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Lower bounds for power moments of L-functions

by

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1. Introduction. Let $s = \sigma + it$ denote a point in the complex plane. Let $g(s) = \sum_{n=1}^{\infty} a_g(n)/n^s$ be a Dirichlet series absolutely convergent for $\sigma > 1$ that can be continued analytically to the region $\sigma \geq 1/2$, $t \geq 1$. For fixed $\sigma \geq 1/2$ and non-negative real k, we define the *kth* (*power*) *moment of* g *at* σ as

$$I_k(g, \sigma, T) = \int_1^T |g(\sigma + it)|^{2k}\, dt.$$

For $\sigma = 1/2$, we denote $I_k(g, 1/2, T)$ by $I_k(g, T)$ and we call it the *kth* (*power*) *moment of* g. We are interested in studying the behaviour of $I_k(g, \sigma, T)$ as $T \to \infty$.

If $(g(s))^k$ has a representation on $\sigma > 1$ as an absolutely convergent Dirichlet series $\sum_{n=1}^{\infty} g_k(n)/n^s$, then from the mean value theorem for Dirichlet series [19, Theorem 7.1] we know that

$$I_k(g, \sigma, T) \sim T \sum_{n=1}^{\infty} \frac{|g_k(n)|^2}{n^{2\sigma}}, \tag{1.1}$$

for $\sigma > 1$, as $T \to \infty$. Very little is known about the behaviour of $I_k(g, \sigma, T)$ for $1/2 \leq \sigma \leq 1$. However for several classes of Dirichlet series arising in number theory it is conjectured that (1.1) holds for $1/2 < \sigma \leq 1$. Another interesting feature is the close connection of the behaviour of $I_k(g, \sigma, T)$ with the size of $I_k(g, T)$. In fact for certain g it is known that if $I_k(g, T) \ll T^{1+\epsilon}$ for any $\epsilon > 0$, then (1.1) holds for $\sigma > 1/2$ (see [18, Section 9.51]).

In this paper, we are interested in finding lower bounds of $I_k(g, T)$ in terms of T for certain number-theoretical L-functions g which have Dirichlet series representations on the half-plane $\sigma > 1$. The special case when g is the Riemann zeta function $\zeta(s)$ is classical and has been under investigation

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for a long time. For $k = 1$ and 2 the asymptotic formulas are known. One of the first results on this subject which treats all integral values of k is due to Titchmarsh [19, Theorem 7.19], who showed that

$$\int_0^\infty \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2k} e^{-t/T}\, dt \gg T(\log T)^{k^2}.$$

The first general lower bound for the kth moment of $\zeta(s)$ was given by Ramachandra. In [14, Theorem 1] he proved that

$$I_k(\zeta, T) \gg T(\log T)^{k^2}(\log\log T)^{-c_k}$$

for real $k \geq 1/2$, where c_k is a constant depending possibly on k. Moreover he proved that under the assumption of the Riemann Hypothesis this lower bound holds for any real $k \geq 0$. Later in [15] Ramachandra studied the general problem of finding lower bounds for the moments of Dirichlet series and in this direction obtained several important results. One of his results is the following.

Theorem 1.1 (Ramachandra). *Let $F(s)$ be a Dirichlet series convergent for $\sigma > 1$ that can be continued analytically to the region $\sigma \geq 1/2$, $t \geq 1$, and in this region $|F(s)| \leq \exp((\log t)^{c_0})$, where c_0 is a fixed positive constant. Moreover assume that $F(s)$ has a representation in the form*

$$F(s) = \left(\sum_{n=1}^\infty \frac{a(n)}{n^s}\right)^2$$

for $\sigma > 1$, where for $1/2 < \sigma < 1$ the series $\sum_{n=1}^\infty |a(n)|^2/n^{2\sigma}$ converges and

$$\frac{1}{(\sigma - 1/2)^\alpha} \ll \sum_{n=1}^\infty \frac{|a(n)|^2}{n^{2\sigma}} \ll \frac{1}{(\sigma - 1/2)^\alpha}$$

for some positive constant α. Then

$$I_{1/2}(F, T) \gg T(\log T)^\alpha.$$

Proof. This is a special case of [15, Theorem 2]. ∎

Letting $F(s) = ((g(s))^{k/2})^2 = (g(s))^k$ in Ramachandra's theorem we get the following.

Corollary 1.2. *Let $g(s)$ be a Dirichlet series convergent for $\sigma > 1$ that can be continued analytically to the region $\sigma \geq 1/2$, $t \geq 1$, and in this region $|g(s)| \leq \exp((\log t)^{c_0})$, where c_0 is a fixed positive constant. Let $k \geq 0$ be an integer. Suppose that $(g(s))^{k/2}$ in $\sigma > 1$ can be represented by a Dirichlet series*

$$(g(s))^{k/2} = \sum_{n=1}^\infty \frac{g_{k/2}(n)}{n^s}.$$

Moreover suppose that for $1/2 < \sigma < 1$ *the series* $\sum_{n=1}^{\infty} |g_{k/2}(n)|^2/n^{2\sigma}$ *converges and*

$$\frac{1}{(\sigma - 1/2)^{\alpha}} \ll \sum_{n=1}^{\infty} \frac{|g_{k/2}(n)|^2}{n^{2\sigma}} \ll \frac{1}{(\sigma - 1/2)^{\alpha}}$$

for some positive constant α. *Then*

$$I_{k/2}(g, T) \gg T(\log T)^{\alpha}.$$

Also, if the above conditions hold for real $k \geq 0$ *and in addition* $g(s)$ *does not have any zero in the region* $\sigma > 1/2$, $t \geq 1$, *then*

$$I_k(g, T) \gg T(\log T)^{\alpha}$$

for any non-negative real number k.

A special case of the above corollary is the following.

Corollary 1.3 (Ramachandra). *For the Riemann zeta function* $\zeta(s)$, *and for a non-negative integer* k *we have*

$$I_{k/2}(\zeta, T) \gg T(\log T)^{k^2/4}.$$

Moreover, under the assumption of the Riemann Hypothesis, for non-negative real k *we have*

$$I_k(\zeta, T) \gg T(\log T)^{k^2}.$$

In [3], Heath-Brown extended the above result to the fractional moments of the Riemann zeta function.

Theorem 1.4 (Heath-Brown). *For the Riemann zeta function* $\zeta(s)$ *and for a non-negative rational number* k *we have*

$$I_k(\zeta, T) \gg T(\log T)^{k^2}.$$

Under the assumption of the Riemann Hypothesis, Heath-Brown's method also establishes the same lower bound for any non-negative real k.

In recent years several authors have employed the Heath-Brown method to study the fractional moments of other L-functions. Let χ be a primitive Dirichlet character mod q and let $I_k(\chi, T)$ denote the kth moment of the Dirichlet L-function $L(\chi, s)$. Then Kačěnas, Laurinčikas and Zamarys [6] proved that for $k = 1/n$, $n \geq 2$, we have

$$I_k(\chi, T) \gg T(\log T)^{k^2}.$$

In [9] Laurinčikas and Steuding proved that if f is a holomorphic cusp form of weight ℓ and level 1, and $I_k(f, T)$ is the kth moment of the modular L-function attached to f, then for $k = 1/n$, $n \geq 2$ one has

$$I_k(f, T) \gg T(\log T)^{k^2}.$$

Zamarys [20] has generalized this result to the case of the kth moment of twisted modular L-functions for positive rational $k \leq 1/2$.

The proofs of the above results closely follow [3]. Our first goal in this paper is to follow the method of Heath-Brown to derive a general theorem in the spirit of Corollary 1.2. More specifically we prove the following:

THEOREM 1.5. *Let* $g(s) = \sum_{n=1}^{\infty} a_g(n)/n^s$ *be a Dirichlet series absolutely convergent for* $\sigma > 1$ *that has an analytic continuation to the half-plane* $\sigma \geq 1/2$ *with a pole of degree* m *at* $s = 1$ (*with* m *possibly equal to* 0) *and suppose that*

$$g(\sigma + it) \ll |t|^A \tag{1.2}$$

for a fixed $A > 0$, $\sigma \geq 1/2$ *and* $|t| \geq 1$ (*the implied constant is independent of* σ *and* t). *For rational* $k \geq 0$ *and* $\sigma > 1$, *we suppose that* $(g(s))^k$ *may be represented by an absolutely convergent Dirichlet series*

$$(g(s))^k = \sum_{n=1}^{\infty} \frac{g_k(n)}{n^s},$$

where

$$g_k(n) \ll n^{\eta} \tag{1.3}$$

for some fixed η *with* $0 \leq \eta < 1/2$. *In addition, assume that there exist fixed constants* $c > 0$ *and* $\alpha \geq 0$ *such that*

$$\frac{1}{(\sigma - 1/2)^{\alpha}} \ll \sum_{n=1}^{T} \frac{|g_k(n)|^2}{n^{2\sigma}} \ll (\log T)^{\alpha} \tag{1.4}$$

uniformly for $1/2 + c/\log T \leq \sigma \leq 1$ *and* $T \geq 2$. *Then for any rational* $k \geq 0$ *we have*

$$I_k(g, T) \gg T(\log T)^{\alpha}. \tag{1.5}$$

Moreover if the above conditions (1.3) *and* (1.4) *hold for real* $k \geq 0$ *and in addition* $g(s)$ *does not have any zeros in the half-plane* $\sigma > 1/2$ *then* (1.5) *holds for any non-negative real* k.

We next employ the above theorem to establish lower bounds for moments of certain number-theoretical L-functions. The L-functions we consider include the following:

(a) *Principal automorphic* L*-functions.* Let π be a principal automorphic L-function whose local parameters $\alpha_\pi(p, j)$ at unramified primes satisfy

$$|\alpha_\pi(p, j)| \leq p^{\gamma}$$

for a fixed $0 \leq \gamma < 1/4$ (see Section 2 for terminology). Then, for rational k, we prove that

$$I_k(\pi, T) \gg T(\log T)^{k^2}.$$

As a corollary of this general result we deduce unconditional lower bounds of the conjectured order of magnitude for the fractional moments of classical Dirichlet L-functions, modular L-functions, and twisted modular L-functions. Our results generalize the lower bound results of Kačėnas–Laurinčikas–Zamarys [6] and Laurinčikas–Steuding [9] to rational values of $k \geq 0$, and the lower bound result of Zamarys [20] to rational values of $k \geq 1/2$.

Moreover, we derive an unconditional lower bound for the fractional kth moment of Maass L-functions. Let $L(f,s)$ be a Maass cusp newform of weight zero and level N with Nebentypus ψ. Then for any rational $k \geq 0$ we prove that

$$I_k(f,T) \gg T(\log T)^{k^2}.$$

To our knowledge this result is the first established unconditional lower bound for the kth moment of an L-function for which the truth of the Generalized Ramanujan Conjecture (GRC) (see Section 2 for terminology) is not known.

REMARK 1.6. Ji [5] has established lower bounds for kth moments of principal automorphic L-functions over short intervals. As a consequence of [5, Theorem 1.1], for real non-negative k, one has

$$I_k(\pi,\sigma,T) \gg T$$

uniformly for $\sigma \geq 1/2$. Our results give conditional improvements of this lower bound for $\sigma = 1/2$.

(b) *Artin L-functions.* Under the assumption of the Artin Holomorphy Conjecture we deduce a lower bound for the fractional kth moment $I_k(K/\mathbb{Q},\rho,T)$ of the Artin L-function $L(K/\mathbb{Q},\rho,s)$ associated to a representation ρ of the Galois group of a number field $K/\mathbb{Q}$. We prove that for any non-negative rational k,

$$I_k(K/\mathbb{Q},\rho,T) \gg T(\log T)^{\langle\varphi,\varphi\rangle k^2},$$

where φ is the character associated to ρ and $\langle\varphi,\varphi\rangle$ is the inner product of φ and φ as defined in Proposition 2.11.

As a direct corollary of this result we derive an unconditional lower bound for the fractional kth moment $I_k(\zeta_K,T)$ of the Dedekind zeta function $\zeta_K(s)$ of a number field K. More precisely, for non-negative rational k we prove that

$$I_k(\zeta_K,T) \gg T(\log T)^{nk^2},$$

where n is the degree of the Galois extension $K/\mathbb{Q}$.

Our result improves the following theorem of Ramachandra [14, Theorem 2] for rational values of k.

THEOREM 1.7 (Ramachandra). *Let K be a degree n Galois extension of $\mathbb{Q}$, and let $\zeta_K(s)$ be the Dedekind zeta function of K. Then for real*

$k \geq 1/2$ *we have*

$$I_k(\zeta_K, T) \gg T \frac{(\log T)^{nk^2}}{(\log\log T)^c},$$

where c *is a positive constant.*

REMARKS 1.8. (a) The Generalized Riemann Hypothesis (GRH) for an L-function implies that the L-function does not have any zeros in the half-plane $\sigma > 1/2$. Thus by Theorem 1.5, upon the assumption of GRH for an L-function g, the result for the fractional kth moments of g can be extended to non-negative real values of k as long as (1.3) and (1.4) hold for $(g(s))^k$.

(b) In 2005, Rudnick and Soundararajan [17] devised a method that establishes lower bounds of the conjectured order of magnitude for integral kth moments of several families of L-functions. They also commented that their method generalizes to the fractional kth moments for $k \geq 1$. This procedure is also applicable to the integral kth (power) moment of the Riemann zeta function. Using this method, Chandee [1, Theorem 1.2] has established a lower bound of the conjectured order of magnitude for the integral shifted (power) moment of the Riemann zeta function.

(c) In [3], Heath-Brown also considered the upper bound of the kth (power) moment of the Riemann zeta function, and by a method similar to the one used in establishing the lower bound, he proved that

$$I_k(\zeta, T) \ll T(\log T)^{k^2}$$

unconditionally for $k = 1/n > 0$, where n is an integer, and under the assumption of the Riemann Hypothesis for all real values of $0 \leq k \leq 2$. It would be possible to generalize this method to obtain upper bounds for the kth (power) moments of general L-functions. However, unlike the case of the Dirichlet L-functions, the results would be conditional upon GRH and only true for some specific values of $k < 1$. See [9, Theorem 1] for an example of such a result.

The structure of this paper is as follows. In Section 2 we describe some applications of Theorem 1.5. To do this we first observe that if the sequence $\{|a_g(p)|^2\}$ has regular distribution then the technical condition (1.4) holds. We prove this result in Section 3. Using this observation in Section 2, we establish a new version (Theorem 2.3) of our main theorem. Our results on lower bounds for moments of principal automorphic L-functions, Artin L-functions and Dedekind zeta functions are simple consequences of Theorem 2.3. Finally in Sections 4 and 5 we prove Theorem 1.5.

NOTATION 1.9. Throughout $s = \sigma + it$ denotes a point in the complex plane and p denotes a prime number. We use the notation $\ll$, $\mathrm{O}(\cdot)$, and $\sim$ with their usual meaning in analytic number theory. All constants implied by the $\ll$ notation may depend on k and the function g. However, the constants are independent of the variables σ and T. By abuse of notation and for simplicity we denote the kth moment of an L-function $L(f,s)$ by $I_k(f,T)$ instead of $I_k(L(f,\cdot),T)$.

2. Applications of Theorem 1.5. We first identify some classes of Dirichlet series for which the technical condition (1.4) holds. To do this we employ a classical theorem on the average values of multiplicative functions and derive the following proposition.

PROPOSITION 2.1. *Let $h(n)$ be a non-negative multiplicative function. Suppose that for some fixed real $\alpha > 0$ we have*

$$\sum_{p\le x} h(p) \sim \alpha \frac{x}{\log x}$$

as $x \to \infty$, and

$$\sum_{p\le x} \frac{h(p)}{p} = \alpha \log\log x + \mathrm{O}(1).$$

Suppose also that

$$h(p^r) \ll (2p)^{r\theta} \quad \text{for some } 0 \le \theta < 1/2 \text{ and for all } r \ge 2.$$

Let $T \ge 2$ and let $0 < c \le \frac{1}{2}\log 2$ be fixed. Then

$$\frac{1}{(\sigma - 1/2)^{\alpha}} \ll \sum_{n=1}^{T} \frac{h(n)}{n^{2\sigma}} \ll (\log T)^{\alpha}$$

uniformly in the range $1/2 + c/\log T \le \sigma \le 1$.

The proof is postponed to Section 3.

As a consequence of this proposition we prove a version of Theorem 1.5 for Dirichlet series with certain Euler products.

To simplify our exposition, from now on we consider the class $\mathcal{C}$ of Dirichlet series $g(s) = \sum_{n=1}^{\infty} a_g(n)/n^s$ that are absolutely convergent for $\sigma > 1$ and satisfy the following: Except for a pole of degree m at $s = 1$ (with m possibly equal to 0), $g(s)$ extends to an entire function for $\sigma \ge 1/2$ and $|g(\sigma + it)| \ll |t|^A$ for a fixed $A > 0$, $\sigma \ge 1/2$ and $|t| \ge 1$.

For any real k we define the multiplicative function $\tau_k(\cdot)$ by

$$\tau_k(p^r) = \frac{\Gamma(k+r)}{r!\,\Gamma(k)} = \frac{k(k+1)(k+2)\cdots(k+r-1)}{r!}.$$

Note that $\tau_k(n) \ge 0$ when $k \ge 0$. The following lemma summarizes some properties of $\tau_k(n)$.

Lemma 2.2.

(i) *For fixed* $k \geq 0$ *and* $\epsilon > 0$ *we have* $\tau_k(n) \ll n^\epsilon$.
(ii) *If* j *is a positive integer then*

$$\tau_{kj}(n) = \sum_{n=n_1\cdots n_j} \tau_k(n_1)\cdots\tau_k(n_j).$$

Proof. See [3, Lemma 1]. ∎

Theorem 2.3. *Let* $g(s) \in \mathcal{C}$. *Suppose* $g(s)$ *has an Euler product of the form*

$$g(s) = \sum_{n=1}^{\infty} \frac{a_g(n)}{n^s} = \prod_p \prod_{j=1}^{d} \Big(1 - \frac{\alpha_g(p,j)}{p^s}\Big)^{-1} \tag{2.1}$$

for $\sigma > 1$, *where* $\alpha_g(p,j) \in \mathbb{C}$ *and*

$$|\alpha_g(p,j)| \leq (2p)^\gamma \quad \text{for a fixed } 0 \leq \gamma < 1/4.$$

Suppose that

$$\sum_{p\leq x} |a_g(p)|^2 \sim \beta \frac{x}{\log x} \tag{2.2}$$

as $x \to \infty$, *and*

$$\sum_{p\leq x} \frac{|a_g(p)|^2}{p} = \beta \log\log x + \mathrm{O}(1) \tag{2.3}$$

for some fixed constant β, *where* $a_g(p) = \sum_{j=1}^{d} \alpha_g(p,j)$. *Then for any rational* $k \geq 0$ *we have*

$$I_k(g,T) \gg T(\log T)^{\beta k^2}.$$

Proof. Since $g(s)$ is represented by an absolutely convergent Euler product and the Euler factors are non-vanishing on $\sigma > 1$, we have $g(s) \neq 0$ for $\sigma > 1$. Letting

$$b_g(p^r) = \sum_{j=1}^{d} \alpha_g(p,j)^r,$$

for $\sigma > 3/2$ we define a branch of $\log g(s)$ by

$$\log g(s) = \sum_p \sum_{r=1}^{\infty} \frac{b_g(p^r)}{rp^{rs}},$$

and a branch of $g^k(s)$ by

$$g^k(s) = \exp(k \log g(s))$$

for any real k. For $\sigma > 3/2$ we have

$$g^k(s) = \exp\left(k\sum_p\sum_{r=1}^{\infty}\frac{b_g(p^r)}{rp^{rs}}\right) = \prod_p \exp\left(-k\sum_{j=1}^{d}\log\left(1-\frac{\alpha_g(p,j)}{p^s}\right)\right)$$
$$= \prod_p\left(1-\frac{\alpha_g(p,1)}{p^s}\right)^{-k}\cdots\left(1-\frac{\alpha_g(p,d)}{p^s}\right)^{-k}.$$

Using the Taylor expansion

$$(1-z)^{-k} = \sum_{r=0}^{\infty}\frac{\Gamma(k+r)}{r!\Gamma(k)}z^r$$

on the region $|z| < 1$ we have

$$g^k(s) = \prod_p\prod_{j=1}^{d}\left(\sum_{r=0}^{\infty}\tau_k(p^r)\frac{\alpha_g(p,j)^r}{p^{rs}}\right) = \prod_p\sum_{r=0}^{\infty}\frac{g_k(p^r)}{p^{rs}} = \sum_{n=1}^{\infty}\frac{g_k(n)}{n^s}.$$

Here $g_k(n)$ is the multiplicative function given by

$$g_k(p^r) = \sum_{p^r=p^{j_1}\cdots p^{j_d}}\tau_k(p^{j_1})\alpha_g(p,1)^{j_1}\tau_k(p^{j_2})\alpha_g(p,2)^{j_2}\cdots\tau_k(p^{j_d})\alpha_g(p,d)^{j_d}.$$

By employing $|\alpha_g(p,j)| \le (2p)^\gamma$ and Lemma 2.2 we have

$$|g_k(p^r)| \le (2p)^{r\gamma}\sum_{p^r=p^{j_1}\cdots p^{j_d}}\tau_k(p^{j_1})\cdots\tau_k(p^{j_d}) = (2p)^{r\gamma}\tau_{kd}(p^r)$$
$$\ll (2p)^{r(\gamma+\epsilon)} \tag{2.4}$$

for any $\epsilon > 0$. Applying this bound we have

$$\sum_p\sum_{r=2}^{\infty}\frac{|g_k(p^r)|}{p^{r\sigma}} < \infty$$

for $\sigma > 1$. On the other hand

$$\sum_p\frac{|g_k(p)|}{p^\sigma} = k\sum_p\frac{|a_g(p)|}{p^\sigma} \le kg(\sigma) < \infty$$

for $\sigma > 1$. Thus $\sum_{n=1}^{\infty} g_k(n)/n^s$ is absolutely convergent for $\sigma > 1$, which together with (2.4) implies that (1.3) holds.

It remains to show that (1.4) holds with $\alpha = \beta k^2$. Let $h(n) = |g_k(n)|^2$, so that $h(p) = k^2|a_g(p)|^2$. We note that (2.2)–(2.4) show that the conditions of Proposition 2.1 hold for $\alpha = \beta k^2$, and thus we may use Proposition 2.1 to see that (1.4) holds.

Therefore, for rational k, Theorem 1.5 holds with $\alpha = \beta k^2$, which completes the proof of Theorem 2.3. ∎

Next we describe several applications of Theorem 2.3 in the cases of automorphic L-functions, Artin L-functions, and Dedekind zeta functions.

Principal automorphic L-functions. Let π be an irreducible unitary cuspidal representation of $\mathrm{GL}_d(\mathbb{Q}_\mathbb{A})$. The global L-function at finite places attached to π is given by the Euler product of local factors for $\sigma > 1$:

$$L(\pi, s) = \sum_{n=1}^{\infty} \frac{a_\pi(n)}{n^s} = \prod_p L_p(\pi_p, s), \tag{2.5}$$

where

$$L_p(\pi_p, s) = \prod_{j=1}^{d} \left(1 - \frac{\alpha_\pi(p, j)}{p^s}\right)^{-1},$$

and $\alpha_\pi(p, j) \in \mathbb{C}$ $(1 \le j \le d)$ are the local parameters at the prime p. It is known [16, formula (2.3)] that

$$|\alpha_p(p, j)| \le p^{1/2 - 1/(d^2+1)}. \tag{2.6}$$

It is conjectured that for all primes except finitely many $|\alpha_p(p, j)| = 1$.

To each π there is a finite set of primes associated that are called the *ramified primes*. The local parameters for primes outside the set of ramified primes (i.e. *unramified primes*) can be given as eigenvalues of a matrix in $\mathrm{GL}_d(\mathbb{C})$.

We say that an irreducible unitary cuspidal representation π of $\mathrm{GL}_d(\mathbb{Q}_\mathbb{A})$ satisfies the *Generalized Ramanujan Conjecture* (GRC) if for local parameters $\alpha_\pi(p, j)$ $(1 \le j \le d)$ at unramified primes p we have $|\alpha_\pi(p, j)| = 1$.

Our aim is to apply Theorem 2.3 to $L(\pi, s)$. We call such an L-function a *principal automorphic L-function*. It is known that $L(\pi, s) \in \mathcal{C}$, and we clearly have an Euler product of the form (2.1). We next investigate sufficient conditions under which formulas analogous to (2.2) and (2.3) hold. To do this, for primes p and integers $r \ge 1$, we define

$$b_\pi(p^r) = \begin{cases} \sum_{j=1}^{d} \alpha_\pi(p, j)^r, & p \text{ unramified}, \\ 0, & p \text{ ramified}. \end{cases}$$

Note that for an unramified prime p,

$$b_\pi(p) = a_\pi(p).$$

We define $b_\pi(n) = 0$ if n is not a prime power. We consider the following hypothesis on the size of the coefficients $b_\pi(p^r)$ for $r \ge 2$:

Hypothesis $\mathbf{H_0}$. There is $\epsilon > 0$ such that for any fixed $r \ge 2$,

$$\sum_{p^r \le x} (\log p) |b_\pi(p^r)|^2 \ll x^{1-\epsilon}.$$

$\mathbf{H_0}$ is a mild hypothesis and expected to be true for all π in general. It is true under the assumption of GRC. More generally if the local parameters satisfy

$$|\alpha_\pi(p,j)| \le p^\gamma \quad \text{for a fixed } 0 \le \gamma < 1/4,$$

then $\mathbf{H_0}$ holds. Employing (2.6) we have

$$\sum_{r\ge\alpha}\sum_{p^r\le x} (\log p)|b_\pi(p^r)|^2 \ll x^{1-\epsilon},$$

where $0 < \epsilon < 2/(d^2+1)$ and $\alpha(2/(d^2+1)-\epsilon) > 1$. Hence, assuming $\mathbf{H_0}$, there is $\epsilon > 0$ such that

$$\sum_{r\ge 2}\sum_{p^r\le x} (\log p)|b_\pi(p^r)|^2 \ll x^{1-\epsilon}. \tag{2.7}$$

Finally observe that $\mathbf{H_0}$ implies the hypothesis $\mathbf{H}$ of Rudnick and Sarnak [16, p. 281]. That is, under the assumption of $\mathbf{H_0}$ we have

$$\sum_{r\ge 2}\sum_{p^r\le x} \frac{(\log p)^2|b_\pi(p^r)|^2}{p^r} < \infty. \tag{2.8}$$

In the proof of the following lemma we use some properties of the Rankin–Selberg L-function $L(\pi\times\tilde{\pi},s)$. See [16, Section 2.4] for a review of these properties.

Lemma 2.4. *Under the assumption of* $\mathbf{H_0}$ *we have*

(i)
$$\sum_{p\le x} |b_\pi(p)|^2 \sim \frac{x}{\log x}$$

as $x\to\infty$, *and*

(ii)
$$\sum_{p\le x} \frac{|b_\pi(p)|^2}{p} = \log\log x + \mathrm{O}(1).$$

Proof. (i) Let S be the product of all ramified primes for π and let $L_S(\pi\times\tilde{\pi},s) = \prod_{p\nmid S} L_p(\pi_p\times\tilde{\pi}_p,s)$ be the partial Rankin–Selberg L-function. Then it is known that for $\sigma > 1$,

$$f(s) = -\frac{L_S'}{L_S}(\pi\times\tilde{\pi},s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)|b_\pi(n)|^2}{n^s},$$

where $\Lambda(n) = \log p$ if $n = p^r$ and zero otherwise (see [16, p. 281, formula (2.22)]). From the properties of the Rankin–Selberg L-functions we deduce that $f(s)$ has a meromorphic continuation to $\sigma \ge 1$ with a simple pole of residue 1 at $s = 1$. By the Wiener–Ikehara Tauberian theorem we have

$$\sum_{n\le x} \Lambda(n)|b_\pi(n)|^2 \sim x. \tag{2.9}$$

For simplicity of exposition and in a similar fashion to the classical number-theoretical functions we define

$$\Psi(x)=\sum_{n\le x}\Lambda(n)|b_\pi(n)|^2,\quad \Theta(x)=\sum_{p\le x}(\log p)|b_\pi(p)|^2,\quad \Pi(x)=\sum_{p\le x}|b_\pi(p)|^2.$$

We have

$$\Theta(x)\le\Psi(x)\le(\log x)\Pi(x)+C_\epsilon x^{1-\epsilon}, \tag{2.10}$$

where C_ϵ is a fixed constant (here we have used (2.7)). On the other hand, from (2.9), for large x we have

$$\Pi(x)\le\Psi(x)\le(1+\epsilon)x.$$

Employing this upper bound for $\Pi(x)$, for $0<\alpha<1$ we have

$$\begin{aligned}\Theta(x)\ge\sum_{x^\alpha\le p\le x}(\log p)|b_\pi(p)|^2 &\ge(\Pi(x)-\Pi(x^\alpha))\log x^\alpha\\ &\ge\alpha(\Pi(x)-(1+\epsilon)x^\alpha)\log x.\end{aligned} \tag{2.11}$$

Since we can choose α arbitrarily close to 1, from (2.10), (2.11), and (2.9) we obtain

$$\lim_{x\to\infty}\frac{\Pi(x)}{x/\log x}=\lim_{x\to\infty}\frac{\Theta(x)}{x}=\lim_{x\to\infty}\frac{\Psi(x)}{x}=1.$$

(ii) Under the assumption of (2.8) Rudnick and Sarnak [16, Proposition 2.3] have shown that

$$\sum_{n\le x}\frac{|\Lambda(n)b_\pi(n)|^2}{n}=\frac{(\log x)^2}{2}+\mathrm{O}(\log x).$$

Employing (2.8) and partial summation in the above formula yields the desired result. ∎

Proposition 2.5. *Let $L(\pi,s)$ be a principal automorphic L-function. Suppose that for the local parameters at unramified primes we have*

$$|\alpha_\pi(p,j)|\le p^\gamma \quad \textit{for a fixed } 0\le\gamma<1/4. \tag{2.12}$$

Then

$$I_k(\pi,T)\gg T(\log T)^{k^2}$$

for rational $k\ge 0$.

Proof. Let S be the product of all ramified primes and $g(s)=L_S(\pi,s)=\prod_{p\nmid S}L_p(s,\pi_p)$ be the partial L-function. It is clear that if $\gcd(n,S)\ne 1$ then the nth coefficient of $L_S(\pi,s)$ is zero; otherwise $a_\pi(n)$ is the nth Dirichlet coefficient of $L_S(\pi,s)$. We have $L_S(\pi,s)\in\mathcal{C}$ (see [4, Section 5.12] for details). Moreover (2.12) implies that $\mathbf{H_0}$ in Lemma 2.4 is satisfied, and so (2.2) and (2.3) hold for $\beta=1$. Thus the conditions of Theorem 2.3 hold and therefore we deduce the desired lower bound for the fractional kth moment of $L_S(\pi,s)$.

Since $L_S(\pi, s)$ and $L(\pi, s)$ differ only by finitely many factors (bounded from below on $s = 1/2$), the same lower bound holds for $I_k(\pi, T)$. ■

COROLLARY 2.6. *The conclusion of Proposition* 2.5 *remains true if GRC holds for* π.

Since Dirichlet L-functions associated to primitive characters, modular L-functions associated to newforms, and twisted modular L-functions associated to newforms and primitive Dirichlet characters all are examples of principal automorphic L-functions that satisfy GRC (see [4, Sections 5.11 and 5.12] for details), as an immediate corollary of the above proposition we have the following extensions of the lower bound results of [6], [9], and [20].

COROLLARY 2.7. *For the Dirichlet L-function $L(\chi, s)$ attached to a primitive Dirichlet character and for any non-negative rational k we have*

$$I_k(\chi, T) \gg T(\log T)^{k^2}.$$

COROLLARY 2.8. *Let $L(f, s)$ be the L-function associated to a newform of weight ℓ, level N, and Nebentypus ψ. Then for any non-negative rational k we have*

$$I_k(f, T) \gg T(\log T)^{k^2}.$$

COROLLARY 2.9. *Let $L(f, \chi, s)$ be the L-function associated to a newform of weight ℓ, level N, Nebentypus ψ, and a primitive Dirichlet character χ mod q where $(q, N) = 1$. Then for any non-negative rational k we have*

$$I_k(f, \chi, T) \gg T(\log T)^{k^2},$$

where $I_k(f, \chi, T)$ denotes the kth (*power*) *moment of $L(f, \chi, s)$.*

The following is also a direct corollary of Proposition 2.5.

COROLLARY 2.10. *Let $L(f, s)$ be a Maass cusp newform of weight zero and level N with Nebentypus ψ. Then for any rational $k \geq 0$ we have*

$$I_k(f, T) \gg T(\log T)^{k^2}.$$

Proof. Let $\alpha_f(p, 1)$ and $\alpha_f(p, 2)$ be the local parameters of f at an unramified prime p. From a result of Kim and Sarnak [7, Proposition 2] we know that

$$|\alpha_f(p, 1)| \leq p^{7/64} \quad \text{and} \quad |\alpha_f(p, 2)| \leq p^{7/64}.$$

The result follows from Proposition 2.5 as $7/64 < 1/4$. ■

Artin L-functions. Let $K/\mathbb{Q}$ be a Galois extension with the Galois group $\mathrm{Gal}(K/\mathbb{Q})$ and let

$$\rho : \mathrm{Gal}(K/\mathbb{Q}) \to \mathrm{GL}_d(\mathbb{C})$$

be a representation of $\mathrm{Gal}(K/\mathbb{Q})$. Let $L(K/\mathbb{Q},\rho,s)$ be the Artin L-function associated to ρ, which is defined as the Euler product $\prod_p L_p(K/\mathbb{Q},\rho,s)$ of the local factors $L_p(K/\mathbb{Q},\rho,s)$ for $\sigma > 1$.

For unramified primes p the local factor of the Artin L-function is defined by

$$L_p(K/\mathbb{Q},\rho,s) = (\det(I_d - p^{-s}\rho(\sigma_p)))^{-1},$$

where I_d denotes the identity matrix, σ_p is the Frobenius conjugacy class, and $\rho(\sigma_p)$ is the value of ρ at any element belonging to σ_p. Letting $\alpha_\rho(p,j)$ $(1 \le j \le d)$ be the eigenvalues of the unitary matrix $\rho(\sigma_p)$, we have

$$L_p(K/\mathbb{Q},\rho,s) = \prod_{j=1}^{d}\left(1 - \frac{\alpha_\rho(p,j)}{p^s}\right)^{-1},$$

where $|\alpha_\rho(p,j)| = 1$.

As a consequence of the Brauer induction theorem it is known that any Artin L-function has a meromorphic continuation to the whole complex plane. Moreover *Artin's Conjecture* asserts that $L(K/\mathbb{Q},\rho,s)$ has an analytic continuation for all s except possibly for a pole at $s = 1$ of order equal to the multiplicity of the trivial representation in ρ.

PROPOSITION 2.11. *Assume Artin's Conjecture for the Artin L-function $L(K/\mathbb{Q},\rho,s)$. Then for any non-negative rational k, we have*

$$I_k(K/\mathbb{Q},\rho,T) \gg T(\log T)^{\langle\varphi,\varphi\rangle k^2},$$

where $I_k(K/\mathbb{Q},\rho,T)$ denotes the kth moment of $L(K/\mathbb{Q},\rho,s)$. Here φ is the character associated to ρ, and

$$\langle\varphi,\varphi\rangle = \frac{1}{|G|}\sum_{g\in G}|\varphi(g)|^2$$

where $G = \mathrm{Gal}(K/\mathbb{Q})$.

Proof. Set

$$L_{\mathrm{ur}}(K/\mathbb{Q},\rho,s) = \prod_{\text{unramified}\,p} L_p(K/\mathbb{Q},\rho,s) = \sum_{n=1}^{\infty}\frac{a_\rho(n)}{n^s}.$$

Under the assumption of the Artin Conjecture we know that $L_{\mathrm{ur}}(K/\mathbb{Q},\rho,s) \in \mathcal{C}$ (see [4, Section 5.13] for details), and for $\sigma > 1$, $L_{\mathrm{ur}}(K/\mathbb{Q},\rho,s)$ has an Euler product satisfying the conditions given in Theorem 2.3. Note that for unramified p we have $a_\rho(p) = \varphi(p)$. To apply Theorem 2.3 we need to study the asymptotic behaviour of $\sum_{p\le x}|a_\rho(p)|^2$. By decomposing this sum according to the conjugacy class C of G that contains σ_p, we have

$$\sum_{p\le x}|a_\rho(p)|^2 = \sum_{\text{unramified}\, p\le x}|\varphi(p)|^2 = \sum_C |\varphi(g_C)|^2(\#\{\text{unramified } p\le x\,;\,\sigma_p\subseteq C\}), \tag{2.13}$$

where g_C is any element of C. From the version of the Chebotarev density theorem with the remainder [8, Theorem 1.3] we get

$$\#\{\text{unramified } p\le x\,;\,\sigma_p\subseteq C\} = \frac{|C|}{|G|}\mathrm{Li}(x) + \mathrm{O}(x\exp(-c_0\sqrt{\log x/n})),$$

where $\mathrm{Li}(x)=\int_2^\infty dt/\log t\sim x/\log x$, c_0 is an absolute constant and n is the degree of K over $\mathbb{Q}$. Applying this in (2.13), we get

$$\sum_{p\le x}|a_\rho(p)|^2 = \langle\varphi,\varphi\rangle\mathrm{Li}(x) + \mathrm{O}\Big(\Big(\sum_C|\varphi(g_C)|^2\Big)x\exp(-c_0\sqrt{\log x/n})\Big).$$

It is clear that this formula implies (2.2) for $\beta=\langle\varphi,\varphi\rangle$. Moreover we can derive (2.3) from this formula by partial summation. Thus the conditions of Theorem 2.3 are satisfied and we get the desired lower bound for the kth moment of $L_{\mathrm{ur}}(K/\mathbb{Q},\rho,s)$. Since $L_{\mathrm{ur}}(K/\mathbb{Q},\rho,s)$ and $L(K/\mathbb{Q},\rho,s)$ differ only by finitely many factors (bounded from below on $s=1/2$), the same lower bound holds for $I_k(K/\mathbb{Q},\rho,T)$. ■

As a direct corollary of Proposition 2.11 we have the following unconditional improvement of Ramachandra's Theorem 1.7 for non-negative rational values of k.

Corollary 2.12. *Let K be a degree n Galois extension of $\mathbb{Q}$, and let $\zeta_K(s)$ be the Dedekind zeta function of K. Then for non-negative rational k we have*

$$I_k(\zeta_K,T)\gg T(\log T)^{nk^2}.$$

Proof. Let ρ be the regular representation of $\mathrm{Gal}(K/\mathbb{Q})$. In this case we have $L(K/\mathbb{Q},\rho,s)=\zeta_K(s)$ and Artin's Conjecture holds. Thus the result follows since $\langle\varphi,\varphi\rangle=n$, where φ is the character of the regular representation of $\mathrm{Gal}(K/\mathbb{Q})$. ■

Remark 2.13. R. Murty [13, Theorem 3.1] has shown that the Selberg Orthogonality Conjecture implies the Artin Conjecture. Thus in Proposition 2.11 we can replace the Artin Conjecture with the Selberg Orthogonality Conjecture.

3. Proof of Proposition 2.1. We start by recalling a classical theorem on the average values of multiplicative functions.

Lemma 3.1 (Levin and Fainleib). *Let $h(n)$ be a complex multiplicative function for which*

$$\sum_{p\le x} h(p) \sim \alpha \frac{x}{\log x}$$

as $x \to \infty$,

$$\sum_{p\le x} |h(p)| = \mathrm{O}\biggl(\frac{x}{\log x}\biggr),$$

and

$$h(p^r) \ll (2p)^{r\theta} \quad \text{for some } 0 \le \theta < 1/2 \text{ and all } r \ge 2.$$

Then, as $x \to \infty$,

$$\sum_{n\le x} h(n) \sim \frac{e^{-\gamma_0\alpha}}{\Gamma(\alpha)} \frac{x}{\log x} \Pi_h(x),$$

where α is a fixed number, $\gamma_0 = 0.5772\ldots$ is Euler's constant and

$$\Pi_h(x) = \prod_{p\le x}\biggl(1 + \sum_{r=1}^{\infty} \frac{h(p^r)}{p^r}\biggr).$$

Proof. See [10, Theorem 3]. ∎

Proof of Proposition 2.1. With the notation of Lemma 3.1 we have

$$\Pi_h(x) = \prod_{p\le x}\biggl(1 + \frac{h(p)}{p}\biggr)\prod_{p\le x}\biggl(1 + \frac{1}{1+h(p)/p}\sum_{r=2}^{\infty}\frac{h(p^r)}{p^r}\biggr).$$

Since $h(n)$ is non-negative and

$$\sum_{p\le x} h(p) \sim \alpha \frac{x}{\log x}$$

as $x \to \infty$, we have $h(p)/p \le 1/2$ for large enough p. This together with the bound given on $h(p^r)$ for $r \ge 2$ implies that

$$\prod_{p\le x}\biggl(1 + \frac{h(p)}{p}\biggr) \ll \Pi_h(x) \ll \prod_{p\le x}\biggl(1 + \frac{h(p)}{p}\biggr).$$

Moreover, by employing the bound $h(p)/p \le 1/2$ (for large p) we can deduce that

$$\exp\biggl(\sum_{p\le x}\frac{h(p)}{p} + \mathrm{O}(1)\biggr) \ll \Pi_h(x) \ll \exp\biggl(\sum_{p\le x}\frac{h(p)}{p} + \mathrm{O}(1)\biggr).$$

Combining

$$\sum_{p\le x}\frac{h(p)}{p} = \alpha\log\log x + \mathrm{O}(1)$$

with the previous inequality results in

$$(\log x)^{\alpha} \ll \mathit{\Pi}_h(x) \ll (\log x)^{\alpha}.$$

Thus by applying these bounds in Lemma 3.1, we have

$$x(\log x)^{\alpha-1} \ll \sum_{n\le x} h(n) \ll x(\log x)^{\alpha-1}. \tag{3.1}$$

Next note that by partial summation we have

$$\sum_{n=1}^{T} \frac{h(n)}{n^{2\sigma}} = \frac{1}{T^{2\sigma}} \sum_{n\le T} h(n) + 2\sigma \int_1^T \Big(\sum_{n\le u} h(n)\Big) \frac{du}{u^{2\sigma+1}}. \tag{3.2}$$

By application of the upper bound of (3.1) in (3.2) and considering that $1/2 + c/\log T \le \sigma \le 1$ we deduce that

$$\begin{aligned}
\sum_{n=1}^{T} \frac{h(n)}{n^{2\sigma}} &\ll e^{-2c}(\log T)^{\alpha-1} + \int_1^T (\log u)^{\alpha-1} \frac{du}{u^{2\sigma}} \\
&\ll (\log T)^{\alpha-1} + (2\sigma-1)^{-\alpha} \int_0^{(2\sigma-1)\log T} e^{-t} t^{\alpha-1}\, dt \\
&\ll (\log T)^{\alpha-1} + (\sigma-1/2)^{-\alpha} \ll (\log T)^{\alpha}.
\end{aligned}$$

Similarly by employing the lower bound of (3.1) in (3.2), and considering that $1/2 + c/\log T \le \sigma \le 1$, we have

$$\begin{aligned}
\sum_{n=1}^{T} \frac{h(n)}{n^{2\sigma}} &\gg (2\sigma-1)^{-\alpha} \int_0^{(2\sigma-1)\log T} e^{-t} t^{\alpha-1}\, dt \\
&\gg (2\sigma-1)^{-\alpha} \int_0^{2c} e^{-t} t^{\alpha-1}\, dt \gg (\sigma-1/2)^{-\alpha}. \ \blacksquare
\end{aligned}$$

4. Proof of Theorem 1.5. In this section, we aim to prove Theorem 1.5. The proof closely follows the method devised by Heath-Brown in [3]. We assume that (1.2) holds for g, and moreover (1.3) and (1.4) hold for rational $k \ge 0$. Note that from (1.4) it follows that

$$(\log T)^{\alpha} \ll \sum_{n=1}^{T} \frac{|g_k(n)|^2}{n} \ll (\log T)^{\alpha}. \tag{4.1}$$

For $T \ge 2$ and $-\infty < t < \infty$, let

$$w(t,T) = \int_T^{2T} \exp(-(t-\tau)^2)\, d\tau.$$

The next lemma summarizes some properties of $w(t,T)$ that will be used later.

LEMMA 4.1. *Let $T \geq 2$ and $-\infty < t < \infty$. We have*

$$\begin{aligned}
&w(t,T) \ll 1 \quad \text{for all } t \text{ and } T,\\
&w(t,T) \ll \exp\Big(-\frac{1}{36}(t^2+T^2)\Big) \quad \text{for } t \leq 0 \text{ and } t \geq 3T,\\
&w(t,T) \gg 1 \quad \text{for all } 4T/3 \leq t \leq 5T/3.
\end{aligned}$$

Proof. See [3]. ∎

Define

$$J_k(g,\sigma,T) = \int_{-\infty}^{\infty} |g(\sigma+it)|^{2k} w(t,T)\, dt.$$

The next lemma shows that when finding a lower bound for $I_k(g,T)$ it is enough to find a lower bound for $J_k(g,1/2,T)$.

LEMMA 4.2. *We have*

$$I_k(g,3T) + e^{-T^2/37} \gg J_k(g,1/2,T).$$

Proof. By employing (1.2) and Lemma 4.1 we have

$$\begin{aligned}
J_k(g,1/2,T) &= \int_{-\infty}^{\infty} |g(1/2+it)|^{2k} w(t,T)\, dt\\
&\ll \int_{0}^{3T} |g(1/2+it)|^{2k}\, dt\\
&\quad + \Big(\int_{-\infty}^{0} + \int_{3T}^{\infty}\Big)\Big((1+|t|)^{2kA} \exp\Big(-\frac{1}{36}(t^2+T^2)\Big)\Big)\, dt\\
&\ll \int_{1}^{3T} |g(1/2+it)|^{2k}\, dt + e^{-T^2/37}. \ \blacksquare
\end{aligned}$$

Let $0 < \lambda < \sqrt{3}/24\pi$ be fixed. We now define

$$s_k(g,s,T) = \sum_{n=1}^{\lambda T} \frac{g_k(n)}{n^s}, \tag{4.2}$$

and let

$$S_k(g,\sigma,T) = \int_{-\infty}^{\infty} |s_k(g,\sigma+it,T)|^2 w(t,T)\, dt.$$

Observe that, by (1.3), for $\sigma \geq 1/2$ we have

$$|s_k(g,s,T)| \leq \sum_{n=1}^{\lambda T} \frac{|g_k(n)|}{n^{1/2}} \ll \sum_{n=1}^{\lambda T} n^{\eta-1/2} \ll T^{\eta+1/2} \ll T. \tag{4.3}$$

We find upper and lower bounds for $S_k(g,\sigma,T)$.

Lemma 4.3. *Let $1/2 \leq \sigma \leq 3/4$ and $0 < \lambda < \sqrt{3}/24\pi$. Then*

$$T\sum_{n=1}^{\lambda T} \frac{|g_k(n)|^2}{n^{2\sigma}} \ll S_k(g,\sigma,T) \ll T\sum_{n=1}^{\lambda T} \frac{|g_k(n)|^2}{n^{2\sigma}}.$$

Proof. Recall the following mean value theorem for Dirichlet polynomials:

$$\int_0^T \Big|\sum_{n=1}^{N} a_n n^{-it}\Big|^2 dt = \Big(T + \xi\frac{4\pi}{\sqrt{3}}N\Big)\sum_{n=1}^{N} |a_n|^2,$$

where $-1 \leq \xi \leq 1$ (see [11, p. 50]).

We use Lemma 4.1, (4.3), and the above mean value theorem for $N = \lambda T$ to deduce that

$$\begin{aligned} S_k(g,\sigma,T) &= \int_0^{3T} |s_k(g,\sigma+it,T)|^2 w(t,T)\,dt + \mathrm{O}(1) \\ &\ll \int_0^{3T} \Big|\sum_{n=1}^{\lambda T} \frac{g_k(n)}{n^{\sigma+it}}\Big|^2 dt + 1 = (3T + \mathrm{O}(T))\sum_{n=1}^{\lambda T} \frac{|g_k(n)|^2}{n^{2\sigma}} \\ &\ll T\sum_{n=1}^{\lambda T} \frac{|g_k(n)|^2}{n^{2\sigma}}. \end{aligned}$$

We again use Lemma 4.1 and the above mean value theorem for $N = \lambda T$ to deduce that

$$\begin{aligned} S_k(g,\sigma,T) &\gg \int_{4T/3}^{5T/3} \Big|\sum_{n=1}^{\lambda T} \frac{g_k(n)}{n^{\sigma+it}}\Big|^2 dt = \Big(\frac{T}{3} + \lambda\xi'\frac{4\pi}{\sqrt{3}}T\Big)\sum_{n=1}^{\lambda T} \frac{|g_k(n)|^2}{n^{2\sigma}} \\ &\gg T\sum_{n=1}^{\lambda T} \frac{|g_k(n)|^2}{n^{2\sigma}}. \end{aligned}$$

Note that in the above formula $-2 \leq \xi' \leq 2$. ∎

Next we obtain an upper bound on $J_k(g,\sigma,T)$.

Lemma 4.4. *Let $1/2 \leq \sigma \leq 3/4$ and $T \geq 2$. Then*

$$J_k(g,\sigma,T) \ll T^{\sigma-1/2} J_k(g,1/2,T)^{3/2-\sigma} + e^{-T^2/7}.$$

The proof of this lemma is given in Section 5.

Write $k = u/v$, where u and v are positive coprime integers. We define

$$d_k(g, s, T) = g^u(s) - s_k^v(g, s, T),$$
$$D_k(g, \sigma, T) = \int_{-\infty}^{\infty} |d_k(g, \sigma + it, T)|^{2/v} w(t, T)\, dt.$$

We find an upper bound for $D_k(g, \sigma, T)$.

Lemma 4.5. *Let $1/2 \le \sigma \le 3/4$, $k = u/v$, and $T \ge 2$. Then*

$$D_k(g, \sigma, T) \ll D_k(g, 1/2, T)^{\frac{5+2\eta-4\sigma}{3+2\eta}} (T^{1-(1+\epsilon)/v})^{\frac{4\sigma-2}{3+2\eta}}$$
$$+ D_k(g, 1/2, T)^{\frac{7+2\eta-8\sigma}{3+2\eta}} e^{-\frac{2\sigma-1}{9+6\eta}T^2},$$

where $0 \le \eta < 1/2$ is given in (1.3) and $0 < \epsilon < 1/2 - \eta$.

The proof of this lemma is given in Section 5.

We now use the bounds from Lemmas 4.4, 4.5 and 4.3 to find a lower bound for $J_k(g, 1/2, T)$.

Proposition 4.6. *We have*

$$J_k(g, 1/2, T) \gg T(\log T)^\alpha.$$

Proof. Notice that by the definition of $d_k(g, s, T)$ we have

$$|s_k(g, s, T)|^2 = |s_k^v(g, s, T)|^{2/v} = |g^u(s) - d_k(g, s, T)|^{2/v}$$
$$\ll |g(s)|^{2k} + |d_k(g, s, T)|^{2/v}.$$

Thus

$$S_k(g, \sigma, T) \ll J_k(g, \sigma, T) + D_k(g, \sigma, T). \tag{4.4}$$

We also have

$$|d_k(g, s, T)|^{2/v} = |g(s)^u - s_k^v(g, s, T)|^{2/v} \ll |g(s)|^{2k} + |s_k(g, s, T)|^2.$$

Thus

$$D_k(g, 1/2, T) \ll J_k(g, 1/2, T) + S_k(g, 1/2, T). \tag{4.5}$$

We now consider two cases, and show that the lemma follows from either of them.

Case $D_k(g, 1/2, T) \le T$. In this case, Lemma 4.3, (4.4) with $\sigma = 1/2$, and (4.1) imply

$$T(\log T)^\alpha \ll T \sum_{n=1}^{\lambda T} \frac{|g_k(n)|^2}{n} \ll J_k(g, 1/2, T) + T,$$

from which the lemma follows.

CASE $D_k(g, 1/2, T) > T$. Then Lemma 4.5 yields

$$D_k(g,\sigma,T) \ll D_k(g,1/2,T)\big(T^{\frac{(1+\epsilon)(2-4\sigma)}{v(3+2\eta)}} + e^{-\frac{2\sigma-1}{9+6\eta}T^2}\big) \ll D_k(g,1/2,T)T^{\frac{1-2\sigma}{2v}}, \tag{4.6}$$

the last line following because $\eta < 1/2$ by (1.3) and $1-2\sigma \le 0$. We see that (4.4) followed by (4.6) and (4.5) implies

$$S_k(g,\sigma,T) \ll J_k(g,\sigma,T) + (J_k(g,1/2,T) + S_k(g,1/2,T))T^{\frac{1-2\sigma}{2v}}.$$

So there exists a constant $c(k) > 0$ such that for any given σ and T we have either

$$S_k(g,\sigma,T) \le c(k)(S_k(g,1/2,T)T^{\frac{1-2\sigma}{2v}}) \tag{4.7}$$

or

$$S_k(g,\sigma,T) \le c(k)(J_k(g,\sigma,T) + J_k(g,1/2,T)T^{\frac{1-2\sigma}{2v}}). \tag{4.8}$$

We claim that (4.7) cannot hold for all σ and T. The reason is that (4.7) together with Lemma 4.3 and (1.4) yields

$$\frac{(\sigma-1/2)^{-\alpha}}{(\log T)^{\alpha}} \ll \frac{\sum_{n=1}^{\lambda T}|g_k(n)|^2/n^{2\sigma}}{\sum_{n=1}^{\lambda T}|g_k(n)|^2/n} \ll \frac{S_k(g,\sigma,T)}{S_k(g,1/2,T)} \ll T^{\frac{1-2\sigma}{2v}}$$

uniformly for $1/2 + c/\log\lambda T \le \sigma \le 3/4$, $\lambda T \ge 2$. Letting $\sigma = 1/2 + \delta/\log T$ in the above inequality yields

$$\delta^{-\alpha} \ll e^{-\delta/v},$$

which cannot be true for large values of δ. Thus (4.7) is false for a value δ_0, and so (4.8) must hold for $\sigma_0 = 1/2 + \delta_0/\log T$ and T large enough. Using these values of σ and T in (4.8) and applying Lemma 4.4 yields

$$S_k(g,\sigma_0,T) \ll T^{\sigma_0-1/2}J_k(g,1/2,T)^{3/2-\sigma_0} + T^{\frac{1-2\sigma_0}{2v}}J_k(g,1/2,T) + e^{-T^2/7}. \tag{4.9}$$

Then by the lower bound for $S_k(g,\sigma_0,T)$ given in Lemma 4.3, (4.9), and (4.1), we deduce that

$$T(\log T)^{\alpha} \ll T\sum_{n=1}^{\lambda T}\frac{|g_k(n)|^2}{n} \ll S_k(g,\sigma_0,T) \ll J_k(g,1/2,T). \quad \blacksquare$$

We are ready to prove our main result.

Proof of Theorem 1.5. First assume that (1.3) and (1.4) hold for rational $k \ge 0$. Lemma 4.2 and Proposition 4.6 yield

$$I_k(g,T)+e^{-(T/3)^2/37} \gg J_k(g,1/2,T/3) \gg T\sum_{n=1}^{\lambda T/3}\frac{|g_k(n)|^2}{n} \gg T(\log T)^{\alpha}.$$

Next assume that (1.3) and (1.4) hold for real $k \geq 0$ and $g(s)$ does not have any zero in the half-plane $\sigma > 1/2$. Then it is clear that Lemmas 4.2–4.4 extend to real $k \geq 0$. Moreover, since $g(s)$ has no zero in the half-plane $\sigma > 1/2$, $(g(s))^k$ is analytic on the half-plane $\sigma > 1/2$ (except possibly at $s = 1$) and so Lemma 4.5 holds for $u = k$ and $v = 1$. Thus the theorem follows similarly to the case for rational k. ∎

5. Proofs of Lemmas 4.4 and 4.5. We state a result which we will utilize in the proofs of this section.

LEMMA 5.1 (Gabriel, [2, Theorem 2]). *Let $f(s)$ be analytic in the infinite strip $\alpha < \sigma < \beta$, and continuous for $\alpha \leq \sigma \leq \beta$. Suppose $f(s) \to 0$ as $|\mathrm{Im}(s)| \to \infty$ uniformly for $\alpha \leq \sigma \leq \beta$. Then for $\alpha \leq \gamma \leq \beta$ and any $q > 0$ we have*

$$\int_{-\infty}^{\infty} |f(\gamma+it)|^q\,dt \leq \Big(\int_{-\infty}^{\infty} |f(\alpha+it)|^q\,dt\Big)^{\frac{\beta-\gamma}{\beta-\alpha}}\Big(\int_{-\infty}^{\infty} |f(\beta+it)|^q\,dt\Big)^{\frac{\gamma-\alpha}{\beta-\alpha}}. \ ∎$$

Proof of Lemma 4.4. Let $f(s) = (s-1)^m g(s)\exp\big(\frac{1}{2k}(s-i\tau)^2\big)$, $\gamma = \sigma$, $\alpha = 1/2$, $\beta = 3/2$, $q = 2k$ and $1/2 \leq \sigma \leq 3/4$.

We use Lemma 5.1 to see that

$$\int_{-\infty}^{\infty} |f(\sigma+it)|^{2k}\,dt \leq \Big(\int_{-\infty}^{\infty} |f(1/2+it)|^{2k}\,dt\Big)^{3/2-\sigma} \times \Big(\int_{-\infty}^{\infty} |f(3/2+it)|^{2k}\,dt\Big)^{\sigma-1/2}. \tag{5.1}$$

Next we consider

$$\int_{-\infty}^{\infty} |f(1/2+it)|^{2k}\,dt = \int_{-\infty}^{\infty} |-1/2+it|^{2mk}|g(1/2+it)|^{2k}|e^{(1/2+it-i\tau)^2}|\,dt. \tag{5.2}$$

Since $|e^{(1/2+it-i\tau)^2}| \leq e^{1/4}e^{-(t-\tau)^2}$, by employing (1.2) we have

$$\Big(\int_{-\infty}^{\tau/2} + \int_{3\tau/2}^{\infty}\Big)(|f(1/2+it)|^{2k})\,dt \ll e^{-\tau^2/5}.$$

Using this with (5.2) yields

$$\int_{-\infty}^{\infty} |f(1/2+it)|^{2k}\,dt \ll \tau^{2mk} \int_{\tau/2}^{3\tau/2} |g(1/2+it)|^{2k} e^{-(t-\tau)^2}\,dt + e^{-\tau^2/5}. \tag{5.3}$$

In a similar fashion, we may show that

$$\int_{-\infty}^{\infty} |f(3/2+it)|^{2k}\,dt \ll \tau^{2mk}. \tag{5.4}$$

Using (5.3) and (5.4) with (5.1) yields

$$\begin{aligned}\int_{-\infty}^{\infty} &|f(\sigma+it)|^{2k}\,dt \\ &\ll \tau^{2mk}\Big(\int_{-\infty}^{\infty} |g(1/2+it)|^{2k} e^{-(t-\tau)^2}\,dt\Big)^{3/2-\sigma} + e^{-\tau^2/6}.\end{aligned} \tag{5.5}$$

Moreover, since $e^{-(t-\tau)^2} \le |e^{(\sigma+it-i\tau)^2}|$, we have

$$\begin{aligned}\int_{-\infty}^{\infty} |g(\sigma+it)|^{2k} e^{-(t-\tau)^2}\,dt &\ll \int_{\tau/2}^{3\tau/2} |g(\sigma+it)|^{2k} e^{-(t-\tau)^2}\,dt + e^{-\tau^2/5} \\ &\ll \tau^{-2mk} \int_{-\infty}^{\infty} |f(\sigma+it)|^{2k}\,dt + e^{-\tau^2/5}.\end{aligned} \tag{5.6}$$

Now (5.5) and (5.6) together imply

$$\int_{-\infty}^{\infty} |g(\sigma+it)|^{2k} e^{-(t-\tau)^2}\,dt \ll \Big(\int_{-\infty}^{\infty} |g(1/2+it)|^{2k} e^{-(t-\tau)^2}\,dt\Big)^{3/2-\sigma} + e^{-\tau^2/6}.$$

Integrating over $T \le \tau \le 2T$ and using Hölder's inequality completes the proof of the lemma. ∎

The subsequent proof will require the use of another theorem due to Gabriel.

LEMMA 5.2 (Gabriel, [2, Theorem 1]). *Let R be a rectangle with vertices s_0, $\overline{s}_0$, $-s_0$ and $-\overline{s}_0$. Let $F(s)$ be continuous on R and analytic on the interior of R. Then*

$$\int_L |F(s)|^q\,|ds| \le \Big(\int_{P_1} |F(s)|^q\,|ds|\Big)^{1/2}\Big(\int_{P_2} |F(s)|^q\,|ds|\Big)^{1/2}$$

for any $q \ge 0$, where L is the line segment from $\frac{1}{2}(\overline{s}_0 - s_0)$ to $\frac{1}{2}(s_0 - \overline{s}_0)$, P_1 consists of the three line segments connecting $\frac{1}{2}(\overline{s}_0 - s_0)$, $\overline{s}_0$, s_0 and $\frac{1}{2}(s_0 - \overline{s}_0)$, and P_2 is the mirror image of P_1 in L.

We also need the following mean value theorem for Dirichlet series.

LEMMA 5.3 (Montgomery–Vaughan, [12, Corollary 3]). *If* $\sum_{n=1}^{\infty} n|a_n|^2 < \infty$, *then*

$$\int_0^T \Big|\sum_{n=1}^{\infty} a_n n^{-it}\Big|^2 dt = \sum_{n=1}^{\infty} |a_n|^2 (T + \mathrm{O}(n)),$$

the implied constant being absolute.

Proof of Lemma 4.5. Using Lemma 5.1 with

$$f(s) = d_k(g, s, T) \exp\bigg(\frac{v}{2}(s - i\tau)^2\bigg),$$

$\gamma = \sigma$, $\alpha = 1/2$, $3/4 < \beta < 1$, and $q = 2/v$, we get

$$\int_{-\infty}^{\infty} |f(\sigma + it)|^{2/v}\, dt \le \Big(\int_{-\infty}^{\infty} |f(1/2 + it)|^{2/v}\, dt\Big)^{\frac{\beta-\sigma}{\beta-1/2}} \times \Big(\int_{-\infty}^{\infty} |f(\beta + it)|^{2/v}\, dt\Big)^{\frac{\sigma-1/2}{\beta-1/2}}. \tag{5.7}$$

By (1.2) and (4.3), we have

$$d_k(g, s, T) \ll T^v + |t|^{Au} \ll (T + |t|^A)^{u+v}. \tag{5.8}$$

Thus (5.8) yields

$$\int_{-\infty}^{\infty} |f(\beta + it)|^{2/v}\, dt = \int_{\tau/2}^{3\tau/2} |f(\beta + it)|^{2/v}\, dt + O\bigg(T^{2+2k}\Big(\int_{-\infty}^{\tau/2} + \int_{3\tau/2}^{\infty}\Big)\bigg(\bigg(1 + \frac{|t|^A}{T}\bigg)^{2+2k} e^{-(t-\tau)^2}\bigg)\, dt\bigg). \tag{5.9}$$

We observe that

$$\Big(\int_{-\infty}^{\tau/2} + \int_{3\tau/2}^{\infty}\Big)\bigg(\bigg(1 + \frac{|t|^A}{T}\bigg)^{2+2k} e^{-(t-\tau)^2}\bigg)\, dt \ll e^{-\tau^2/5}. \tag{5.10}$$

Putting (5.9) and (5.10) together, we have

$$\int_{-\infty}^{\infty} |f(\beta + it)|^{2/v}\, dt = \int_{\tau/2}^{3\tau/2} |f(\beta + it)|^{2/v}\, dt + \mathrm{O}(T^{2+2k} e^{-\tau^2/5}). \tag{5.11}$$

We will now use Lemma 5.2 with $F(s) = f(s+\beta+i\tau)$, $s_0 = \beta-1/2+(1/2)i\tau$ and $q = 2/v$. This allows us to avoid the possible pole of $g(s)$. On one hand, we have

$$\int_L |F(s)|^q \, |ds| = \int_{\tau/2}^{3\tau/2} |f(\beta+it)|^{2/v} \, dt. \tag{5.12}$$

Also,

$$\begin{aligned} \int_{P_1} |F(s)|^q \, |ds| &= \int_{\tau/2}^{3\tau/2} |f(2\beta-1/2+it)|^{2/v} \, dt \\ &\quad + \int_{\beta}^{2\beta-1/2} \left(\left| f\left(\mu+\frac{1}{2}i\tau\right) \right|^{2/v} + \left| f\left(\mu+\frac{3}{2}i\tau\right) \right|^{2/v} \right) d\mu \end{aligned} \tag{5.13}$$

and

$$\begin{aligned} \int_{P_2} |F(s)|^q \, |ds| &= \int_{\tau/2}^{3\tau/2} |f(1/2+it)|^{2/v} \, dt \\ &\quad + \int_{1/2}^{\beta} \left(\left| f\left(\mu+\frac{1}{2}i\tau\right) \right|^{2/v} + \left| f\left(\mu+\frac{3}{2}i\tau\right) \right|^{2/v} \right) d\mu. \end{aligned} \tag{5.14}$$

However, (5.8) yields

$$\begin{aligned} f\left(\mu+\frac{1}{2}i\tau\right) &= d_k\left(g, \mu+\frac{1}{2}i\tau, T\right) \exp\left(\frac{v}{2}\left(\mu-\frac{1}{2}i\tau\right)^2\right) \\ &\ll (T+\tau^A)^{u+v} e^{-v\tau^2/8} \end{aligned} \tag{5.15}$$

and similarly

$$f\left(\mu+\frac{3}{2}i\tau\right) \ll (T+\tau^A)^{u+v} e^{-v\tau^2/8}. \tag{5.16}$$

Thus (5.15) and (5.16) show that the second integrals in (5.13) and (5.14) are $\ll T^{2+2k} e^{-\tau^2/5}$. Using this, along with Lemma 5.2 and (5.12)–(5.14), we have

$$\begin{aligned} \int_{\tau/2}^{3\tau/2} |f(\beta+it)|^{2/v} \, dt &\ll \Big(\int_{-\infty}^{\infty} |f(1/2+it)|^{2/v} \, dt \Big)^{1/2} \\ &\quad \times \Big(\int_{\tau/2}^{3\tau/2} |f(2\beta-1/2+it)|^{2/v} \, dt \Big)^{1/2} + T^{2+2k} e^{-\tau^2/11}. \end{aligned} \tag{5.17}$$

From (5.11) and (5.17) together with (5.7) and the definition of $f(s)$ we obtain

$$\int_{-\infty}^{\infty} |d_k(g,\sigma+it,T)|^{2/v} e^{-(t-\tau)^2}\,dt \tag{5.18}$$
$$\ll \Big(\int_{-\infty}^{\infty} |d_k(g,1/2+it,T)|^{2/v} e^{-(t-\tau)^2}\,dt\Big)^{\frac{\beta-\sigma/2-1/4}{\beta-1/2}}$$
$$\times \Big(\int_{\tau/2}^{3\tau/2} |d_k(g,2\beta-1/2+it,T)|^{2/v} e^{-(t-\tau)^2}\,dt\Big)^{\frac{\sigma/2-1/4}{\beta-1/2}}$$
$$+ \Big(\int_{-\infty}^{\infty} |d_k(g,1/2+it,T)|^{2/v} e^{-(t-\tau)^2}\,dt\Big)^{\frac{\beta-\sigma}{\beta-1/2}}$$
$$\times (T^{2+2k} e^{-\tau^2/11})^{\frac{\sigma-1/2}{\beta-1/2}}.$$

Integrating (5.18) for $T \le \tau \le 2T$ and applying Hölder's inequality yields

$$D_k(g,\sigma,T) \ll D_k(g,1/2,T)^{\frac{\beta-\sigma/2-1/4}{\beta-1/2}} \tag{5.19}$$
$$\times \Big(\int_{T}^{2T}\int_{\tau/2}^{3\tau/2} |d_k(g,2\beta-1/2+it,T)|^{2/v} e^{-(t-\tau)^2}\,dt\,d\tau\Big)^{\frac{\sigma/2-1/4}{\beta-1/2}}$$
$$+ D_k(g,1/2,T)^{\frac{\beta-\sigma}{\beta-1/2}} e^{-\frac{2\sigma-1}{12(2\beta-1)}T^2}.$$

Since $w(t,T) \ll 1$, we have

$$\int_{T}^{2T}\int_{\tau/2}^{3\tau/2} |d_k(g,2\beta-1/2+it,T)|^{2/v} e^{-(t-\tau)^2}\,dt\,d\tau$$
$$\ll \int_{T/2}^{3T} |d_k(g,2\beta-1/2+it,T)|^{2/v}\,dt.$$

Now choose $\beta > 3/4$ so that $2\beta - 1/2 > 1$. By this choice of β, we see that $d_k(g, 2\beta-1/2+it,T)$ is represented by an absolutely convergent Dirichlet series. Also note that

$$g_u(n) = g_{kv}(n) = \sum_{n=n_1\cdots n_v} g_k(n_1)\cdots g_k(n_v).$$

Thus, for $\sigma > 1$, we have

$$d_k(g,s,T) = g^u(s) - s_k^v(g,s,T) \tag{5.20}$$
$$= \sum_{n=1}^{\infty} \frac{g_u(n)}{n^s} - \Big(\sum_{n=1}^{\lambda T} \frac{g_k(n)}{n^s}\Big)^v = \sum_{n=\lambda T}^{\infty} \frac{a(n)}{n^s},$$

where

$$|a(n)| = \Big|g_u(n) - \sum_{\substack{n=n_1\cdots n_v \\ n_j \le \lambda T \text{ for } j=1,\dots,v}} g_k(n_1)\cdots g_k(n_v)\Big| \tag{5.21}$$

$$= \Big|\sum_{\substack{n=n_1\cdots n_v \\ \exists j \text{ such that } n_j > \lambda T}} g_k(n_1)\cdots g_k(n_v)\Big|$$

$$\le \sum_{n=n_1\cdots n_v} |g_k(n_1)|\cdots|g_k(n_v)| \le n^\eta \tau_v(n) \ll n^{\eta+\epsilon_0}.$$

In the last inequality, ϵ_0 is a positive constant that will be chosen appropriately later. Thus, (5.20) and (5.21) allow us to use Lemma 5.3 to get

$$\int_{T/2}^{3T} |d_k(g, 2\beta - 1/2 + it, T)|^2\,dt = \sum_{n=\lambda T}^{\infty} \frac{|a(n)|^2}{n^{4\beta-1}}\left(\frac{5}{2}T + \mathrm{O}(n)\right) \tag{5.22}$$

$$\ll T\int_{\lambda T}^{\infty} t^{2\eta+2\epsilon_0-4\beta+1}\,dt + \int_{\lambda T}^{\infty} t^{2\eta+2\epsilon_0-4\beta+2}\,dt$$

$$\ll T^{2(\eta+\epsilon_0)-4\beta+3}.$$

The last inequality holds if $\beta > (\eta+\epsilon_0)/2 + 3/4$. We now use Hölder's inequality and (5.22) to see that

$$\int_{T/2}^{3T} |d_k(g, 2\beta - 1/2 + it, T)|^{2/v}\,dt \le \left(\frac{5}{2}T\right)^{1-1/v} (T^{2(\eta+\epsilon_0)-4\beta+3})^{1/v} \tag{5.23}$$

$$\ll T^{1+(2(\eta+\epsilon_0)-4\beta+2)/v}.$$

Applying the bound (5.23) in (5.19) and choosing $\epsilon_0 = (1/2)(1/2-\eta) - \epsilon/2$ and $\beta = 7/8 + (1/4)\eta$ completes the proof of the lemma. ■

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References

[1] V. Chandee, *On the correlation of shifted values of the Riemann zeta function*, Quart. J. Math., to appear.

[2] R. M. Gabriel, *Some results concerning the integrals of moduli of regular functions along certain curves*, J. London Math. Soc. 2 (1927), 112–117.

[3] D. R. Heath-Brown, *Fractional moments of the Riemann zeta-function*, ibid. 24 (1981), 65–78.

[4] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, Amer. Math. Soc. Colloq. Publ. 53, Amer. Math. Soc., 2004.
[5] G. H. Ji, *Lower bounds for moments of automorphic L-functions over short intervals*, Proc. Amer. Math. Soc. 137 (2009), 3569–3574.
[6] A. Kačėnas, A. Laurinčikas and S. Zamarys, *On fractional moments of Dirichlet L-functions*, Lithuanian Math. J. 45 (2005), 173–191.
[7] H. H. Kim and P. Sarnak, *Refined estimates towards the Ramanujan and Selberg conjectures*, Appendix 2 to: H. H. Kim, *Functoriality for the exterior square of* GL_4 *and the symmetric fourth of* GL_2, J. Amer. Math. Soc. 16 (2003), 139–183.
[8] J. C. Lagarias and A. M. Odlyzko, *Effective versions of the Chebotarev density theorem*, in: Algebraic Number Fields: L-Functions and Galois Properties, A. Fröhlich (ed.), Academic Press, 1977, 409–464.
[9] A. Laurinčikas and J. Steuding, *On fractional power moments of zeta-functions associated with certain cusp forms*, Acta Appl. Math. 97 (2007), 25–39.
[10] B. V. Levin and A. S. Fainleib, *On a method of summation of multiplicative functions*, Izv. Akad. Nauk SSSR Ser. Mat. 31 (1967), 697–710 (in Russian).
[11] H. L. Montgomery, *Topics in Multiplicative Number Theory*, Lecture Notes in Math. 227, Springer, 1971.
[12] H. L. Montgomery and R. C. Vaughan, *Hilbert's inequality*, J. London Math. Soc. (2) 8 (1974), 73–82.
[13] M. R. Murty, *Selberg's conjectures and Artin L-functions*, Bull. Amer. Math. Soc. 31 (1994), 1–14.
[14] K. Ramachandra, *Some remarks on the mean value of the Riemann zeta-function and other Dirichlet series. I*, Hardy–Ramanujan J. 1 (1978), 1–15.
[15] —, *Some remarks on the mean value of the Riemann zeta-function and other Dirichlet series. II*, ibid. 3 (1980), 1–24.
[16] Z. Rudnick and P. Sarnak, *Zeros of principal L-functions and random matrix theory*, Duke Math. J. 81 (1996), 269–322.
[17] Z. Rudnick and K. Soundararajan, *Lower bounds for moments of L-functions*, Proc. Nat. Acad. Sci. USA 102 (2005), 6837–6838.
[18] E. C. Titchmarsh, *The Theory of Functions*, 2nd ed., Oxford Univ. Press, 1939.
[19] —, *The Theory of the Riemann Zeta-Function*, Clarendon Press, 1951.
[20] S. Zamarys, *On fractional moments of twists of L-functions attached to cusp forms*, Šiauliai Math. Semin. 4 (12) (2009), 211–231.

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