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On the Diophantine equation $\binom{n}{k_1,\ldots,k_s} = x^l$

by

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1. Introduction. Consider the Diophantine equation involving the binomial coefficient,

$$\binom{n}{k} = x^l, \tag{1.1}$$

with integers n, k, x, l satisfying $n \geq k \geq 2$, $x, l \geq 2$. For $k = 2$, $l = 2$, there are infinitely many solutions of (1.1) given e.g. by the recursion formula $\binom{(2n-1)^2}{2} = 4(2n-1)^2\binom{n}{2}$ (cf. [1]). For $k = 3$, $l = 2$, when n is odd Meyl [8] proved that (1.1) has only the trivial solution $(n, x) = (3, 1)$, when n is even Watson [11] proved that (1.1) has only the solutions $(n, x) = (4, 2)$, $(50, 140)$. In 1939, Erdős [3] proved that (1.1) has no solution if $l = 3$ or $k \geq 2^l$ and he conjectured that (1.1) has no solution if $l > 2$. In 1951, Erdős [4] proved his conjecture for $k \geq 4$. For $k = 2$, $l > 2$, it can be deduced from Darmon and Merel's result on the equation $x^l + y^l = 2z^l$ with $x, y, z \in \mathbb{Z}$ and $(x, y, z) = 1$ (cf. [2]) that (1.1) has no solution [5], [6]. For $k = 3$, $l > 2$, Győry [5] proved that (1.1) has no solution, thereby completing the proof of Erdös' conjecture.

In this note, we consider the multinomial coefficient form of (1.1), i.e.

$$\binom{n}{k_1, \ldots, k_s} = x^l,\ s \geq 3,\ l \geq 2,\ k_1 + \cdots + k_s = n,\ k_1 \geq \cdots \geq k_s \geq 1, \tag{1.2}$$

where $\binom{n}{k_1,\ldots,k_s} = \frac{n!}{k_1!\cdots k_s!}$.

Our main result is:

THEOREM. (1.2) *has no solution for* $n \geq 3$, $s \geq 3$, $l \geq 2$.

Together with the results concerning (1.1), our Theorem gives the complete solution of (1.2) for $s \geq 2$ as well.

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2. Proof of the Theorem. The well-known Chebyshev's theorem says that for any positive integer n, there is a prime p which satisfies $n < p \le 2n$. We first give a consequence of Chebyshev's theorem:

LEMMA 2.1. *For any integer $n > 1$, there is a prime $p \le n$ satisfying $n = p + r$, $0 \le r < p$.*

Proof. Lemma 2.1 is obviously true when $n = 2$. Let $n > 2$, and take the largest prime $p_1 \le n$. We write $n = t_1 p_1 + r_1$ with $0 \le r_1 < p_1$, $t_1 \ge 1$. If $t_1 \ge 2$, by Chebyshev's theorem, we have a prime p_2 satisfying $p_1 < p_2 < 2p_1 \le n$, a contradiction. ∎

The following lemma is a generalized form of Chebyshev's theorem.

LEMMA 2.2 (Sylvester [10]). *For positive integers n, k with $2k \le n$, there is a prime $p > k$ with $p \mid \binom{n}{k}$.*

LEMMA 2.3 (Saradha [9], Győry [7]). *For any positive integer b, denote by $P(b)$ the greatest prime factor of b. Apart from $k = b = l = 2$, consider the equation*

$$n(n+1)\cdots(n+k-1) = bx^l$$

with positive integers n, k, b, x, l satisfying $k \ge 2$, $l \ge 2$, $P(b) \le k$, b lth power free. When $P(x) > k$, the equation has only the solution $(n, k, b, x, l) = (48, 3, 6, 140, 2)$.

LEMMA 2.4. *If $n/2 \ge k_1 \ge \cdots \ge k_s \ge 1$, then there is a prime $p > k_1$ such that*

$$p \parallel \binom{n}{k_1, \ldots, k_s}.$$

Proof. By Lemma 2.1, there is a prime p such that $n = p + r$, $0 \le r < p$, which means $k_1 \le n/2 < p \le n < 2p$. Since

$$\binom{n}{k_1, \ldots, k_s} = \frac{n(n-1)\cdots(k_1+1)}{k_2!\cdots k_s!}$$

and $P(k_2!\ldots k_s!) < k_1$, we have $v_p\big(\binom{n}{k_1,\ldots,k_s}\big) = 1$, where $v_p(n)$ means the exponent of p in the factorization of n. ∎

Proof of the Theorem. By Lemma 2.4, we only need to consider $k_1 > n/2 > k_2 \ge \cdots \ge k_s \ge 1$. Suppose in this case that (1.2) has an integer solution. We can write (1.2) in the form

$$n(n-1)\cdots(n-k_2-\cdots-k_s+1) = k_2!\cdots k_s! x^l = bt^l x^l \tag{2.1}$$

where $k_2!\cdots k_s! = bt^l$, b is lth power free. By assumption, we have $P(b) \le P(k_2!\cdots k_s!) < k_1$. Let $k = k_2 + \cdots + k_s$, $n' = k_1 + 1$. Then (2.1) takes the form

$$n'(n'+1)\cdots(n'+k-1) = b(tx)^l. \tag{2.2}$$

Note that $k = n - k_1 < n/2$ and by Lemma 2.2, we have a prime $p > k$

with $p \mid \binom{n}{k}$. Since $\binom{n}{k_1,\dots,k_s} = \binom{n}{k}\binom{k}{k_2,\dots,k_s}$, we have $p \mid \binom{n}{k_1,\dots,k_s}$, whence $p \mid x$, $P(tx) \ge P(x) > k$, which contradicts Lemma 2.3 if $k > 3$. Therefore (2.2) has no integer solution if $k > 3$. Now we only need to consider the cases $k = 2$ and $k = 3$.

For $k = 2$, equation (1.2) takes the form $\binom{n}{n-2,1,1} = n(n-1) = x^l$. If this has a solution, then as $(n, n-1) = 1$, both n and $n-1$ should be full lth powers, which is impossible. Hence there is no positive integer solution when $k = 2$.

For $k = 3$, (1.2) leads to $\binom{n}{n-3,2,1} = n(n-1)(n-2)/2 = x^l$ or $\binom{n}{n-3,1,1,1} = n(n-1)(n-2) = x^l$. By Lemma 2.3, the above equations have no positive integer solution. This completes the proof of the Theorem.

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