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On determination of GL_3 cusp forms

by

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1. Introduction. In 1997, Luo and Ramakrishnan [LR] studied to what extent modular forms can be characterized by central values of twisted L-functions and their derivatives. Since then, this subject has been studied by many authors for various modular forms (see Luo [L] and Ganguly, Hoffstein and Sengupta [GHS]). The determination of GL_3 cusp forms by central values of twisted L-functions was first studied by Chinta and Diaconu [CD] as a generalization of the results in [LR]. Recently, Liu [Liu] studied the question of determining a GL_3 self-dual Hecke–Maass cusp form by central values of its GL_2 twisted L-functions. Precisely, let f be a fixed self-dual Hecke–Maass cusp form for $SL_3(\mathbb{Z})$ and let $A(m,n)$ denote its (m,n)th Fourier coefficient. Liu proved that f is uniquely determined by the family $\{L(1/2, f\times g) : g \in \mathcal{H}_k\}$, where $\mathcal{H}_k$ is an orthogonal basis of holomorphic cusp forms of weight $k \equiv 0 \pmod 4$ for $SL_2(\mathbb{Z})$.

In this paper, we will show that f is also uniquely determined by the family $\{L'(1/2, f\times g) : g \in \mathcal{B}_k\}$, where $\mathcal{B}_k$ is an orthogonal basis of holomorphic cusp forms of weight $k \equiv 2 \pmod 4$ for $SL_2(\mathbb{Z})$.

THEOREM 1.1. *Let f and f' be fixed self-dual Hecke–Maass cusp forms for $SL_3(\mathbb{Z})$. Let $c \neq 0$ be a constant. If*

$$L'(1/2, f\times g) = cL'(1/2, f'\times g) \tag{1.1}$$

for all $g \in \mathcal{B}_k$, then $f = f'$.

As in [Liu], we will prove Theorem 1.1 by establishing an asymptotic formula for the first twisted moment of $L'(s, f\times g)$ at $s = 1/2$, where g runs over $\mathcal{B}_k$. Let $\lambda_g(n)$ be the normalized nth Fourier coefficient of $g \in \mathcal{B}_k$ and

$$\omega_g = \frac{k-1}{2\pi^2} L(1, \mathrm{sym}^2 g),$$

where $L(s, \mathrm{sym}^2 g)$ is the symmetric-square L-function associated to g.

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THEOREM 1.2. *Let h be a fixed positive valued, smooth function of compact support on $[1,2]$, with derivatives satisfying $h^{(j)} \ll_j 1$. Given any prime p, we have*

$$\sum_{k\equiv 2\,(\mathrm{mod}\,4)} h\left(\frac{k-1}{K}\right)\sum_{g\in\mathcal{B}_k}\omega_g^{-1}L'(1/2,f\times g)\lambda_g(p)$$
$$= \frac{3(A(1,p)-p^{-1})L(1,f)}{2\sqrt{p}}\widehat{h}(0)K\log K$$
$$+ \frac{(A(1,p)-p^{-1})(2L'(1,f)+c_0L(1,f))}{2\sqrt{p}}\widehat{h}(0)K$$
$$+ \frac{\log p}{p\sqrt{p}}L(1,f)\widehat{h}(0)K + O_{\epsilon,f,p}(K^{\epsilon})$$

for any $\epsilon > 0$ and $K > 0$ large enough, where $c_0 = -6\log 2 - 3\log\pi - \log p$.

The proof of Theorem 1.2 starts from an approximate functional equation and then an application of Petersson's trace formula. Subsequently, we need to deal with a diagonal term and an off-diagonal term as expected. For the diagonal term, we use the analytic continuation of a Dirichlet series which may be of interest in other problems. For the off-diagonal term, we apply a result of Iwaniec, Luo and Sarnak [ILS] to deal with an averaging of J-Bessel functions. Goldfeld and Li's Voronoĭ formula for GL_3 in [GL] plays an important role in estimating the off-diagonal term. Theorem 1.2 yields the following non-vanishing result.

COROLLARY 1.3. *For each K large enough, there exists $g \in \mathcal{B}_k$ with $K \le k \le 2K$ such that for any prime p,*

$$L'(1/2,f\times g)\lambda_g(p) \neq 0.$$

Proof. Let p be a fixed prime. By Jacquet and Shalika [JS], $L(1,f)\neq 0$. Then by Theorem 1.2, if $A(1,p)\neq p^{-1}$,

$$\sum_{k\equiv 2\,(\mathrm{mod}\,4)} h\left(\frac{k-1}{K}\right)\sum_{g\in\mathcal{B}_k}\omega_g^{-1}L'(1/2,f\times g)\lambda_g(p)$$
$$\sim \frac{3(A(1,p)-p^{-1})L(1,f)}{2\sqrt{p}}\widehat{h}(0)K\log K,$$

and if $A(1,p) = p^{-1}$,

$$\sum_{k\equiv 2\,(\mathrm{mod}\,4)} h\left(\frac{k-1}{K}\right)\sum_{g\in\mathcal{B}_k}\omega_g^{-1}L'(1/2,f\times g)\lambda_g(p) \sim \frac{\log p}{p\sqrt{p}}L(1,f)\widehat{h}(0)K.$$

It follows that there exists $g\in\mathcal{B}_k$ such that $L'(1/2,f\times g)\lambda_g(p)\neq 0$. ∎

Now we can prove Theorem 1.1. Let $A(1,p)$ and $A'(1,p)$ be the normalized $(p,1)$th Fourier coefficients of f and f', respectively. By the strong multiplicity one theorem (see Theorem 12.6.1 in Goldfeld [G]), we only need to prove $A(1,p) = A'(1,p)$ for all but finitely many primes p. If $A(1,p) = A'(1,p) = p^{-1}$, then we are done. In the following, we assume that $A(1,p) \neq p^{-1}$ and $A'(1,p) \neq p^{-1}$. In [S], the author proved that

$$\sum_{k\equiv 2\,(\mathrm{mod}\,4)} h\Big(\frac{k-1}{K}\Big) \sum_{g\in\mathcal{B}_k} \omega_g^{-1} L'(1/2, f\times g) \sim \frac{3L(1,f)}{2}\widehat{h}(0) K\log K.$$

Thus under the condition (1.1), we have

$$L(1,f) = cL(1,f'). \tag{1.2}$$

On the other hand, by (1.1) and Theorem 1.2, we have

$$\frac{3(A(1,p)-p^{-1})L(1,f)}{2\sqrt{p}}\widehat{h}(0) = c\frac{3(A'(1,p)-p^{-1})L(1,f')}{2\sqrt{p}}\widehat{h}(0). \tag{1.3}$$

By (1.2) and (1.3) we obtain $A(1,p) = A'(1,p)$. This proves Theorem 1.1.

In Section 2, we recall some basic facts about Maass cusp forms for $SL_3(\mathbb{Z})$. In Section 3, we study the properties of $GL_3\times GL_2$ L-functions. We will prove Theorem 1.2 in Sections 4–6.

2. Maass cusp forms for $SL_3(\mathbb{Z})$. Let f be a Maass cusp form of type (ν_1,ν_2) for $SL_3(\mathbb{Z})$ and let $A(m_1,m_2)$ denote the (m_1,m_2)th Fourier coefficient of f. Assume f is normalized so that $A(1,1)=1$. We have (see Remark 12.1.8 in [G])

$$\sum_{m_2\le N} |A(m_1,m_2)| \ll_f N|m_1|. \tag{2.1}$$

Let $\widetilde{f}$ denote the dual Maass form of f. Then $\widetilde{f}$ is of type (ν_2,ν_1) and the (m_1,m_2)th Fourier coefficient of $\widetilde{f}$ is the corresponding (m_2,m_1)th Fourier coefficient of f. If f is self-dual, then the Fourier coefficients are all real and $A(m_1,m_2) = A(m_2,m_1)$.

For $\Re s>1$, we define the Godement–Jacquet L-function associated to f,

$$L(s,f) = \sum_{n\ge 1} A(1,n)n^{-s},$$

which has a holomorphic continuation to all $s\in\mathbb{C}$ and satisfies the functional equation

$$\gamma(s,f)L(s,f) = \widetilde{\gamma}(1-s,f)L(1-s,\widetilde{f}) \tag{2.2}$$

where

$$\gamma(s,f)=\pi^{-3s/2}\Gamma\Big(\frac{s-\alpha}{2}\Big)\Gamma\Big(\frac{s-\beta}{2}\Big)\Gamma\Big(\frac{s-\gamma}{2}\Big), \tag{2.3}$$

$$\widetilde{\gamma}(s,f)=\pi^{-3s/2}\Gamma\Big(\frac{s+\alpha}{2}\Big)\Gamma\Big(\frac{s+\beta}{2}\Big)\Gamma\Big(\frac{s+\gamma}{2}\Big), \tag{2.4}$$

with

$$\alpha=-\nu_1-2\nu_2+1,\quad \beta=-\nu_1+\nu_2,\quad \gamma=2\nu_1+\nu_2-1.$$

Here $L(s,\widetilde{f})$ is the L-function associated to the dual Maass form $\widetilde{f}$. By Luo, Rudnick and Sarnak [LRS] we have $|\Re\alpha|,|\Re\beta|,|\Re\gamma|\le 1/2-1/10$.

Let p be a fixed prime. For $\Re s>2$, we define

$$L_p(s,f)=\sum_{m\ge1}\frac{A(p,m)}{m^s}, \tag{2.5}$$

$$L_p(s,\widetilde{f})=\sum_{m\ge1}\frac{A(m,p)}{m^s}. \tag{2.6}$$

The following result shows $L_p(s,f)$ and $L_p(s,\widetilde{f})$ have holomorphic continuations to all $s\in\mathbb{C}$.

Lemma 2.1. *Let p be a fixed prime. Then $L_p(s,f)$ and $L_p(s,\widetilde{f})$ defined in* (2.5) *and* (2.6) *have holomorphic continuations to all $s\in\mathbb{C}$ and satisfy the functional equation*

$$(A(1,p)-p^{s-1})\gamma(s,f)L_p(s,f)=(A(p,1)-p^{-s})\widetilde{\gamma}(1-s,f)L_p(1-s,\widetilde{f}),$$

where $\gamma(s,f)$ and $\widetilde{\gamma}(s,f)$ are defined in (2.3) *and* (2.4), *respectively.*

Proof. Applying the multiplicative property

$$A(m_1,1)A(1,m_2)=\sum_{d|(m_1,m_2)}A\Big(\frac{m_1}{d},\frac{m_2}{d}\Big),\quad m_1,m_2\ge1,$$

we have

$$\begin{aligned}L_p(s,f)&=(A(p,1)-p^{-s})L(s,f),\\ L_p(s,\widetilde{f})&=(A(1,p)-p^{-s})L(s,\widetilde{f}).\end{aligned} \tag{2.7}$$

Then the lemma follows from the functional equation (2.2). ∎

Let $\psi(x)$ be a smooth function compactly supported on $(0,\infty)$ and denote the Mellin transform of $\psi(x)$ by

$$\widetilde{\psi}(s):=\int_0^\infty\psi(x)x^s\,\frac{dx}{x}.$$

For $k = 0, 1$, we set

$$\Psi_k(x) = \int_{\Re s=\sigma} (\pi^3 x)^{-s} \frac{\Gamma\left(\frac{1+s+2k+\alpha}{2}\right)\Gamma\left(\frac{1+s+2k+\beta}{2}\right)\Gamma\left(\frac{1+s+2k+\gamma}{2}\right)}{\Gamma\left(\frac{-s-\alpha}{2}\right)\Gamma\left(\frac{-s-\beta}{2}\right)\Gamma\left(\frac{-s-\gamma}{2}\right)} \times \widetilde{\psi}(-s-k)\, ds \tag{2.8}$$

with $\sigma > \max\{-1-\Re\alpha, -1-\Re\beta, -1-\Re\gamma\}$,

$$\Psi_{0,1}^0(x) = \Psi_0(x) + \frac{\pi^{-3}c^3 m}{n_1^2 n_2 i}\Psi_1(x), \tag{2.9}$$

$$\Psi_{0,1}^1(x) = \Psi_0(x) - \frac{\pi^{-3}c^3 m}{n_1^2 n_2 i}\Psi_1(x). \tag{2.10}$$

We have the following Voronoĭ formula for GL_3 (see Goldfeld and Li [GL]):

Lemma 2.2. *Let $\psi \in C_c^\infty(0,\infty)$. Let $d, \overline{d}, c \in \mathbb{Z}$ with $c \neq 0$, $(d,c) = 1$ and $d\overline{d} \equiv 1 \pmod{c}$. Then*

$$\sum_{n>0} A(m,n) e\left(\frac{n\overline{d}}{c}\right)\psi(n)$$
$$= \frac{c\pi^{-5/2}}{4i} \sum_{n_1 \mid cm} \sum_{n_2>0} \frac{A(n_2,n_1)}{n_1 n_2} S(md, n_2; mcn_1^{-1}) \Psi_{0,1}^0\left(\frac{n_2 n_1^2}{c^3 m}\right)$$
$$+ \frac{c\pi^{-5/2}}{4i} \sum_{n_1 \mid cm} \sum_{n_2>0} \frac{A(n_2,n_1)}{n_1 n_2} S(md, -n_2; mcn_1^{-1}) \Psi_{0,1}^1\left(\frac{n_2 n_1^2}{c^3 m}\right).$$

As pointed out in Li [Li2], $x^{-1}\Psi_1(x)$ has similar asymptotic behavior to $\Psi_0(x)$. Therefore, we only need to consider $\Psi_0(x)$. The following result is Lemma 6.1 of Li [Li1]. For $\alpha = \beta = \gamma = 0$, it was proved by Ivić [I].

Lemma 2.3. *Suppose ψ is a smooth function compactly supported on $[X, 2X]$. Let $\Psi_0(x)$ be defined as in (2.8). Then for any fixed integer $M \geq 1$ and $xX \gg 1$, we have*

$$\Psi_0(x) = 2\pi^4 x i \int_0^\infty \psi(y) \sum_{j=1}^{M} \frac{c_j \cos(6\pi x^{1/3} y^{1/3}) + d_j \sin(6\pi x^{1/3} y^{1/3})}{(\pi^3 x y)^{j/3}}\, dy$$
$$+ O((xX)^{(-M+2)/3}),$$

where c_j and d_j are constants depending on α, β and γ. In particular, $c_1 = 0$ and $d_1 = -2/\sqrt{3\pi}$.

3. Rankin–Selberg L-functions. Let f be a self-dual Hecke–Maass cusp form of type (ν,ν) for $SL_3(\mathbb{Z})$ and $\mathcal{B}_k$ be an orthogonal basis of holomorphic cusp forms of weight $k \equiv 2 \pmod 4$ for $SL_2(\mathbb{Z})$. The Rankin–Selberg

L-function of f and $g \in \mathcal{B}_k$ defined by

$$L(s, f \times g) = \sum_{m \geq 1} \sum_{n \geq 1} \frac{\lambda_g(n) A(m,n)}{(m^2 n)^s}$$

is entire and satisfies the functional equation

$$\Lambda(s, f \times g) = -\Lambda(1-s, f \times g)$$

where $\Lambda(s, f \times g) = \gamma(s,k) L(s, f \times g)$ and for $\alpha = -3\nu + 1$,

$$\gamma(s,k) = \pi^{-3s} \Gamma\Big(\frac{s + \frac{k-1}{2} - \alpha}{2}\Big) \Gamma\Big(\frac{s + \frac{k-1}{2}}{2}\Big) \Gamma\Big(\frac{s + \frac{k-1}{2} + \alpha}{2}\Big) \times \Gamma\Big(\frac{s + \frac{k+1}{2} - \alpha}{2}\Big) \Gamma\Big(\frac{s + \frac{k+1}{2}}{2}\Big) \Gamma\Big(\frac{s + \frac{k+1}{2} + \alpha}{2}\Big). \tag{3.1}$$

Set $G(u) = e^{u^2}$. We define

$$V(y,k) = \frac{1}{2\pi i} \int_{(3)} y^{-u} G(u) \frac{\gamma(1/2 + u, k)}{\gamma(1/2, k)} \frac{du}{u^2}. \tag{3.2}$$

One has the following approximate functional equation for $L'(1/2, f \times g)$ (see Iwaniec and Kowalski [IK]).

Lemma 3.1. *For a self-dual Hecke–Maass cusp form f of type (ν, ν) for $SL_3(\mathbb{Z})$ and g in an orthogonal basis of holomorphic cusp forms of weight $k \equiv 2 \pmod 4$ for $SL_2(\mathbb{Z})$, we have*

$$L'(1/2, f \times g) = 2 \sum_{m \geq 1} \sum_{n \geq 1} \frac{\lambda_g(n) A(m,n)}{(m^2 n)^{1/2}} V(m^2 n, k)$$

where $V(y,k)$ is defined in (3.2).

$V(y,k)$ has the following properties (see Lemma 4.2 in Sun [S]).

Lemma 3.2. *For $y > 0$ and k large enough, we have*

$$V(y,k) \ll_{f,A} (k^3/y)^A,$$

and

$$V(y,k) = \log(k^3/y) + c_0 + O_f(y/k^3 + k^{-1}),$$

where $c_0 = -6 \log 2 - 3 \log \pi$.

4. Proof of Theorem 1.2. Petersson's trace formula states that for any $m, n \geq 1$,

$$\sum_{g \in \mathcal{B}_k} \omega_g^{-1} \lambda_g(m) \lambda_g(n) = \delta_{mn} + 2\pi i^k \sum_{c \geq 1} \frac{S(m,n;c)}{c} J_{k-1}\Big(\frac{4\pi\sqrt{mn}}{c}\Big), \tag{4.1}$$

where $\delta_{mn} = 1$ if $m = n$, and is 0 otherwise, $J_{k-1}(x)$ is the J-Bessel function and $S(m,n;c)$ is the classical Kloosterman sum defined by

$$S(m,n;c) = \sum_{d\overline{d}\equiv 1\,(\mathrm{mod}\,c)} e\Big(\frac{md + n\overline{d}}{c}\Big).$$

Applying Lemma 3.1 and Petersson's trace formula (4.1) we have

$$\begin{aligned}
&\sum_{k\equiv 2\,(\mathrm{mod}\,4)} h\Big(\frac{k-1}{K}\Big) \sum_{g\in\mathcal{B}_k} \omega_g^{-1} L'(1/2, f\times g)\lambda_g(p)\\
&= \sum_{k\equiv 2\,(\mathrm{mod}\,4)} h\Big(\frac{k-1}{K}\Big) \sum_{g\in\mathcal{B}_k} \omega_g^{-1}\lambda_g(p)\Big\{2\sum_{m\geq 1}\sum_{n\geq 1} \frac{\lambda_g(n)A(m,n)}{(m^2n)^{1/2}} V(m^2n,k)\Big\}\\
&= 2\sum_{k\equiv 2\,(\mathrm{mod}\,4)} h\Big(\frac{k-1}{K}\Big) \sum_{m\geq 1}\sum_{n\geq 1} \frac{A(m,n)}{(m^2n)^{1/2}} V(m^2n,k)\Big\{\sum_{g\in\mathcal{B}_k} \omega_g^{-1}\lambda_g(n)\lambda_g(p)\Big\}\\
&= \mathcal{D} + N\mathcal{D},
\end{aligned}$$

where

$$\mathcal{D} = 2p^{-1/2} \sum_{k\equiv 2\,(\mathrm{mod}\,4)} h\Big(\frac{k-1}{K}\Big) \sum_{m\geq 1} \frac{A(m,p)}{m} V(m^2p,k), \tag{4.2}$$

$$\begin{aligned}
N\mathcal{D} = -4\pi \sum_{m\geq 1}\sum_{n\geq 1} \frac{A(m,n)}{(m^2n)^{1/2}} \sum_{c\geq 1} c^{-1} S(n,p;c)\\
\times \sum_{k\equiv 2\,(\mathrm{mod}\,4)} h\Big(\frac{k-1}{K}\Big) V(m^2n,k) J_{k-1}\Big(\frac{4\pi\sqrt{np}}{c}\Big).
\end{aligned}\tag{4.3}$$

Then Theorem 1.2 follows from

$$\begin{aligned}
\mathcal{D} = &\frac{3(A(1,p)-p^{-1})L(1,f)}{2\sqrt{p}} \widehat{h}(0) K\log K\\
&+ \frac{(A(1,p)-p^{-1})(2L'(1,f)+c_0L(1,f))}{2\sqrt{p}} \widehat{h}(0) K\\
&+ \frac{\log p}{p\sqrt{p}} L(1,f)\widehat{h}(0)K + O_{f,p}(1),
\end{aligned}\tag{4.4}$$

$$N\mathcal{D} = O_{\epsilon,f,p}(K^{\epsilon}). \tag{4.5}$$

We will establish (4.4) and (4.5) in Sections 5 and 6, respectively.

5. Estimation of $\mathcal{D}$. By (4.2) we have

$$\mathcal{D} = 2p^{-1/2} \sum_{k\equiv 2\,(\mathrm{mod}\,4)} h\Big(\frac{k-1}{K}\Big) \triangle(k), \tag{5.1}$$

where

$$\triangle(k) = \sum_{m\geq 1} \frac{A(m,p)}{m} V(m^2 p, k).$$

By the definition of $V(y,k)$ in (3.2),

$$\begin{aligned}(5.2)\qquad \triangle(k) &= \sum_{m\geq 1} \frac{A(m,p)}{m} \frac{1}{2\pi i} \int_{(3)} (m^2 p)^{-u} G(u) \frac{\gamma(1/2+u,k)}{\gamma(1/2,k)} \frac{du}{u^2} \\ &= \frac{1}{2\pi i} \int_{(3)} \Big(\sum_{m\geq 1} \frac{A(m,p)}{m^{1+2u}} \Big) p^{-u} G(u) \frac{\gamma(1/2+u,k)}{\gamma(1/2,k)} \frac{du}{u^2} \\ &= \frac{1}{2\pi i} \int_{(3)} L_p(1+2u,f) p^{-u} G(u) \frac{\gamma(1/2+u,k)}{\gamma(1/2,k)} \frac{du}{u^2},\end{aligned}$$

where $L_p(s,f)$ is defined in (2.5). By Lemma 2.1, we can move the line of integration in (5.2) to $\Re u = -1/2$, picking up a double pole at $u = 0$,

$$\begin{aligned}(5.3)\qquad \triangle(k) &= \operatorname{res}_{u=0} \Big(L_p(1+2u,f) \frac{G(u)}{p^u u^2} \frac{\gamma(1/2+u,k)}{\gamma(1/2,k)} \Big) \\ &\quad + \frac{1}{2\pi i} \int_{(-1/2)} L_p(1+2u,f) p^{-u} G(u) \frac{\gamma(1/2+u,k)}{\gamma(1/2,k)} \frac{du}{u^2}.\end{aligned}$$

First we compute the residue in (5.3). By the duplication formula, $\gamma(s,k)$ in (3.1) is

$$\begin{aligned}\gamma(s,k) &= \pi^{3/2-3s} 2^{3-3(s+(k-1)/2)} \Gamma\Big(s + \frac{k-1}{2} - \alpha\Big) \Gamma\Big(s + \frac{k-1}{2}\Big) \\ &\quad \times \Gamma\Big(s + \frac{k-1}{2} + \alpha\Big).\end{aligned}$$

Thus,

$$(5.4)\qquad \frac{\gamma(1/2+u,k)}{\gamma(1/2,k)} = (2\pi)^{-3u} \frac{\Gamma(u+k/2-\alpha)\Gamma(u+k/2)\Gamma(u+k/2+\alpha)}{\Gamma(k/2-\alpha)\Gamma(k/2)\Gamma(k/2+\alpha)},$$

and

$$\begin{aligned}(5.5)\qquad &\lim_{u\to 0} \frac{d}{du} \frac{\gamma(1/2+u,k)}{\gamma(1/2,k)} \\ &\quad = -3\log(2\pi) + \frac{\Gamma'(k/2-\alpha)}{\Gamma(k/2-\alpha)} + \frac{\Gamma'(k/2)}{\Gamma(k/2)} + \frac{\Gamma'(k/2+\alpha)}{\Gamma(k/2+\alpha)}.\end{aligned}$$

By Stirling's formula, for $|\arg z| \leq \pi - \delta$, $\delta > 0$,

$$\frac{\Gamma'(z)}{\Gamma(z)} = \log z - \frac{1}{2z} + O_\delta\Big(\frac{1}{|z|^2}\Big).$$

Thus by (5.5),

$$\lim_{u\to 0}\frac{d}{du}\frac{\gamma(1/2+u,k)}{\gamma(1/2,k)} = -3\log(2\pi)+3\log(k/2)+O_f(k^{-1}),$$

and the residue in (5.3) is

$$(5.6)\qquad \lim_{u\to 0}\frac{d}{du}\left(L_p(1+2u,f)\frac{G(u)}{p^u}\frac{\gamma(1/2+u,k)}{\gamma(1/2,k)}\right)$$
$$= 2L_p'(1,f)-L_p(1,f)\log p+L_p(1,f)\lim_{u\to 0}\frac{d}{du}\frac{\gamma(1/2+u,k)}{\gamma(1/2,k)}$$
$$= 3L_p(1,f)\log k+2L_p'(1,f)+c_0L_p(1,f)+O_f(k^{-1}),$$

where $c_0=-6\log 2-3\log\pi-\log p$. Here we have used the fact that $G(0)=1$ and $G'(0)=0$. By (2.7), we have

$$L_p(1,f)=(A(1,p)-p^{-1})L(1,f),$$
$$L_p'(1,f)=(A(1,p)-p^{-1})L'(1,f)+\frac{\log p}{p}L(1,f).$$

So by (5.6), the residue in (5.3) is

$$(5.7)\qquad \mathrm{res}_{u=0}\left(L_p(1+2u,f)\frac{G(u)}{p^u u^2}\frac{\gamma(1/2+u,k)}{\gamma(1/2,k)}\right)$$
$$= 3(A(1,p)-p^{-1})L(1,f)\log k+(A(1,p)-p^{-1})(2L'(1,f)+c_0L(1,f))$$
$$+\frac{2\log p}{p}L(1,f)+O_f(k^{-1}).$$

Next, we compute the integral in (5.3). By Stirling's formula, for $|\arg z|\le\pi-\delta$, $\delta>0$,

$$\log\Gamma(z)=\left(z-\frac{1}{2}\right)\log z-z+\frac{1}{2}\log(2\pi)+O_\delta\left(\frac{1}{|z|}\right).$$

Thus, for $u=-1/2+iv$, we have

$$\log\frac{\Gamma(u+k/2-\alpha)}{\Gamma(k/2-\alpha)}$$
$$=\left(-1+\frac{k}{2}-\Re\alpha+i(v-\Im\alpha)\right)\log\left(-\frac{1}{2}+\frac{k}{2}-\Re\alpha+i(v-\Im\alpha)\right)$$
$$-\left(-\frac{1}{2}+\frac{k}{2}-\Re\alpha+i(v-\Im\alpha)\right)+\frac{1}{2}\log(2\pi)+o_f(1)$$
$$-\left(-\frac{1}{2}+\frac{k}{2}-\Re\alpha-i\Im\alpha\right)\log\left(\frac{k}{2}-\Re\alpha-i\Im\alpha\right)$$
$$+\left(\frac{k}{2}-\Re\alpha-i\Im\alpha\right)-\frac{1}{2}\log(2\pi)+o_f(1)$$

$$= \left(-1+\frac{k}{2}-\Re\alpha+i(v-\Im\alpha)\right)\log\left(\frac{k}{2}+iv\right)$$
$$-\left(-\frac{1}{2}+\frac{k}{2}-\Re\alpha-i\Im\alpha\right)\log\left(\frac{k}{2}\right)+\left(\frac{1}{2}-iv\right)$$
$$+\left(-1+\frac{k}{2}-\Re\alpha+i(v-\Im\alpha)\right)\log\left(1+\frac{-1/2-\Re\alpha-i\Im\alpha}{k/2+iv}\right)$$
$$-\left(-\frac{1}{2}+\frac{k}{2}-\Re\alpha-i\Im\alpha\right)\log\left(1+\frac{-\Re\alpha-i\Im\alpha}{k/2}\right)+o_f(1).$$

For $-1 < x \le 1$, we have

$$\log(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots+(-1)^{n+1}\frac{x^n}{n}+\cdots.$$

Thus for k sufficiently large, we have

$$\log\left(1+\frac{-1/2-\Re\alpha-i\Im\,\alpha}{k/2+iv}\right)=O\left(\left|\frac{-1/2-\Re\alpha-i\Im\alpha}{k/2+iv}\right|\right)=O_f\left(\frac{1}{k}\right),$$
$$\log\left(1+\frac{-\Re\alpha-i\Im\alpha}{k/2}\right)=O\left(\left|\frac{-\Re\alpha-i\Im\alpha}{k/2}\right|\right)=O_f\left(\frac{1}{k}\right).$$

Therefore,

$$\text{(5.8)}\qquad \log\frac{\Gamma(u+k/2-\alpha)}{\Gamma(k/2-\alpha)}$$
$$=\left(-1+\frac{k}{2}-\Re\alpha\right)\log\left(\frac{k^2}{4}+v^2\right)^{1/2}-(v-\Im\alpha)\arctan\left(\frac{2v}{k}\right)$$
$$-\left(-\frac{1}{2}+\frac{k}{2}-\Re\alpha\right)\log\left(\frac{k}{2}\right)+i\theta+O_f(1),$$

where

$$\theta=(v-\Im\alpha)\log\left(\frac{k^2}{4}+v^2\right)^{1/2}+\left(-1+\frac{k}{2}-\Re\alpha\right)\arctan\left(\frac{2v}{k}\right)$$
$$-(\Im\alpha)\log\left(\frac{k}{2}\right)-v.$$

By (5.8), we obtain

$$\text{(5.9)}\qquad \left|\frac{\Gamma(u+k/2-\alpha)}{\Gamma(k/2-\alpha)}\right|\ll_f\frac{(k/2+|v|)^{-1+k/2-\Re\alpha}e^{\frac{\pi}{2}(|v|+|\Im\alpha|)}}{(k/2)^{-1/2+k/2-\Re\alpha}}$$
$$\ll_f k^{-1/2}\left(1+\frac{2|v|}{k}\right)^{-1+k/2-\Re\alpha}e^{\pi|v|/2}.$$

Similarly,

$$\left|\frac{\Gamma(u+k/2)}{\Gamma(k/2)}\right| \ll_f k^{-1/2}\left(1+\frac{2|v|}{k}\right)^{-1+k/2} e^{\pi|v|/2}, \tag{5.10}$$

$$\left|\frac{\Gamma(u+k/2+\alpha)}{\Gamma(k/2+\alpha)}\right| \ll_f k^{-1/2}\left(1+\frac{2|v|}{k}\right)^{-1+k/2+\Re\alpha} e^{\pi|v|/2}, \tag{5.11}$$

By (5.4) and (5.9)–(5.11), we have

$$\left|\frac{\gamma(1/2+u,k)}{\gamma(1/2,k)}\right| \ll_f k^{-3/2}\left(1+\frac{2|v|}{k}\right)^{-3+3k/2} e^{3\pi|v|/2}. \tag{5.12}$$

By (2.7), (5.12) and the convexity bound for $L(s,f)$:

$$L(\sigma+iv,f) \ll_f (1+|v|)^{3(1-\sigma)/2+\epsilon}, \quad 0\le\sigma\le 1,$$

for any $\epsilon>0$, we have

$$\begin{aligned} &\frac{1}{2\pi i}\int_{(-1/2)} L_p(1+2u,f)p^{-u}G(u)\frac{\gamma(1/2+u,k)}{\gamma(1/2,k)}\,\frac{du}{u^2} \\ &\ll_{f,p} \int_{(-1/2)} |L(1+2u,f)|\,|G(u)|\left|\frac{\gamma(1/2+u,k)}{\gamma(1/2,k)}\right| \frac{du}{|u|^2} \\ &\ll_{f,p} k^{-3/2}\int_{-\infty}^{\infty} (1+|v|)^{3/2+\epsilon}e^{-v^2}\left(1+\frac{2|v|}{k}\right)^{-3+3k/2} e^{3\pi|v|/2}\,\frac{dv}{1/4+v^2} \\ &\ll_{f,p} k^{-3/2} \end{aligned} \tag{5.13}$$

for k sufficiently large. By (5.3), (5.7) and (5.13), we have

$$\begin{aligned} \triangle(k) &= 3(A(1,p)-p^{-1})L(1,f)\log k \\ &\quad + (A(1,p)-p^{-1})(2L'(1,f)+c_0L(1,f)) \\ &\quad + \frac{2\log p}{p}L(1,f)+O_{f,p}(k^{-1}). \end{aligned} \tag{5.14}$$

Then (4.4) follows from (5.1) and (5.14). Here we have used the fact that

$$4\sum_{k\equiv 2\,(\mathrm{mod}\,4)} h\left(\frac{k-1}{K}\right) = K\widehat{h}(0)+O_A(K^{-A})$$

for any $A>0$.

6. Estimation of $\mathcal{ND}$. In this section, we estimate $\mathcal{ND}$ of (4.3). Let ω be a smooth function of compact support on $[1,2]$. By Lemma 3.2, we only need to estimate

$$(6.1)\qquad N\mathcal{D}^* = \sum_{m\geq 1}\sum_{n\geq 1}\frac{A(m,n)}{(m^2n)^{1/2}}\omega\Big(\frac{m^2n}{N}\Big)\sum_{c\geq 1}c^{-1}S(n,p;c) \times \sum_{k\equiv 2\,(\mathrm{mod}\,4)} h\Big(\frac{k-1}{K}\Big)V(m^2n,k)J_{k-1}\Big(\frac{4\pi\sqrt{np}}{c}\Big)$$

with $N\leq K^{3+\epsilon}$ for any $\epsilon>0$.

The following result is Proposition 8.1 in Iwaniec, Luo and Sarnak [ILS].

Lemma 6.1. *Fix a real valued function $h\in C_0^\infty(\mathbb{R}^+)$ and $K\geq 1$. Then*

$$4\sum_{k\equiv 2\,(\mathrm{mod}\,4)} h\Big(\frac{k-1}{K}\Big)J_{k-1}(x) = h\Big(\frac{x}{K}\Big)+\frac{K}{\sqrt{x}}\Im\Big(e^{ix-i\pi/4}\mathcal{H}\Big(\frac{K^2}{2x}\Big)\Big)+O\Big(\frac{x}{K^3}\Big),$$

where

$$\mathcal{H}(v)=\int_0^\infty \frac{h(\sqrt{u})}{\sqrt{2\pi u}}e^{iuv}\,du.$$

Applying Lemma 6.1 for $x=4\pi\sqrt{np}/c$ we have

$$(6.2)\qquad 4\sum_{k\equiv 2\,(\mathrm{mod}\,4)} h\Big(\frac{k-1}{K}\Big)V(m^2n,k)J_{k-1}(x) = h\Big(\frac{x}{K}\Big)V(m^2n,x+1)+\frac{K}{\sqrt{x}}\Im\Big(e^{ix-i\pi/4}\mathcal{H}\Big(\frac{K^2}{2x}\Big)\Big)+O_f\Big(\frac{x}{K^3}\Big),$$

where

$$\mathcal{H}(v)=\int_0^\infty \frac{h(\sqrt{u})}{\sqrt{2\pi u}}V(m^2n,\sqrt{u}K+1)e^{iuv}\,du.$$

By multiple partial integration, we have

$$(6.3)\qquad \mathcal{H}(v)\ll_{f,A,B}|v|^{-A}\Big(\frac{K^3}{m^2n}\Big)^B$$

for any $A,B>0$. By Weil's bound for Kloosterman sums,

$$|S(n,p;c)|\leq c^{1/2}(n,p,c)^{1/2}\tau(c).$$

Thus the contribution from the error term in (6.2) to $N\mathcal{D}^*$ in (6.1) is

$$(6.4)\qquad \ll_f \sum_{m\geq 1}\sum_{n\geq 1}\frac{|A(m,n)|}{(m^2n)^{1/2}}\omega\Big(\frac{m^2n}{N}\Big)\sum_{c\geq 1}c^{-1}c^{1/2}(n,p,c)^{1/2}\tau(c)K^{-3}\frac{4\pi\sqrt{np}}{c}$$
$$\ll_{\epsilon,f,p} K^{-3}\sum_{m\geq 1}m^{-1}\sum_{n\geq 1}|A(m,n)|\omega\Big(\frac{m^2n}{N}\Big)\sum_{c\geq 1}c^{-3/2+\epsilon}$$

$$\ll_{\epsilon,f,p} K^{-3} \sum_{m\leq\sqrt{2N}} m^{-1} \sum_{n\leq 2N/m^2} |A(m,n)|$$
$$\ll_{\epsilon,f,p} K^{-3} \sum_{m\leq\sqrt{2N}} m^{-1}\cdot Nm^{-1} \ll_{\epsilon,f,p} NK^{-3} \ll_{\epsilon,f,p} K^{\epsilon}$$

for any $\epsilon > 0$. Here we have used the bound (2.1). By (6.1), (6.2) and (6.4), we need to estimate the quantities

$$N\mathcal{D}_1 = \sum_{m\geq 1}\sum_{n\geq 1} \frac{A(m,n)}{(m^2n)^{1/2}}\omega\Big(\frac{m^2n}{N}\Big)\sum_{c\geq 1} c^{-1}S(n,p;c) \times h\Big(\frac{4\pi\sqrt{np}}{cK}\Big)V\Big(m^2n, \frac{4\pi\sqrt{np}}{c}+1\Big), \tag{6.5}$$

$$N\mathcal{D}_2 = \sum_{m\geq 1}\sum_{n\geq 1} \frac{A(m,n)}{(m^2n)^{1/2}}\omega\Big(\frac{m^2n}{N}\Big)\sum_{c\geq 1} c^{-1}S(n,p;c) \times \frac{K\sqrt{c}}{(pn)^{1/4}}\Im\Big(e^{i4\pi\sqrt{np}/c-i\pi/4}\mathcal{H}\Big(\frac{K^2c}{8\pi\sqrt{pn}}\Big)\Big). \tag{6.6}$$

Note that

$$\frac{K^2c}{8\pi\sqrt{n}} \gg_p K^2N^{-1/2} \gg_p K^{1/2-\epsilon}$$

for any $\epsilon > 0$, so by (6.3), $N\mathcal{D}_2$ in (6.6) is negligible.

It remains to estimate $N\mathcal{D}_1$. Note that $1 \leq 4\pi\sqrt{np}/(cK) \leq 2$ and $1 \leq m^2n/N \leq 2$. Thus

$$\frac{2\pi\sqrt{pN}}{Km} \leq c \leq \frac{4\pi\sqrt{2pN}}{Km}. \tag{6.7}$$

Opening the Kloosterman sum in (6.5), we have

$$\begin{aligned} N\mathcal{D}_1 &= \sum_{m\geq 1}\sum_{n\geq 1} \frac{A(m,n)}{(m^2n)^{1/2}}\omega\Big(\frac{m^2n}{N}\Big)\sum_{c} c^{-1} \\ &\quad\times \sum_{d\bar{d}\equiv 1 \,(\mathrm{mod}\, c)} e\Big(\frac{pd+n\bar{d}}{c}\Big)h\Big(\frac{4\pi\sqrt{np}}{cK}\Big)V\Big(m^2n, \frac{4\pi\sqrt{np}}{c}+1\Big) \\ &= \sum_{m\geq 1} m^{-1}\sum_{c} c^{-1} \sum_{d\bar{d}\equiv 1 \,(\mathrm{mod}\, c)} e\Big(\frac{pd}{c}\Big)\Big\{\sum_{n\geq 1} A(m,n)e\Big(\frac{n\bar{d}}{c}\Big)\psi(n)\Big\}, \end{aligned} \tag{6.8}$$

where

$$\psi(y) = y^{-1/2}\omega\Big(\frac{m^2y}{N}\Big)h\Big(\frac{4\pi\sqrt{py}}{cK}\Big)V\Big(m^2y, \frac{4\pi\sqrt{py}}{c}+1\Big).$$

Applying the Voronoĭ formula in Lemma 2.2 for the n-sum, we have

$$(6.9)\quad \sum_{n\geq 1} A(m,n)e\left(\frac{n\bar{d}}{c}\right)\psi(n)$$
$$= \frac{c\pi^{-5/2}}{4i}\sum_{n_1|cm}\sum_{n_2>0}\frac{A(n_2,n_1)}{n_1n_2}S(md,n_2;mcn_1^{-1})\Psi_{0,1}^0\left(\frac{n_2n_1^2}{c^3m}\right)$$
$$+ \frac{c\pi^{-5/2}}{4i}\sum_{n_1|cm}\sum_{n_2>0}\frac{A(n_2,n_1)}{n_1n_2}S(md,-n_2;mcn_1^{-1})\Psi_{0,1}^1\left(\frac{n_2n_1^2}{c^3m}\right),$$

where $\Psi_{0,1}^0(x)$ and $\Psi_{0,1}^1(x)$ are defined in (2.9) and (2.10), respectively. By (6.8) and (6.9), we only need to estimate

$$N\mathcal{D}_1^0 = \frac{\pi^{-5/2}}{4i}\sum_{m\geq 1} m^{-1}\sum_{c\geq 1}\sum_{d\bar{d}\equiv 1\,(\mathrm{mod}\,c)} e\left(\frac{pd}{c}\right)$$
$$\times \sum_{n_1|cm}\sum_{n_2>0}\frac{A(n_2,n_1)}{n_1n_2}S(md,n_2;mcn_1^{-1})\Psi_0\left(\frac{n_2n_1^2}{c^3m}\right).$$

By (6.7),

$$\frac{n_2n_1^2}{c^3m}\,\frac{N}{m^2} = N\frac{n_2n_1^2}{(cm)^3} \gg_p N\left(\frac{K}{\sqrt{N}}\right)^3 = K^3N^{-1/2} \gg K^{3/2-\epsilon}$$

for any $\epsilon > 0$. Thus by Lemma 2.3 for $x = n_2n_1^2/(c^3m)$,

$$\Psi_0(x) = 2\pi^4xi\int_0^\infty \psi(y)\sum_{j=1}^{M}\frac{c_j\cos(6\pi x^{1/3}y^{1/3}) + d_j\sin(6\pi x^{1/3}y^{1/3})}{(\pi^3xy)^{j/3}}\,dy$$
$$+ O\left(\left(\frac{n_2n_1^2}{c^3m}\,\frac{N}{m^2}\right)^{(-M+2)/3}\right),$$

where c_j and d_j are constants depending only on f. In particular, $c_1 = 0$ and $d_1 = -2/\sqrt{3\pi}$. Denote

$$\Psi_0^j(x) = 2\pi^4xi\int_0^\infty \psi(y)\frac{c_j\cos(6\pi x^{1/3}y^{1/3}) + d_j\sin(6\pi x^{1/3}y^{1/3})}{(\pi^3xy)^{j/3}}\,dy.$$

Then

$$\Psi_0(x) = \sum_{j=1}^{M}\Psi_0^j(x) + O\left(\left(\frac{n_2n_1^2}{c^3m}\,\frac{N}{m^2}\right)^{(-M+2)/3}\right).$$

Take $M = 8$. Then the contribution from the O-term above to $N\mathcal{D}_1^0$ is negligible. Now we estimate $\Psi_0^1(x)$:

$$\Psi_0^1(x) = 2\pi^4 xi \int_0^\infty \psi(y) \frac{d_1 \sin(6\pi x^{1/3} y^{1/3})}{(\pi^3 xy)^{1/3}}\, dy = 2\pi^3 x^{2/3} d_1 \int_0^\infty b(y) \sin(a(y))\, dy,$$

where $a(y) = 6\pi x^{1/3} y^{1/3}$ and

$$b(y) = y^{-5/6} \omega\left(\frac{m^2 y}{N}\right) h\left(\frac{4\pi\sqrt{py}}{cK}\right) V\left(m^2 y, \frac{4\pi\sqrt{py}}{c} + 1\right).$$

Since $a'(y)y \gg K^{1/2-\epsilon}$, by multiple partial integration, one shows that the contribution from $\Psi_0^1(x)$ to $N\mathcal{D}_1^0$ is negligible. Repeating the above arguments for $\Psi_0^j(x)$, $j = 2, \dots, M$, one shows that the other terms are also negligible. Thus $N\mathcal{D}_1^0$ is negligible. This proves (4.5).

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