 15 (1985), 47–58.

[25] S. Pilipović, B. Stanković and A. Takači, *Asymptotic Behaviour and Stieltjes Transformation of Distributions*, Teubner-Texte zur Math. 116, Teubner, Leipzig, 1990.

[26] S. Pilipović, B. Stanković and J. Vindas, *Asymptotic Behavior of Generalized Functions*, Ser. Anal. Appl. Comput. 5, World Sci., Hackensack, NJ, 2011.

[27] L. Schwartz, *Théorie des Distributions*, Hermann, Paris, 1966.
[28] G. Tenenbaum, *Introduction to Analytic and Probabilistic Number Theory*, Cambridge Stud. Adv. Math. 46, Cambridge Univ. Press, Cambridge, 1995.
[29] E. C. Titchmarsh, *The Theory of Functions*, 2nd ed., Oxford Univ. Press, Oxford, 1939.
[30] F. Trèves, *Topological Vector Spaces, Distributions and Kernels*, Academic Press, New York, 1967.
[31] J. Vindas, *Structural theorems for quasiasymptotics of distributions at infinity*, Publ. Inst. Math. (Beograd) (N.S.) 84 (98) (2008), 159–174.
[32] —, *Local Behavior of Distributions and Applications*, Dissertation, Louisiana State University, 2009.
[33] —, *The structure of quasiasymptotics of Schwartz distributions*, in: Linear and Nonlinear Theory of Generalized Functions and its Applications, Banach Center Publ. 88, Inst. Math., Polish Acad. Sci., Warszawa, 2010, 297–314.
[34] J. Vindas and R. Estrada, *A quick distributional way to the prime number theorem*, Indag. Math. (N.S.) 20 (2009), 159–165.
[35] J. Vindas and S. Pilipović, *Structural theorems for quasiasymptotics of distributions at the origin*, Math. Nachr. 282 (2009), 1584–1599.
[36] V. S. Vladimirov, *Methods of the Theory of Generalized Functions*, Taylor & Francis, London, 2002.
[37] V. S. Vladimirov, Yu. N. Drozzinov and B. I. Zavialov, *Tauberian Theorems for Generalized Functions*, Kluwer, Dordrecht, 1988.
[38] W.-B. Zhang, *A generalization of Halász's theorem to Beurling's generalized integers and its application*, Illinois J. Math. 31 (1987), 645–664.
[39] —, *Chebyshev type estimates for Beurling generalized prime numbers. II*, Trans. Amer. Math. Soc. 337 (1993), 651–675.
[40] —, *Mean-value theorems and extensions of the Elliott–Daboussi theorem on Beurling's generalized integers. I*, Math. Z. 261 (2009), 201–232.

Jan-Christoph Schlage-Puchta, Jasson Vindas
Department of Mathematics
Ghent University
Krijgslaan 281 Gebouw S22
B 9000 Gent, Belgium
E-mail: jcsp@cage.Ugent.be
jvindas@cage.Ugent.be

Received on 26.3.2011
and in revised form on 17.10.2011 (6652)