Twisted inhomogeneous Diophantine approximation and badly approximable sets

by

STEPHEN HARRAP (York)

1. Introduction

1.1. Background—the homogeneous theory. A classical result due to Dirichlet states that for any real number x there exist infinitely many natural numbers q such that

$$\|qx\| \leq 1/q, \tag{1.1}$$

where $\|\cdot\|$ denotes the distance to the nearest integer. This result can easily be generalised to higher dimensions. In particular, the following "weighted" simultaneous version of the above statement is valid. Choose any positive real numbers i and j satisfying

$$i, j \geq 0 \quad \text{and} \quad i + j = 1. \tag{1.2}$$

Then for any vector $\mathbf{x} \in \mathbb{R}^2$ there exist infinitely many natural numbers q such that

$$\max\{\|qx_1\|^{1/i}, \|qx_2\|^{1/j}\} \leq 1/q. \tag{1.3}$$

Here, without loss of generality, if $i = 0$ we employ the convention that $\|x\|^{1/i} = 0$ and so the above statement reduces to Dirichlet's original result. It is natural to ask whether the right hand side of inequality (1.3) can in general be tightened, that is, whether $1/q$ may be replaced by c/q for some absolute constant $c \in (0, 1)$ whilst still allowing (1.3) to hold infinitely often for all real vectors. It is still an open problem as to whether there exists an "optimal" constant in the sense that the statement holds only finitely often for at least one real vector if it is replaced by any smaller constant. Conversely, in the one-dimensional setting, concerning statement (1.1), such

[55]

an "optimal" constant (namely $1/\sqrt{5}$) was found by Hurwitz (e.g., Theorems 193 & 194 in [10, Chapter XI]).

The above discussion motivates the study of real vectors $\mathbf{x}$ for which the right hand side of (1.3) cannot be improved by an arbitrary positive constant. Throughout, we will impose the following natural restriction on these vectors. We say $\mathbf{x} := (x_1, x_2)$ is *irrational* (abbreviated *irr.*) if its components x_i together with 1 are linearly independent over the rationals.

DEFINITION 1.1. An irrational vector $\mathbf{x}$ is (i,j)-*badly approximable* if there exists a constant $c(\mathbf{x}) > 0$ such that

$$\max\{\|qx_1\|^{1/i}, \|qx_2\|^{1/j}\} > c(\mathbf{x})/q \quad \forall q \in \mathbb{N}.$$

The set of all such vectors will be denoted $\mathbf{Bad}(i,j)$.

The results of this paper (for $i, j > 0$) do remain true when $\mathbf{x}$ is not assumed to be irrational in the above and later definitions. However, we choose to avoid this degenerate case for the sake of clarity. Furthermore, all of the sets and arguments considered in this paper are invariant under integer translation, so there will be no loss of generality in assuming throughout that all vectors are confined to the unit square (or the unit n-cube when in higher dimensions) unless otherwise stated. Accordingly, for example, if $i = 0$ then the set $\mathbf{Bad}(0,1)$ will be identified with $[0,1] \times \mathbf{Bad}$, where $\mathbf{Bad}$ is the standard one-dimensional set of badly approximable numbers. In other words, $\mathbf{Bad}(0,1)$ consists of vectors $\mathbf{x}$ with $x_1 \in [0,1]$ and

$$x_2 \in \mathbf{Bad} := \{\text{irr. } x \in [0,1] : \exists c(x) > 0 \; \forall q \in \mathbb{N} \; \|qx\| > c(x)/q\}.$$

DEFINITION 1.2. A mapping $\psi : \mathbb{N} \to \mathbb{R}$ is an *approximating function* if ψ is strictly positive and non-increasing.

DEFINITION 1.3. For any approximating function ψ, define $\mathbf{W}_{(i,j)}(\psi)$ to be the set of vectors $\mathbf{x} \in [0,1]^2$ such that the inequality

$$\max\{\|qx_1\|^{1/i}, \|qx_2\|^{1/j}\} \leq \psi(q)$$

holds for infinitely many natural numbers q.

Application of the following classical theorem of Khintchine [13] shows that for every pair of reals i, j satisfying (1.2) the set $\mathbf{Bad}(i,j)$ is of two-dimensional Lebesgue measure zero. Lebesgue measure will hereafter be denoted μ.

KHINTCHINE'S THEOREM (1924). *For any pair of reals i, j satisfying* (1.2) *and any approximating function ψ we have*

$$\mu(\mathbf{W}_{(i,j)}(\psi)) = \begin{cases} 0 & \text{if } \sum_{r=1}^{\infty} \psi(r) < \infty, \\ 1 & \text{if } \sum_{r=1}^{\infty} \psi(r) = \infty. \end{cases}$$

It is worth emphasising here that the choice of approximating function ψ is completely irrelevant once the reals i, j have been fixed. We also mention that in the $i = j = 1/2$ case the monotonicity restriction imposed on ψ can be relaxed (see [9] for details). However, whether this is true in general is still an open problem.

The question of whether each null set $\mathbf{Bad}(i, j)$ is non-empty was formally (*) answered by Pollington & Velani [21] who showed that for every choice of reals i, j satisfying (1.2) we have

$$(1.4) \qquad \dim(\mathbf{Bad}(i, j) \cap \mathbf{Bad}(1, 0) \cap \mathbf{Bad}(0, 1)) = \dim([0, 1]^2) = 2.$$

Here, and throughout, "dim" denotes standard Hausdorff dimension. With this result in mind, the aim of this paper is to obtain an expression for $\mathbf{Bad}(i, j)$ in terms of "well-approximable" vectors in the area of "twisted" inhomogeneous Diophantine approximation.

1.2. Background—the "twisted" inhomogeneous theory. Another result of Khintchine (see for example [11, Chapter 10, Theorem 10.2]) states that for any irrational x and any real γ there exist infinitely many natural numbers q such that

$$(1.5) \qquad \qquad \|qx - \gamma\| \leq \frac{1 + \epsilon}{\sqrt{5}q},$$

where $\epsilon > 0$ is an arbitrary constant. The inequality is "optimal" and differs from Hurwitz's homogeneous "$\gamma = 0$" theorem by only the constant ϵ. When certain restrictions are placed on the choice of γ, a tighter "optimal" inequality was found to hold by Minkowski [18]: The right hand side of (1.5) can be replaced with $1/(4q)$ if it is assumed that γ is not of the form $\gamma = mx + n$ for some integers m and n. Both of these statements lead to the implication that the sequence $\{qx\}_{q \in \mathbb{N}}$ modulo one is dense in the unit interval for any irrational x. Moreover, Kronecker's Theorem (see [16]) implies that the sequence $\{q\mathbf{x}\}_{q \in \mathbb{Z}}$ modulo one is dense in $[0, 1]^2$ for any irrational vector $\mathbf{x}$. Furthermore, the sequence is uniformly distributed. This naturally leads to the concept of approximating real vectors $\boldsymbol{\gamma}$ in $[0, 1]^2$ by the sequence $\{q\mathbf{x}\}_{q \in \mathbb{N}}$ modulo one with increasing degrees of accuracy. For obvious reasons we call this approach "twisted" Diophantine approximation.

DEFINITION 1.4. For each fixed approximating function ψ, any irrational vector $\mathbf{x}$ and each pair i, j satisfying (1.2) define $\mathbf{W}^{\mathbf{x}}_{(i,j)}(\psi)$ to be the set of

---

(*) The arguements used by Davenport in [5] to show that $\mathbf{Bad}(1/2, 1/2)$ is uncountable can easily be adapted to show that $\mathbf{Bad}(i, j)$ is uncountable for every choice of reals i, j satisfying (1.2).

vectors $\boldsymbol{\gamma} := (\gamma_1, \gamma_2) \in [0,1]^2$ such that the inequality

$$\max\{\|qx_1 - \gamma_1\|^{1/i}, \|qx_2 - \gamma_2\|^{1/j}\} \leq \psi(|q|)$$

holds for infinitely many non-zero integers q.

Establishing a Khintchine-type law (an analogue to Khintchine's Theorem) for the Lebesgue measure of $\mathbf{W}^{\mathbf{x}}_{(i,j)}(\psi)$ is more difficult than in the homogeneous case. That said, by utilising the Borel–Cantelli lemma from probability theory it is easy to show that for every i, j satisfying (1.2), any irrational $\mathbf{x}$ and every approximating function ψ we have

$$\mu(\mathbf{W}^{\mathbf{x}}_{(i,j)}(\psi)) = 0 \quad \text{if} \quad \sum_{r=1}^{\infty} \psi(r) < \infty.$$

One might therefore expect that no matter what the choice of reals i, j, irrational $\mathbf{x}$ or approximating function ψ we should be able to conclude that $\mu(\mathbf{W}^{\mathbf{x}}_{(i,j)}(\psi)) = 1$ if the above sum diverges. However, the following statement, a consequence of Theorem 6.1 (see Appendix), suggests that once the reals i, j have been fixed, the set of irrational vectors for which we do obtain a set of full measure is dependent on the choice of approximating function. This subtle distinction is what makes the metrical theory in the "twisted" setting more delicate, and sophisticated, than its standard homogeneous counterpart.

THEOREM 1.5 (Twisted Khintchine-type theorem). *Let ψ be a fixed approximating function. Then, for μ-almost all irrational vectors $\mathbf{x} \in [0,1]^2$,*

$$\mu(\mathbf{W}^{\mathbf{x}}_{(i,j)}(\psi)) = 1 \quad \textit{if} \quad \sum_{r=1}^{\infty} \psi(r) = \infty.$$

Approximating functions whose sum diverges will hereafter simply be referred to as *divergent* and the set of all divergent approximating functions will be denoted by $\mathcal{D}$.

DEFINITION 1.6. Fix a pair of reals i, j satisfying (1.2). Then, for each $\psi \in \mathcal{D}$, we define

$$\mathbf{V}_{(i,j)}(\psi) := \{\text{irr. } \mathbf{x} : \mu(\mathbf{W}^{\mathbf{x}}_{(i,j)}(\psi)) = 1\}.$$

Note that Theorem 1.5 is equivalent to the statement "$\mu(\mathbf{V}_{(i,j)}(\psi)) = 1$ for each $\psi \in \mathcal{D}$". In view of this theorem we ask whether there exist irrational vectors $\mathbf{x}$ such that a set of full measure is obtained regardless of the choice of divergent approximating function. In other words, we wish to characterise the set

$$\bigcap_{\psi \in \mathcal{D}} \mathbf{V}_{(i,j)}(\psi).$$

It is certainly not obvious as to whether the intersection is non-empty. Almost all activity in the past has been centred on the specific $i = j = 1/2$ case, where elements of $\mathbf{Bad}(1/2, 1/2)$ are commonly referred to as *simultaneously badly approximable pairs*. The most notable breakthrough was made by Kurzweil [17], who proved the following remarkable result.

KURZWEIL'S THEOREM (1955).
$$\bigcap_{\psi \in \mathcal{D}} \mathbf{V}_{(1/2, 1/2)}(\psi) = \mathbf{Bad}(1/2, 1/2).$$

In fact, Kurzweil's result was more general than the above (see §2.2 for further discussion) but did not touch upon the weighted route with which we are interested. His work has since been extended in various directions by Fayad [7] (who gave a shorter proof of the above result from a dynamical systems viewpoint), Tseng [24] and Chaika [4].

The work of Kim [14] in a similar vein inspired activity concerning real vectors that are badly approximable in the "twisted" inhomogeneous sense.

DEFINITION 1.7. Fix an irrational vector $\mathbf{x} \in [0,1]^2$ and two real numbers i and j satisfying (1.2). Define $\mathbf{Bad}^{\mathbf{x}}(i, j)$ as the set of vectors $\boldsymbol{\gamma} \in [0,1]^2$ for which there exists a constant $c(\boldsymbol{\gamma}) > 0$ such that
$$\max\{\|qx_1 - \gamma_1\|^{1/i}, \|qx_2 - \gamma_2\|^{1/j}\} > c(\boldsymbol{\gamma})/|q| \quad \text{ for all } q \in \mathbb{Z}_{\neq 0}.$$

The set $\mathbf{Bad}^{\mathbf{x}}(i, j)$ represents the twisted inhomogeneous analogue of $\mathbf{Bad}(i, j)$ introduced in §1.1. Previous work has again been confined to the $i = j = 1/2$ setting. In particular, Bugeaud et al. [2] proved the following result (also see the work of Tseng [23] and Moshchevitin [19] for more recent extensions).

THEOREM BHKV (2010). *For any irrational* $\mathbf{x} \in [0,1]^2$,
$$\dim(\mathbf{Bad}^{\mathbf{x}}(1/2, 1/2)) = 2.$$

Once more, the statement proved was more general than the above, which has been simplified for our present needs. At the time of writing there were no known results concerning the Hausdorff dimension of $\mathbf{Bad}^{\mathbf{x}}(i, j)$ for a general pair i and j.

2. The main results

2.1. Statements of results. The following statement represents our main theorem and generalises Kurzweil's Theorem from the classical "1/2-1/2" statement to all "(i, j)-weightings".

THEOREM 2.1. *For every pair of reals i and j satisfying* (1.2),
$$\bigcap_{\psi \in \mathcal{D}} \mathbf{V}_{(i,j)}(\psi) = \mathbf{Bad}(i, j).$$

In view of Khintchine's Theorem and statement (1.4), Theorem 2.1 immediately implies that the intersection on the LHS above is of 2-dimensional Lebesgue measure zero and of full Hausdorff dimension two.

Our next result makes a contribution towards determining the Hausdorff dimension of $\mathbf{Bad^x}(i,j)$.

THEOREM 2.2. *For any real i and j satisfying* (1.2) *and any* $\mathbf{x} \in \mathbf{Bad}(i,j)$,
$$\dim(\mathbf{Bad^x}(i,j)) = 2.$$

The proof of this theorem makes use of a general framework developed by Kristensen, Thorn & Velani [15]. This framework was designed for establishing dimension results for large classes of badly approximable sets and the above statement constitutes one further application. In all likelihood the above result is true without the assumption on $\mathbf{x}$.

CONJECTURE 2.3. *For any real i and j satisfying* (1.2) *and any irrational vector* $\mathbf{x} \in [0,1]^2$,
$$\dim(\mathbf{Bad^x}(i,j)) = 2.$$

It seems that the ideas of [2], which also make use of the framework in [15], are not extendable to the full weighted setting of Conjecture 2.3; a new approach may be required. Note that Theorem 2.2 together with (1.4) trivially implies that the conjecture is true for a set of irrational vectors $\mathbf{x}$ of full dimension.

REMARK. Since submission, Nikolay Moshchevitin and the named author have strengthened Theorem 2.2 from a statement implying full Hausdorff dimension to the statement that $\mathbf{Bad^x}(i,j)$ is "winning" under the given conditions. However, obtaining a solution to Conjecture 2.3 still remains out of reach.

2.2. Higher dimensions. We describe the n-dimensional generalisation of the sets $\mathbf{Bad}(i,j)$ and $\mathbf{V}_{(i,j)}(\psi)$ along with the higher-dimensional analogue of the statements in §2.1. Fix any n-tuple of reals $\mathbf{i} := i_1, \ldots, i_n \geq 0$ such that $\sum_{j=1}^{n} i_j = 1$. We naturally define $\mathbf{Bad}(\mathbf{i})$ to be the set of vectors $\mathbf{x} := (x_1, \ldots, x_n) \in [0,1]^n$ for which there exists a constant $c(\mathbf{x}) > 0$ such that
$$\max\{\|qx_1\|^{1/i_1}, \ldots, \|qx_n\|^{1/i_n}\} > c(\mathbf{x})/q \quad \forall q \in \mathbb{N}.$$
For any approximating function ψ and any irrational vector $\mathbf{x} \in [0,1]^n$, we denote by $\mathbf{W_i^x}(\psi)$ the set of vectors $\boldsymbol{\gamma} := (\gamma_1, \ldots, \gamma_n) \in [0,1]^n$ such that
$$\max\{\|qx_1 - \gamma_1\|^{1/i_1}, \ldots, \|qx_n - \gamma_n\|^{1/i_n}\} \leq \psi(|q|)$$
for infinitely many non-zero integers q. Also, set
$$\mathbf{V_i}(\psi) := \{\mathbf{x} \in [0,1]^n : \mu_n(\mathbf{W_i^x}(\psi)) = 1\},$$

where μ_n denotes the standard n-dimensional Lebesgue measure, and once more denote by $\mathcal{D}$ the set of approximating functions for which

$$\sum_{r=1}^{\infty} \psi(r) = \infty.$$

The proof of Theorem 2.1 can be extended in the obvious way, with no new ideas or difficulties, allowing us to establish the following statement. For every real n-tuple $\mathbf{i}$ such that $i_1, \ldots, i_n \geq 0$ and $\sum_{j=1}^{n} i_j = 1$,

$$(2.1) \qquad \bigcap_{\psi \in \mathcal{D}} \mathbf{V_i}(\psi) = \mathbf{Bad}(\mathbf{i}).$$

Khintchine's Theorem and statement (1.4) can also be generalised and show that the above intersection is of n-dimensional Lebesgue measure zero and of full Hausdorff dimension n. As alluded to above, Kurzweil proved in [17] that equality (2.1) holds in the case that $i_1 = \cdots = i_n = 1/n$ for every natural number n. This includes the one-dimensional formulation of the problem corresponding to the set $\mathbf{Bad}$. However, in these generalisations the notation gets rather awkward and so for the sake of clarity (and relevance to the material in §1) we will prove the $n = 2$ case only.

The set $\mathbf{Bad^x}(\mathbf{i})$ can be defined in the obvious way and analogues of Theorem 2.2 and Conjecture 2.3 can easily be established. The framework and proof of Theorem 2.2 in §5 can easily be modified to establish the corresponding result in higher dimensions.

3. Multiplicative Diophantine approximation. This section comprises a brief discussion of related problems in the area of multiplicative Diophantine approximation, where loosely speaking the supremum norm is replaced by the geometric mean. For example, one could consider the set of vectors that are "well-approximable" in a multiplicative sense.

DEFINITION 3.1. Let ψ be any approximating function. Then define

$$\mathbf{W}_M(\psi) := \{\mathbf{x} \in [0,1]^2 : \|qx_1\| \|qx_2\| \leq \psi(q) \text{ for infinitely many } q \in \mathbb{N}\}.$$

The relevant measure-theoretic result concerning $\mathbf{W}_M(\psi)$ was found by Gallagher [8] who proved a theorem implying the following.

GALLAGHER'S THEOREM (1962). *For any approximating function ψ,*

$$\mu(\mathbf{W}_M(\psi)) = \begin{cases} 0 & \text{if } \sum_{r=1}^{\infty} \psi(r) \log(1/\psi(r)) < \infty, \\ 1 & \text{if } \sum_{r=1}^{\infty} \psi(r) \log(1/\psi(r)) = \infty. \end{cases}$$

It is natural to develop a twisted theory for the multiplicative setup.

DEFINITION 3.2. Fix any approximating function ψ and any irrational vector $\mathbf{x}$ in $[0,1]^2$. Then define

$$\mathbf{W}_M^{\mathbf{x}}(\psi) := \{\boldsymbol{\gamma} \in [0,1]^2 :$$
$$\|qx_1 - \gamma_1\| \, \|qx_2 - \gamma_2\| \leq \psi(|q|) \text{ for infinitely many } q \in \mathbb{Z}_{\neq 0}\}.$$

The following statement is a consequence of Theorem 6.1 (see the Appendix).

THEOREM 3.3. *Fix any approximating function ψ. Then for μ-almost all irrational vectors $\mathbf{x} \in [0,1]^2$ we have*

$$\mu(\mathbf{W}_M^{\mathbf{x}}(\psi)) = \begin{cases} 0 & \text{if } \sum_{r=1}^{\infty} \psi(r) \log(1/\psi(r)) < \infty, \\ 1 & \text{if } \sum_{r=1}^{\infty} \psi(r) \log(1/\psi(r)) = \infty. \end{cases}$$

Once more one could ask whether there exist irrational vectors $\mathbf{x}$ such that a set of full measure is obtained irrespective of the choice of approximating function. Accordingly, let $\mathcal{D}_M$ denote the set of approximating functions for which $\sum_{r=1}^{\infty} \psi(r) \log(1/\psi(r))$ diverges and define

$$\mathbf{V}_M(\psi) := \{\text{irr. } \mathbf{x} : \mu(\mathbf{W}_M^{\mathbf{x}}(\psi)) = 1\}.$$

Consider the intersection

$$(3.1) \qquad\qquad \bigcap_{\psi \in \mathcal{D}_M} \mathbf{V}_M(\psi).$$

In view of Theorem 2.1, one might expect that (3.1) is equivalent to the multiplicative analogue of the set of badly approximable pairs. However, quite how such an analogue should be defined is up for debate.

One could argue that a valid choice for a set of *multiplicatively badly approximable numbers* might be

$$\mathbf{Bad}_L := \{\mathbf{x} \in [0,1]^2 : \exists c(\mathbf{x}) > 0 \ \forall q \in \mathbb{N} \ \|qx_1\| \, \|qx_2\| > c(\mathbf{x})/q\}.$$

The famous Littlewood conjecture states that the set $\mathbf{Bad}_L$ is empty. For recent developments and background concerning the Littlewood conjecture see [6], [20] and the references therein.

Another candidate for the multiplicatively badly approximable numbers is the larger set

$$\mathbf{Mad} := \left\{\mathbf{x} \in [0,1]^2 : \exists c(\mathbf{x}) > 0 \ \forall q \in \mathbb{N} \ \|qx_1\| \, \|qx_2\| > \frac{c(\mathbf{x})}{q \log q}\right\},$$

recently introduced in [1]. Hence, the following question arises:

$$\text{Can } \bigcap_{\psi \in \mathcal{D}_M} \mathbf{V}_M(\psi) \text{ be characterised as } \mathbf{Bad}_L \text{ or } \mathbf{Mad}?$$

Even establishing that $\mathbf{Bad}_L \subseteq \bigcap_{\psi \in \mathcal{D}_M} \mathbf{V}_M(\psi)$ seems non-trivial.

4. Proof of Theorem 2.1

4.1. Proof of Theorem 2.1 (Part 1). If either $i = 0$ or $j = 0$ then the theorem simplifies to a one-dimensional $n = 1$ version of Kurzweil's Theorem corresponding to **Bad**. Therefore, we can and will assume hereafter that $i, j > 0$. The proof of Theorem 2.1 takes the form of two inclusion propositions, the first of which is proved in this section.

PROPOSITION 4.1. *For every real $i, j > 0$ such that $i + j = 1$,*

$$\bigcap_{\psi \in \mathcal{D}} \mathbf{V}_{(i,j)}(\psi) \subseteq \mathbf{Bad}(i,j).$$

Proof. We will show that if $\mathbf{x} \notin \mathbf{Bad}(i,j)$ then $\mathbf{x} \notin \bigcap_{\psi \in \mathcal{D}} \mathbf{V}_{(i,j)}(\psi)$ and prove the result via a contrapositive argument. In particular, we will show that for every such $\mathbf{x}$ there exists an approximating function $\psi_0 \in \mathcal{D}$ for which

$$(4.1) \qquad \mu(\mathbf{W}^{\mathbf{x}}_{(i,j)}(\psi_0)) = 0,$$

i.e., the points $\boldsymbol{\gamma} := (\gamma_1, \gamma_2) \in [0,1]^2$ that satisfy the inequality

$$\max\{\|qx_1 - \gamma_1\|^{1/i}, \|qx_2 - \gamma_2\|^{1/j}\} \leq \psi_0(|q|)$$

for infinitely many non-zero integers q form a null set with respect to the Lebesgue measure.

First, if $\mathbf{x} \notin \mathbf{Bad}(i,j)$ then by definition there exists a sequence $\{q_k\}_{k \in \mathbb{N}}$ of non-zero integers such that

$$(4.2) \qquad \max\{\|q_k x_1\|^{1/i}, \|q_k x_2\|^{1/j}\} < \frac{c_k}{|q_k|}, \qquad |q_k| < |q_{k+1}| \quad \forall k \in \mathbb{N},$$

where $c_k > 0$ and $c_k \to 0$ as $k \to \infty$. Furthermore, it can be assumed that

$$(4.3) \qquad 1 > c_k > 2^{3/(2\min\{i,j\})} c_{k+1} \quad \forall k \in \mathbb{N}.$$

If this were not the case then we could simply choose a suitable subsequence of $\{q_k\}$. In addition, it may also be assumed that the sequence $\{(c_k)^{-1/3}\}_{k \in \mathbb{N}}$ takes integer values for every index k. Note that the latter assumption, along with condition (4.3), guarantees that for every k,

$$(4.4) \qquad c_k^{-1/3} \geq 2.$$

We wish to construct a divergent approximating function ψ_0 for which equation (4.1) is satisfied. To that end, we introduce some useful notation. For each $k \geq 1$, let $n_k := |q_k| c_k^{-1/3}$. In view of the above assumptions the sequence $\{n_k\}_{k \in \mathbb{N}}$ is increasing and takes strictly positive integer values for each index k. That said, we set $n_0 := 0$ for future conciseness. Next, for each natural number r define

$$\psi_0(r) := \begin{cases} 1, & r \leq n_1, \\ |q_{k+1}|^{-1} c_{k+1}^{1/3}, & n_k < r \leq n_{k+1}, \end{cases}$$

for every $k \geq 1$. It is obvious that ψ_0 is a decreasing and strictly positive function. To show $\psi_0 \in \mathcal{D}$, note that

$$\sum_{r=1}^{\infty} \psi_0(r) > \sum_{k=1}^{\infty} \sum_{r=n_k+1}^{n_{k+1}} \psi_0(r) = \sum_{k=1}^{\infty} (n_{k+1} - (n_k + 1) + 1)\psi_0(n_{k+1})$$

$$= \sum_{k=1}^{\infty} (|q_{k+1}|c_{k+1}^{-1/3} - |q_k|c_k^{-1/3})|q_{k+1}|^{-1}c_{k+1}^{1/3}$$

$$= \sum_{k=1}^{\infty} \left(1 - \frac{|q_k|}{|q_{k+1}|}\left(\frac{c_{k+1}}{c_k}\right)^{1/3}\right)$$

$$> \sum_{k=1}^{\infty} \left(1 - \left(\frac{c_{k+1}}{c_k}\right)^{1/3}\right) \quad \text{(since } |q_k| < |q_{k+1}|\text{)}$$

$$\overset{(4.3)}{>} \sum_{k=1}^{\infty} (1 - 2^{-1/(2\min\{i,j\})}) \geq \sum_{k=1}^{\infty} \frac{1}{2} = \infty,$$

as required.

Finally, we endeavour to show (4.1) holds for our choice of divergent function. To that end, for each non-zero integer q let

$$\mathbf{R}_{\psi_0}(q) := \{\boldsymbol{\gamma} \in [0,1]^2 : \max\{\|qx_1 - \gamma_1\|^{1/i}, \|qx_2 - \gamma_2\|^{1/j}\} \leq \psi_0(|q|)\}$$

denote the closed rectangular region in the plane centred at the point $q\mathbf{x}$ (mod 1) of sidelengths $2\psi_0^i(|q|)$ and $2\psi_0^j(|q|)$. When using the notation $\mathbf{R}_{\psi_0}(q)$ it will be understood that i, j and $\mathbf{x}$ are fixed. In addition, all such closed rectangular regions will be referred to throughout as simply a "rectangle" and all points within any such rectangle will tacitly be modulo one. It follows that

$$(4.5) \qquad \mathbf{W}_{(i,j)}^{\mathbf{x}}(\psi_0) = \{\boldsymbol{\gamma} \in [0,1]^2 : \boldsymbol{\gamma} \in \mathbf{R}_{\psi_0}(q) \text{ for infinitely many } q \in \mathbb{Z}_{\neq 0}\}$$

$$= \left\{\boldsymbol{\gamma} \in [0,1]^2 : \boldsymbol{\gamma} \in \bigcup_{|q|=n_{k-1}+1}^{n_k} \mathbf{R}_{\psi_0}(q) \text{ for infinitely many } k \in \mathbb{N}\right\}.$$

In view of the Borel–Cantelli lemma, to show that equation (4.1) holds it is enough to show that

$$(4.6) \qquad \sum_{k=1}^{\infty} \mu\left(\bigcup_{|q|=n_{k-1}+1}^{n_k} \mathbf{R}_{\psi_0}(q)\right) < \infty.$$

We will estimate the LHS by estimating the measure of each union of rectangles of the form

$$\mathbf{R}_{\psi_0}^*(k) := \bigcup_{|q|=n_{k-1}+1}^{n_k} \mathbf{R}_{\psi_0}(q) \qquad \text{for } k \in \mathbb{N}.$$

We will hereafter refer to any union of rectangles as a "collection". For each k, the collection $\mathbf{R}^*_{\psi_0}(k)$ consists of $2(n_k - n_{k-1})$ rectangles in $[0,1]^2$ each centred at some point $q\mathbf{x}$ for which $n_{k-1} < |q| \leq n_k$. By definition, every rectangle in a collection is of the same measure, in particular each has sidelengths $2\psi_0^i(n_k)$ and $2\psi_0^j(n_k)$.

To estimate the measure of $\mathbf{R}^*_{\psi_0}(k)$ we will cover it with a collection of larger rectangles whose measure will in some sense increase at a more controllable rate than those of $\mathbf{R}^*_{\psi_0}(k)$. This will allow us to calculate a finite upper bound for the sum (4.6) as required. With these aims in mind, for each index k set

$$\mathbf{S}^*_{\psi_0}(k) := \bigcup_{|q|=1}^{|q_k|} \left\{ \boldsymbol{\gamma} \in [0,1]^2 : \|qx_1 - \gamma_1\| \leq \frac{n_k}{|q_k|}\left(\frac{c_k}{|q_k|}\right)^i + \psi_0^i(n_k) \right.$$

$$\left. \text{and } \|qx_2 - \gamma_2\| \leq \frac{n_k}{|q_k|}\left(\frac{c_k}{|q_k|}\right)^j + \psi_0^j(n_k) \right\}.$$

Each collection $\mathbf{S}^*_{\psi_0}(k)$ now consists of $2|q_k|$ rectangles in $[0,1]^2$, one centred at each point $q\mathbf{x}$ with $1 \leq |q| < |q_k|$. The sidelengths of each of these rectangles are

$$2\left(\frac{n_k}{|q_k|}\left(\frac{c_k}{|q_k|}\right)^i + \psi_0^i(n_k)\right) \quad \text{and} \quad 2\left(\frac{n_k}{|q_k|}\left(\frac{c_k}{|q_k|}\right)^j + \psi_0^j(n_k)\right).$$

An upper bound for the Lebesgue measure of $\mathbf{S}^*_{\psi_0}(k)$ can be easily deduced. We have

$$(4.7) \quad \mu(\mathbf{S}^*_{\psi_0}(k)) \leq 2^3 |q_k| \left(\frac{n_k}{|q_k|}\left(\frac{c_k}{|q_k|}\right)^i + \psi_0^i(n_k)\right)\left(\frac{n_k}{|q_k|}\left(\frac{c_k}{|q_k|}\right)^j + \psi_0^j(n_k)\right)$$

for every index $k \geq 1$.

We wish to show that $\mathbf{S}^*_{\psi_0}(k)$ covers $\mathbf{R}^*_{\psi_0}(k)$ for each k. As the rectangles of $\mathbf{S}^*_{\psi_0}(k)$ are larger than those of $\mathbf{R}^*_{\psi_0}(k)$, any rectangle of $\mathbf{R}^*_{\psi_0}(k)$ centred at a point $q'\mathbf{x}$ with $n_{k-1} < |q'| \leq |q_k|$ will automatically be contained in the corresponding rectangle of $\mathbf{S}^*_{\psi_0}(k)$. Hence, it will suffice to check that any rectangle of $\mathbf{R}^*_{\psi_0}(k)$ centred at a point $q'\mathbf{x}$ with $|q_k| < |q'| \leq n_k$ is covered by some rectangle of $\mathbf{S}^*_{\psi_0}(k)$. It is clear by construction and inequality (4.4) that $|q_k| < n_k$ and so rectangles of this type are present in every $\mathbf{R}^*_{\psi_0}(k)$. For each of these integers q' we can find a natural number m such that $|q' - mq_k| \leq |q_k|$. This implies there must be a rectangle of the collection $\mathbf{S}^*_{\psi_0}(k)$ that is centred at the point $(q' - mq_k)\mathbf{x}$. It is also clear that m can always be chosen in a way such that $|mq_k| < |q'|$. Therefore

$$(4.8) \qquad\qquad |m| < \frac{|q'|}{|q_k|} \leq \frac{n_k}{|q_k|}.$$

Now, consider the distance between the points $q'\mathbf{x}$ and $(q' - mq_k)\mathbf{x}$. We have

$$\|q'x_1 - (q' - mq_k)x_1\| \ = \ \|-mq_kx_1\| \leq |m| \, \|q_kx_1\|$$

$$\overset{(4.2)}{<} |m|\left(\frac{c_k}{|q_k|}\right)^i \overset{(4.8)}{<} \frac{n_k}{|q_k|}\left(\frac{c_k}{|q_k|}\right)^i,$$

and similarly

$$\|q'x_2 - (q' - mq_k)x_2\| < \frac{n_k}{|q_k|}\left(\frac{c_k}{|q_k|}\right)^j.$$

Combining the above two inequalities implies that any rectangle of $\mathbf{R}^*_{\psi_0}(k)$ centred at a point $q'\mathbf{x}$ with $|q_k| < |q'| \leq n_k$ is covered by the rectangle of $\mathbf{S}^*_{\psi_0}(k)$ centred at $(q' - mq_k)\mathbf{x}$. This shows that $\mathbf{S}^*_{\psi_0}(k)$ is a cover for $\mathbf{R}^*_{\psi_0}(k)$ and so

$$\sum_{k=1}^{\infty}\mu(\mathbf{R}^*_{\psi_0}(k)) \leq \sum_{k=1}^{\infty}\mu(\mathbf{S}^*_{\psi_0}(k)).$$

Estimate (4.7) justifies that the RHS is bounded above by

$$\sum_{k=1}^{\infty}8|q_k|\left(\frac{n_k}{|q_k|}\left(\frac{c_k}{|q_k|}\right)^i + \psi_0^i(n_k)\right)\left(\frac{n_k}{|q_k|}\left(\frac{c_k}{|q_k|}\right)^j + \psi_0^j(n_k)\right)$$

$$= \sum_{k=1}^{\infty}8|q_k|(c_k^{-1/3}c_k^i|q_k|^{-i} + |q_k|^{-i}c_k^{i/3})$$

$$\times (c_k^{-1/3}c_k^j|q_k|^{-j} + |q_k|^{-j}c_k^{j/3})$$

$$= 8\sum_{k=1}^{\infty}|q_k|\,|q_k|^{-i-j}(c_k^{i-1/3} + c_k^{i/3})(c_k^{j-1/3} + c_k^{j/3}).$$

However, we have $i + j = 1$ and so this reduces to

$$8\sum_{k=1}^{\infty}(c_k^{i+j-2/3} + c_k^{(i+j)/3} + c_k^{i/3+j-1/3} + c_k^{i+j/3-1/3})$$

$$= \ 8\sum_{k=1}^{\infty}(2c_k^{1/3} + c_k^{2i/3} + c_k^{2j/3}) \leq 8\sum_{k=1}^{\infty}4c_k^{2\min\{i,j\}/3}$$

$$\overset{(4.3)}{<} 32\sum_{k=1}^{\infty}c_1^{2\min\{i,j\}/3}2^{-(k-1)} = 64c_1^{2\min\{i,j\}/3} < \infty,$$

as required. This completes the proof of Proposition 4.1. ∎

4.2. Proof of Theorem 2.1 (Part 2). In this section we prove the complementary inclusion to that of Proposition 4.1.

PROPOSITION 4.2. *For every real $i, j > 0$ such that $i + j = 1$,*

$$\mathbf{Bad}(i,j) \subseteq \bigcap_{\psi \in \mathcal{D}} \mathbf{V}_{(i,j)}(\psi).$$

Proof. We are required to show that if $\mathbf{x} \in \mathbf{Bad}(i,j)$ then for every divergent approximating function ψ we have

$$\mu(\mathbf{W}^{\mathbf{x}}_{(i,j)}(\psi)) = 1.$$

To do this we first prove the intermediary result that for every $\mathbf{x} \in \mathbf{Bad}(i,j)$ we have

(4.9) $$\mu(\mathbf{W}^{\mathbf{x}}_{(i,j)}(\psi)) > 0$$

for every $\psi \in \mathcal{D}$.

Fix $\mathbf{x} \in \mathbf{Bad}(i,j)$. By definition there exists a constant $c(\mathbf{x}) > 0$ such that for all natural numbers q,

$$\max\{\|qx_1\|^{1/i}, \|qx_2\|^{1/j}\} > c(\mathbf{x})/q.$$

Next, choose any function $\psi \in \mathcal{D}$. To ensure that certain technical conditions required later in the proof are met we will work with a refinement of ψ. Let

$$a^* := 2^{-1/\max\{i,j\}} \quad \text{and} \quad a_* := 2^{-1/\min\{i,j\}},$$

then for each $r \in \mathbb{N}$ set

$$\psi_1(r) := \min\left\{\psi(r), \frac{a^*}{2}, \frac{a_* c(\mathbf{x})}{2|r|}\right\}.$$

Finally, choose any integer k such that

(4.10) $$k > 4$$

and for each natural number r and each $t \in \mathbb{N}$ define

$$\psi_2(r) := \begin{cases} \psi_1(k), & r \le k, \\ \psi_1(k^{t+1}), & k^t < r \le k^{t+1}. \end{cases}$$

It is easy to see that for each $r \in \mathbb{N}$,

(4.11) $$\psi_2(r) \le \psi_1(r) \le \psi(r)$$

and that $\psi_1 \in \mathcal{D}$. It is also clear that ψ_2 is decreasing and strictly positive.

Furthermore,

$$\sum_{r=1}^{\infty} \psi_2(r) \geq \sum_{t=1}^{\infty} \sum_{r=k^t+1}^{k^{t+1}} \psi_2(r) = \sum_{t=1}^{\infty} (k^{t+1} - k^t)\psi_2(k^{t+1})$$

$$= \frac{1}{k} \sum_{t=1}^{\infty} (k^{t+2} - k^{t+1})\psi_1(k^{t+1}) \geq \frac{1}{k} \sum_{t=1}^{\infty} \sum_{r=k^{t+1}+1}^{k^{t+2}} \psi_1(r)$$

$$= \frac{1}{k} \sum_{r=k^2+1}^{\infty} \psi_1(r) = \infty,$$

and so ψ_2 too is a divergent approximating function.

With reference to §4.1, we see that inequality (4.11) and the characterisation of $\mathbf{W}_{(i,j)}^{\mathbf{x}}(\psi)$ in terms of the rectangles $\mathbf{R}_\psi(q)$ given by (4.5) now guarantee that the following statement is sufficient to prove (4.9) for every choice of function ψ. For every integer $r \geq 1$,

(4.12)
$$\mu\Big(\bigcup_{\substack{|q|=r+1}}^{\infty} \mathbf{R}_{\psi_2}(q) \Big) \geq \frac{a_* c(\mathbf{x})}{8}.$$

Note that this statement is in terms of the constructed function ψ_2. To prove (4.12) we will show that there cannot exist a natural number t_0 such that the inequality fails to hold when $r = k^{t_0}$. Assume that such a t_0 exists and consider the collection of rectangles defined by

$$\mathbf{R}_t := \mathbf{R}(\psi_2, t) := \bigcup_{\substack{|q|=k^{t_0}+1}}^{k^t} \mathbf{R}_{\psi_2}(q) \quad \text{for } t = t_0 + 1, t_0 + 2, \ldots.$$

We will demonstrate that the measure of the set $\mathbf{R}_t$ is unbounded as t increases and in doing so reach a contradiction, as each $\mathbf{R}_t$ is contained in $[0,1]^2$. We will do this by estimating the size of a suitable sum of the measure of set differences of the form $\mathbf{R}_{t+1} \setminus \mathbf{R}_t$.

By construction each $\mathbf{R}_{t+1}$ is obtained from $\mathbf{R}_t$ by adding $2(k^{t+1} - k^t)$ new rectangles to those of $\mathbf{R}_t$. These new rectangles are centred at the points $q\mathbf{x}$ for which $k^t < |q| \leq k^{t+1}$. To estimate $\mu(\mathbf{R}_{t+1} \setminus \mathbf{R}_t)$ we will find an upper bound to the number of the new rectangles that intersect any existing rectangle of $\mathbf{R}_t$. In particular, we will find an upper bound on the cardinality of the set $\mathbf{J}_{t+1} \cap 2\mathbf{R}_t$, where $\mathbf{J}_{t+1}$ denotes the set of points $q\mathbf{x}$ for which $k^t < |q| \leq k^{t+1}$ and

$$2\mathbf{R}_t := \bigcup_{\substack{|q|=k^{t_0}+1}}^{k^t} \mathbf{R}_{2\psi_2}(q) \quad \text{for } t = t_0 + 1, t_0 + 2, \ldots.$$

This will suffice as ψ_2 is non-increasing. Before proceeding we first notice

that, since the vector $\mathbf{x}$ was chosen from $\mathbf{Bad}(i, j)$, if $q\mathbf{x}$ and $q'\mathbf{x}$ are members of $\mathbf{J}_{t+1}$ then

$$(4.13) \qquad \max\{\|qx_1 - q'x_1\|^{1/i}, \|qx_2 - q'x_2\|^{1/j}\} \geq \frac{c(\mathbf{x})}{|q - q'|} \geq \frac{c(\mathbf{x})}{2k^{t+1}},$$

provided that the integers q and q' are distinct.

The collection $\mathbf{2R}_t$ can be partitioned into two exhaustive subcollections (which without loss of generality we will assume are non-empty). Recalling that $a_* := 2^{-1/\min\{i,j\}}$, define

$$\mathbf{2R}_t^{(1)} := \bigcup \mathbf{R}_{2\psi_2}(q),$$

where the union runs over all non-zero q with $k^{t_0} < |q| \leq k^t$ such that

$$2\psi_2(|q|) < \frac{a_* c(\mathbf{x})}{2k^{t+1}}.$$

In turn, let

$$\mathbf{2R}_t^{(2)} := \bigcup \mathbf{R}_{2\psi_2}(q),$$

where this time the union runs over q with $k^{t_0} < |q| \leq k^t$ such that

$$2\psi_2(|q|) \geq \frac{a_* c(\mathbf{x})}{2k^{t+1}}.$$

The intersections $\mathbf{J}_{t+1} \cap \mathbf{2R}_t^{(1)}$ and $\mathbf{J}_{t+1} \cap \mathbf{2R}_t^{(2)}$ will now be dealt with independently.

The subcollection $\mathbf{2R}_t^{(1)}$ consists of rectangles of sidelengths

$$2(2\psi_2(|q|))^i \quad \text{and} \quad 2(2\psi_2(|q|))^j,$$

and we have both

$$2(2\psi_2(|q|))^i < \left(\frac{c(\mathbf{x})}{2k^{t+1}}\right)^i \quad \text{and} \quad 2(2\psi_2(|q|))^j < \left(\frac{c(\mathbf{x})}{2k^{t+1}}\right)^j.$$

This follows upon noticing that $\max\{a_*^i, a_*^j\} = 1/2$. Thus, statement (4.13) implies at most one element of $\mathbf{J}_{t+1}$ can lie in each rectangle of $\mathbf{2R}_t^{(1)}$ and so $\mathbf{J}_{t+1} \cap \mathbf{2R}_t^{(1)}$ contains at most $2(k^t - k^{t_0}) < 2k^t$ elements.

Estimating the cardinality of $\mathbf{J}_{t+1} \cap \mathbf{2R}_t^{(2)}$ requires more work and we argue as follows. If a point $\boldsymbol{\gamma}_0$ lies in the subcollection $\mathbf{2R}_t^{(2)}$ then it must lie in a rectangle of the form $\mathbf{R}_{2\psi_2}(q_0) \subseteq \mathbf{2R}_t^{(2)}$ for some integer q_0 with $k^{t_0} < |q_0| \leq k^t$. This rectangle must have sidelengths $2(2\psi_2(|q_0|))^i$ and $2(2\psi_2(|q_0|))^j$ and by definition we have

$$2(2\psi_2(|q_0|))^i \geq 2\left(\frac{a_* c(\mathbf{x})}{2k^{t+1}}\right)^i \quad \text{and} \quad 2(2\psi_2(|q_0|))^j \geq 2\left(\frac{a_* c(\mathbf{x})}{2k^{t+1}}\right)^j.$$

It is now clear that there must exist a point $\mathbf{y}(\boldsymbol{\gamma}_0) \in \mathbf{R}_{2\psi_2}(q_0)$ such that $\boldsymbol{\gamma}_0$ is contained in a subrectangle, say $\mathbf{S}(\boldsymbol{\gamma}_0)$, of $\mathbf{R}_{2\psi_2}(q_0)$ centred at $\mathbf{y}(\boldsymbol{\gamma}_0)$ and of sidelengths $(a_* c(\mathbf{x})/2k^{t+1})^i$ and $(a_* c(\mathbf{x})/2k^{t+1})^j$. The fact that $\max\{a_*^i, a_*^j\}$ $= 1/2$, twinned with equation (4.13), once more guarantees that only one point of $\mathbf{J}_{t+1}$ may lie in any subrectangle of this type. Moreover, any two such subrectangles containing the respective points $q\mathbf{x}$ and $q'\mathbf{x}$, both in $\mathbf{J}_{t+1}$, must be disjoint. Thus, the cardinality of $\mathbf{J}_{t+1} \cap 2\mathbf{R}_t^{(2)}$ cannot exceed $\mu(2\mathbf{R}_t^{(2)})/\mu(\mathbf{S}(\boldsymbol{\gamma}_0))$. We estimate the size of $\mu(2\mathbf{R}_t^{(2)})$ by utilising the following lemma.

LEMMA 4.3. *For every* $t = t_0 + 1, t_0 + 2, \ldots,$
$$\mu(2\mathbf{R}_t) \leq 2\mu(\mathbf{R}_t).$$

Proof. For $s \in \mathbb{N}$, let
$$\mathbf{R}^s := \bigcup_{|q|=k^{t_0}+1}^{k^{t_0}+s} \mathbf{R}_{\psi_2}(q) \quad \text{and} \quad 2\mathbf{R}^s := \bigcup_{|q|=k^{t_0}+1}^{k^{t_0}+s} \mathbf{R}_{2\psi_2}(q).$$

To prove the lemma it suffices to show that $\mu(2\mathbf{R}^s) \leq 2\mu(\mathbf{R}^s)$ for all s. We proceed by induction. If $s = 1$, then
$$\mu(\mathbf{R}^1) = 2\psi_2^i(k^{t_0} + 1) \cdot 2\psi_2^j(k^{t_0} + 1) = 4\psi_2(k^{t_0} + 1).$$

Further,
$$\mu(2\mathbf{R}^1) = 2(2\psi_2(k^{t_0} + 1))^i \cdot 2(2\psi_2(k^{t_0} + 1))^j = 2 \cdot 4\psi_2(k^{t_0} + 1) = 2\mu(\mathbf{R}^1)$$
and the statement holds.

Next, assume the hypothesis holds when $s = s'$ and define a transformation T on the torus $[0,1]^2$ by
$$T(\boldsymbol{\gamma}) := (2^i\gamma_1, 2^j\gamma_2) \quad \forall \boldsymbol{\gamma} \in [0,1]^2.$$

For any subset $\mathbf{D} \subseteq [0,1]^2$, we denote by $T(\mathbf{D})$ the set of all points $T(\boldsymbol{\gamma})$ where $\boldsymbol{\gamma} \in \mathbf{D}$. Let $\mathbf{D}^{s'+1} := \mathbf{R}^{s'+1} \setminus \mathbf{R}^{s'}$; then, since by definition ψ_2 does not exceed $a^*(i,j)/2$,
$$(4.14) \qquad \mu(T(\mathbf{D}^{s'+1})) = 2^i \cdot 2^j \cdot \mu(\mathbf{D}^{s'+1}) = 2\mu(\mathbf{D}^{s'+1}).$$

It is also clear that
$$2\mathbf{R}^{s'+1} = 2\mathbf{R}^{s'} \cup T(\mathbf{D}^{s'+1}),$$

from which it follows that
$$\begin{aligned}
\mu(2\mathbf{R}^{s'+1}) = \mu(2\mathbf{R}^{s'} \cup T(\mathbf{D}^{s'+1})) &\leq \mu(2\mathbf{R}^{s'}) + \mu(T(\mathbf{D}^{s'+1})) \\
&\leq 2\mu(\mathbf{R}^{s'}) + 2\mu(\mathbf{D}^{s'+1}) \quad \text{(by assumption and (4.14) resp.)} \\
&= 2\mu(\mathbf{R}^{s'} \cup \mathbf{D}^{s'+1}) \qquad \text{(since } \mathbf{R}^{s'} \text{ and } \mathbf{D}^{s'+1} \text{ are disjoint)} \\
&= 2\mu(\mathbf{R}^{s'+1}),
\end{aligned}$$

as required. Lemma 4.3 is proved. ∎

We return to our calculation. Since we are assuming that statement (4.12) is false, Lemma 4.3 now yields

$$\mu(\mathbf{2R}_t^{(2)}) \leq \mu(\mathbf{2R}_t) \leq 2\mu(\mathbf{R}_t) < a_* c(\mathbf{x})/4.$$

Thus,

$$\#(\mathbf{J}_{t+1} \cap \mathbf{2R}_t^{(2)}) \leq \frac{\mu(\mathbf{2R}_t^{(2)})}{\mu(\mathbf{S}(\gamma_0))} < \frac{a_* c(\mathbf{x})}{4(a_* c(\mathbf{x})/2k^{t+1})^{i+j}} = \frac{k^{t+1}}{2}$$

and we have found our second upper bound.

Recalling our intention to estimate $\mu(\mathbf{R}_{t+1} \setminus \mathbf{R}_t)$, we can now write down an upper bound for the number of rectangles added to $\mathbf{R}_t$ to make $\mathbf{R}_{t+1}$ that do intersect existing rectangles of $\mathbf{R}_t$. Indeed, this number cannot exceed

$$(4.15) \qquad \#(\mathbf{J}_{t+1} \cap \mathbf{2R}_t) \leq 2k^t + k^{t+1}/2,$$

which follows upon noticing that

$$\mathbf{J}_{t+1} \cap \mathbf{2R}_t = (\mathbf{J}_{t+1} \cap \mathbf{2R}_t^{(1)}) \cup (\mathbf{J}_{t+1} \cap \mathbf{2R}_t^{(2)}).$$

To complete our argument we require one final piece of notation. Let

$$\mathbf{L}_{t+1} := \{q \in \mathbb{Z}_{\neq 0} : q\mathbf{x} \in \mathbf{J}_{t+1}, \, q\mathbf{x} \notin \mathbf{2R}_t\}.$$

The integers $q \in \mathbf{L}_{t+1}$ each correspond to a rectangle of $\mathbf{R}_{t+1}$ that does not intersect any rectangle of $\mathbf{R}_t$. So, by (4.15),

$$(4.16) \quad \#(\mathbf{L}_{t+1}) \ \geq \ 2(k^{t+1} - k^t) - (2k^t + k^{t+1}/2) = (2 - 4/k - 1/2)k^{t+1}$$
$$\overset{(4.10)}{>} \ (2 - 1 - 1/2)k^{t+1} = k^{t+1}/2.$$

We will now estimate $\mu(\mathbf{R}_{t+1} \setminus \mathbf{R}_t)$ by considering the inclusion

$$(4.17) \qquad \mathbf{R}_{t+1} \setminus \mathbf{R}_t \supset \bigcup_{q \in \mathbf{L}_{t+1}} \mathbf{R}_{\psi_2}(q).$$

The rectangles $\mathbf{R}_{\psi_2}(q)$ in the above union have sidelengths $2\psi_2^i(|q|)$ and $2\psi_2^j(|q|)$. Further, if $q, q' \in \mathbf{L}_{t+1}$ then $k^t < |q|, |q'| \leq k^{t+1}$ and so

$$(4.18) \qquad \max\{\|qx_1 - q'x_1\|^{1/i}, \|qx_2 - q'x_2\|^{1/j}\} \overset{(4.13)}{\geq} \frac{c(\mathbf{x})}{2k^{t+1}}.$$

Recall that ψ_2 is constant on each $\mathbf{L}_{t+1}$ by definition, taking the value $\psi_2(k^{t+1})$, and also that

$$\psi_2(r) \leq \frac{a_* c(\mathbf{x})}{2|r|}.$$

Therefore, we have both

$$2\psi_2^i(|q|) = 2\psi_2^i(k^{t+1}) < \left(\frac{c(\mathbf{x})}{2k^{t+1}}\right)^i,$$

$$2\psi_2^j(|q|) = 2\psi_2^j(k^{t+1}) < \left(\frac{c(\mathbf{x})}{2k^{t+1}}\right)^j.$$

Combining these inequalities with statement (4.18) yields that the rectangles $\mathbf{R}_{\psi_2}(q)$ on the RHS of (4.17) are disjoint. Hence,

$$\begin{aligned}
\mu(\mathbf{R}_{t+1} \setminus \mathbf{R}_t) \;&\geq\; \sum_{q\in\mathbf{L}_{t+1}} \mu(\mathbf{R}_{\psi_2}(q)) = 2^2 \sum_{q\in\mathbf{L}_{t+1}} \psi_2(|q|) \\
&\overset{(4.16)}{>}\; 2k^{t+1}\psi_2(k^{t+1}) > 2(k^{t+1}-k^t)\psi_1(k^{t+1}) \\
&=\; \sum_{|q|=k^t+1}^{k^{t+1}} \psi_1(k^{t+1}) = \sum_{|q|=k^t+1}^{k^{t+1}} \psi_1(|q|).
\end{aligned}$$

Finally, ψ_1 is divergent, i.e.

$$\sum_{|q|=1}^{\infty} \psi_1(|q|) = \infty,$$

whence $\sum_{t>t_0} \mu(\mathbf{R}_{t+1} \setminus \mathbf{R}_t) = \infty$. Since $\mathbf{R}_t \subseteq \mathbf{R}_{t+1}$ for any $t > t_0$, this implies that $\mu(\mathbf{R}_t) \to \infty$ as $t \to \infty$. However, each set R_t is contained in $[0,1]^2$ and so a contradiction is reached. This means the assumption that (4.12) fails for some $r = k^{t_0}$ is indeed false, and consequently

$$\mu(\mathbf{W}_{(i,j)}^{\mathbf{x}}(\psi)) > 0$$

for every $\psi \in \mathcal{D}$ as desired.

To complete the proof of Proposition 4.2 we must now show that if $\mathbf{x} \in \mathbf{Bad}(i,j)$ then

$$\mu(\mathbf{W}_{(i,j)}^{\mathbf{x}}(\psi)) = 1$$

for every $\psi \in \mathcal{D}$. Our method will be through the application of two lemmas, the first of which is due to Kurzweil ([17, Lemma 13]).

LEMMA 4.4 (Kurzweil). *Let U and V be subsets of $[0,1]^2$. If $\mu(U) > 0$ and V is dense in $[0,1]^2$ then $\mu(U \oplus V) = 1$, where $U \oplus V := \{u + v \pmod 1 : u \in U, \, v \in V\}$.*

LEMMA 4.5. *For every $\psi \in \mathcal{D}$ and for every natural number s we have*

$$\sum_{r=1}^{\infty} \psi(sr) = \infty.$$

Proof. Suppose $s \geq 1$ and for the ease of notation set $\psi(0) := \psi(1)$. Consider the s-subseries $\sum_{r=0}^{\infty} \psi(sr + k)$ for each $k = 0, \ldots, s-1$. Every

term $\psi(r')$, $r' \in \mathbb{N}$, appears exactly once in exactly one s-subseries. If every s-subseries had a finite sum then the original series $\sum_{r=1}^{\infty} \psi(r)$ would also have a finite sum (precisely equal to the sum of the sums of the s-subseries). Since the original series does not have a finite sum, at least one of the s-subseries must diverge, say $\sum_{r=0}^{\infty} \psi(sr + k_0) = \infty$. Since ψ is decreasing, $\psi(sr) \geq \psi(sr + k_0)$ and so $\sum_{r=0}^{\infty} \psi(sr) = \infty$, and Lemma 4.5 holds. ∎

Returning to the proof of Proposition 4.2, fix a divergent approximating function ψ and a vector $\mathbf{x} \in \mathbf{Bad}(i, j)$. Once again, we will refine ψ before proceeding. Firstly, we will construct a function $\psi_3 \in \mathcal{D}$ such that

$$(4.19) \qquad \lim_{r \to \infty} \frac{\psi_3(r)}{\psi(r)} = 0.$$

Let $r_0 = 0$ and choose $r_1 \geq 1$ such that the inequality $\sum_{r=1}^{r_1} \psi(r) \geq 1$ holds. Then in general construct inductively a strictly increasing sequence $\{r_k\}_{k=0}^{\infty}$ such that for each k,

$$(4.20) \qquad \sum_{r=r_{k-1}+1}^{r_k} \psi(r) \geq k.$$

This is always possible since $\sum_{r=1}^{\infty} \psi(r)$ diverges, so the partial sums from any starting point must tend to infinity. Next, define $c_r := 1/\sqrt{k}$ if $r_{k-1} < r \leq r_k$ and $\psi_3(r) := c_r \psi(r)$. Equation (4.19) therefore holds as $\psi_3(r)/\psi(r) = c_r$ tends to zero. Both ψ and $\{c_r\}$ are strictly positive and decreasing, hence ψ_3 is strictly positive and decreasing. Also, by construction, inequality (4.20) guarantees that

$$\sum_{r=r_{k-1}+1}^{r_k} \psi_3(r) = \frac{1}{k} \sum_{r=r_{k-1}+1}^{r_k} \psi(r) \geq 1,$$

and so

$$\sum_{r=1}^{r_k} \psi_3(r) \geq k.$$

This shows that the sum of ψ_3 diverges and we have verified that $\psi_3 \in \mathcal{D}$.

By Lemma 4.5,

$$\sum_{r=1}^{\infty} \psi_3(sr) = \infty$$

for every natural number s. Consequently, there must exist a strictly increasing sequence of natural numbers $\{s_r\}_{r \in \mathbb{N}}$ with $s_r \to \infty$ as $r \to \infty$ such that

$$\sum_{r=1}^{\infty} \psi_3(s_r r) = \infty.$$

Accordingly, we define $\psi_4(r) := \psi_3(s_r r)$. Hence, for any fixed non-zero integer q' we have

$$(4.21) \qquad \lim_{|q| \to \infty} \frac{\psi_4(|q|)}{\psi(|q + q'|)} = 0.$$

It is also clear that ψ_4 is a divergent approximating function and therefore we know by intermediary result (4.9) that

$$(4.22) \qquad \mu(\mathbf{W}_{(i,j)}^{\mathbf{x}}(\psi_4)) > 0.$$

In addition, if we choose some vector $\mathbf{y}$ such that

$$\mathbf{y} \in \mathbf{W}_{(i,j)}^{\mathbf{x}}(\psi_4) \stackrel{(4.5)}{=} \bigcap_{k=1}^{\infty} \bigcup_{|q|=k}^{\infty} \mathbf{R}_{\psi_4}(q),$$

then for every natural number k there are infinitely many integers q with $|q| \geq k$ such that $\mathbf{y} \in \mathbf{R}_{\psi_4}(q)$. It follows that $\mathbf{y} + q'\mathbf{x}$ is a member of the set of $\boldsymbol{\gamma} \in [0,1]^2$ for which

$$\max\{\|(q+q')x_1 - \gamma_1\|^{1/i}, \|(q+q')x_2 - \gamma_2\|^{1/j}\} \leq \psi_4(|q|)$$

for infinitely many integers q satisfying $|q| \geq k$. For large enough k, equation (4.21) implies that for each q with $|q| \geq k$ the set of $\boldsymbol{\gamma}$ defined above is contained in the rectangle $\mathbf{R}_\psi(q + q')$. It follows that $\mathbf{y} + q'\mathbf{x}$ is contained in infinitely many rectangles of the form $\mathbf{R}_\psi(q)$, i.e.,

$$(4.23) \qquad \mathbf{y} + q'\mathbf{x} \in \bigcap_{k=1}^{\infty} \bigcup_{|q|=k}^{\infty} \mathbf{R}_\psi(q) = \mathbf{W}_{(i,j)}^{\mathbf{x}}(\psi)$$

for every natural number q'.

We are now in a position to apply Lemma 4.4. With reference to the lemma, set

$$U := \mathbf{W}_{(i,j)}^{\mathbf{x}}(\psi_4) \quad \text{and} \quad V := \{q\mathbf{x} : q \in \mathbb{Z}_{\neq 0}\}.$$

By (4.22) we have $\mu(U) > 0$ and, as mentioned in §1.2, Kronecker's Theorem implies that V is dense in $[0,1]^2$ if $\mathbf{x}$ is irrational. Hence, Lemma 4.4 implies that $\mu(U \oplus V) = 1$, from which (4.23) gives

$$\mu(\mathbf{W}_{(i,j)}^{\mathbf{x}}(\psi)) = 1$$

and the proof of Proposition 4.2, and indeed that of Theorem 2.1, is complete. ∎

5. Proof of Theorem 2.2. This proof makes use of the framework developed in [15], which was specifically designed to provide dimension results for a broad range of badly approximable sets. In this section we show that $\mathbf{Bad}^{\mathbf{x}}(i,j)$ falls into this category when $\mathbf{x}$ is chosen from $\mathbf{Bad}(i,j)$. First, we provide a simplification of the framework tailored to our needs.

Let $\mathcal{R} := \{R_\alpha \subset \mathbb{R}^2 : \alpha \in J\}$, where J is an infinite countable index set. We will refer to the R_α as *resonant sets*. Furthermore, it will be assumed that each resonant set takes the form of a cartesian product, i.e., that each R_α can be split into the images $R_{\alpha,t} \subset \mathbb{R}$, $t = 1, 2$, of its two projection maps along the two coordinate axes. Next, let $\beta : J \rightarrow \mathbb{R}_{>0} : \alpha \mapsto \beta_\alpha$ be a positive function on J such that the number of $\alpha \in J$ with β_α bounded above is finite. Thus, as α runs through J the function β_α tends to infinity. Also, for $t = 1, 2$, let $\rho_t : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0} : r \mapsto \rho_t(r)$ be any real, positive, decreasing function such that $\rho_t(r) \rightarrow 0$ as $r \rightarrow \infty$. We assume that either $\rho_1(r) \geq \rho_2(r)$ or $\rho_2(r) \geq \rho_1(r)$ for large enough r. Finally, for each resonant set R_α define a rectangular neighbourhood $\mathcal{F}_\alpha(\rho_1, \rho_2)$ by

$$\mathcal{F}_\alpha(\rho_1, \rho_2) := \{\mathbf{x} \in \mathbb{R}^2 : |x_t - R_{\alpha,t}| \leq \rho_t(\beta_\alpha) \text{ for } t = 1, 2\},$$

where $|x_t - R_{\alpha,t}| := \inf_{a \in \mathcal{R}_{\alpha,t}} |x_t - a|$.

We now introduce the general badly approximable set to which the results of [15] relate. Define $\mathbf{Bad}(\mathcal{R}, \beta, \rho_1, \rho_2)$ to be the set of $\mathbf{x} \in [0, 1]^2$ for which there exists a constant $c(\mathbf{x}) > 0$ such that

$$\mathbf{x} \notin c(\mathbf{x})\mathcal{F}_\alpha(\rho_1, \rho_2) \quad \forall \alpha \in J.$$

That is, $\mathbf{x} \in \mathbf{Bad}(\mathcal{R}, \beta, \rho_1, \rho_2)$ if there exists a constant $c(\mathbf{x}) > 0$ such that for all $\alpha \in J$,

$$|x_t - R_{\alpha,t}| \geq c(\mathbf{x})\rho_t(\beta_\alpha) \quad (t = 1, 2).$$

The aim of the framework is to determine conditions under which the set $\mathbf{Bad}(\mathcal{R}, \beta, \rho_1, \rho_2)$ has full Hausdorff dimension. With this in mind, we begin with some useful notation. For any fixed integers $k > 1$ and $n \geq 1$, define

$$F_n := \{\mathbf{x} \in [0, 1]^2 : |x_t - c_t| \leq \rho_t(k^n) \text{ for } t = 1, 2\}$$

to be the generic closed rectangle in $[0, 1]^2$ with centre $\mathbf{c} := (c_1, c_2)$ and of side lengths given by $2\rho_1(k^n)$ and $2\rho_2(k^n)$. Next, for any $\theta \in \mathbb{R}_{>0}$, let

$$\theta F_n := \{\mathbf{x} \in [0, 1]^2 : |x_t - c_t| \leq \theta\rho_t(k^n) \text{ for } t = 1, 2\}$$

denote the rectangle F_n scaled by θ. Finally, let

$$J(n) := \{\alpha \in J : k^{n-1} \leq \beta_\alpha < k^n\}.$$

The following statement is a simplification of Theorem 2 of [15], made possible by the properties of the 2-dimensional Lebesgue measure μ.

THEOREM KTV (2006). *Let k be sufficiently large. Suppose there exists some $\theta \in \mathbb{R}_{>0}$ such that for any $n \geq 1$ and any rectangle F_n there exists a collection $\mathcal{C}(\theta F_n)$ of disjoint rectangles $2\theta F_{n+1}$ contained within θF_n such that*

(5.1)
$$\#\mathcal{C}(\theta F_n) \geq \kappa_1 \frac{\mu(\theta F_n)}{\mu(\theta F_{n+1})}$$

and

(5.2) $\#\{2\theta F_{n+1} \subset \mathcal{C}(\theta F_n) : R_\alpha \cap 2\theta F_{n+1} \neq \emptyset$ *for some* $\alpha \in J(n+1)\}$

$$\leq \kappa_2 \frac{\mu(\theta F_n)}{\mu(\theta F_{n+1})},$$

where $0 < \kappa_2 < \kappa_1$ *are absolute constants independent of* k *and* n. *Furthermore, suppose*

(5.3) $$\dim\left(\bigcup_{\alpha \in J} R_\alpha\right) < 2.$$

Then

$$\dim(\mathbf{Bad}(\mathcal{R}, \beta, \rho_1, \rho_2)) = 2.$$

Proof of Theorem 2.2. Fix two positive reals i, j with $i+j = 1$ and some $\mathbf{x} \in \mathbf{Bad}(i,j)$. It is once more assumed that $i, j > 0$, for in this case the theorem would otherwise follow immediately from Corollary 1 of [2]. With reference to the above framework, set

$$J := \{q \in \mathbb{Z}_{\neq 0}\}, \quad \alpha := q \in J, \quad R_\alpha := R_q = \{q\mathbf{x} + \mathbf{p} : \mathbf{p} \in \mathbb{Z}^2\},$$
$$\beta_\alpha := \beta_q = |q|, \quad \rho_1(r) := 1/r^i, \quad \rho_2(r) := 1/r^j.$$

By design we then have

$$\mathbf{Bad}(\mathcal{R}, \beta, \rho_1, \rho_2) = \mathbf{Bad}^{\mathbf{x}}(i, j)$$

and so the proof is reduced to showing that the conditions of Theorem KTV are satisfied.

For $k > 1$ and $m \geq 1$, let F_m be a generic closed rectangle with centre in $[0,1]^2$ and of side lengths $2k^{-mi}$ and $2k^{-mj}$. For k sufficiently large and any $\theta \in \mathbb{R}_{>0}$ it is clear that there exists a collection $\mathcal{C}(\theta B_m)$ of closed rectangles $2\theta F_{m+1}$ within θF_m each, of side lengths $4\theta k^{-(m+1)i}$ and $4\theta k^{-(m+1)j}$. Moreover, the number of rectangles in this collection exceeds

$$\left\lfloor \frac{2\theta k^{-mi}}{4\theta k^{-(m+1)i}} \right\rfloor \cdot \left\lfloor \frac{2\theta k^{-mj}}{4\theta k^{-(m+1)j}} \right\rfloor.$$

Here, the symbol $\lfloor \cdot \rfloor$ denotes the integer part. For large enough k the above is strictly positive and is bounded below by

$$\frac{1}{2} \cdot \frac{2\theta k^{-mi}}{4\theta k^{-(m+1)i}} \cdot \frac{1}{2} \cdot \frac{2\theta k^{-mj}}{4\theta k^{-(m+1)j}} = \frac{1}{16} \cdot \frac{4\theta^2 k^{-m(i+j)}}{4\theta^2 k^{-(m+1)(i+j)}} = \frac{1}{16} \cdot \frac{\mu(\theta F_m)}{\mu(\theta F_{m+1})}.$$

Hence, inequality (5.1) holds with $\kappa_1 := 1/16$.

We endeavour to show that the additional condition (5.2) on the collection $\mathcal{C}(\theta F_m)$ is satisfied. To this end, we fix $m \geq 1$ and proceed as follows. Choose two members of distinct moduli from the set $J(m+1)$, i.e., choose

two integers q and q' such that

(5.4) $$k^m \leq |q'| < |q| < k^{m+1}.$$

Associated with the integers q and q' are the resonant sets R_q and $R_{q'}$, whose elements take the form $q\mathbf{x} + \mathbf{p}$ and $q'\mathbf{x} + \mathbf{p}'$ respectively (for some $\mathbf{p}, \mathbf{p}' \in \mathbb{Z}^2$). Consider the minimum distance between a point in R_q and one in $R_{q'}$. For $t = 1, 2$,

$$|(qx_t + p_t) - (q'x_t + p_t')| = |(q - q')x_t + p_t - p_t'| \geq \|(q - q')x_t\|.$$

Since $\mathbf{x} \in \mathbf{Bad}(i, j)$, either

$$\|(q - q')x_1\| \geq \left(\frac{c(\mathbf{x})}{|q - q'|} \right)^i \stackrel{(5.4)}{>} \left(\frac{c(\mathbf{x})}{2k^{m+1}} \right)^i$$

or

$$\|(q - q')x_2\| \geq \left(\frac{c(\mathbf{x})}{|q - q'|} \right)^j \stackrel{(5.4)}{>} \left(\frac{c(\mathbf{x})}{2k^{m+1}} \right)^j.$$

Therefore, if we set

$$\theta := \frac{1}{2} \min\left\{ \left(\frac{c(\mathbf{x})}{2k} \right)^i, \left(\frac{c(\mathbf{x})}{2k} \right)^j \right\}$$

then the rectangle θF_m has side lengths

$$2\theta k^{-mi} = \min\left\{ \left(\frac{c(\mathbf{x})}{2k} \right)^i, \left(\frac{c(\mathbf{x})}{2k} \right)^j \right\} k^{-mi} \leq \left(\frac{c(\mathbf{x})}{2k^{m+1}} \right)^i,$$

$$2\theta k^{-mj} = \min\left\{ \left(\frac{c(\mathbf{x})}{2k} \right)^i, \left(\frac{c(\mathbf{x})}{2k} \right)^j \right\} k^{-mj} \leq \left(\frac{c(\mathbf{x})}{2k^{m+1}} \right)^j.$$

So, for any two integers q, q' of distinct moduli in $J(m + 1)$, if a member of R_q lies in θF_m then no members of $R_{q'}$ may lie in θF_m. Only one point of R_q may lie in θF_m (since $\mu(\theta F_m) < 1$) and so only two points over all possible resident sets may lie in any rectangle θF_m: those corresponding to q and $-q$. Hence,

$$\#\{2\theta F_{m+1} \subset \mathcal{C}(\theta F_m) : R_q \cap 2\theta F_{m+1} \neq \emptyset \text{ for some } q \in J(m + 1)\} \leq 2,$$

which for large enough k is certainly less than

$$\frac{k}{32} = \frac{1}{32} \cdot \frac{\mu(\theta F_m)}{\mu(\theta F_{m+1})}.$$

So, with θ as defined above and with $\kappa_2 := 1/32 < \kappa_1$, the collection $\mathcal{C}(\theta F_m)$ satisfies inequality (5.2).

Finally, note that the family $\mathcal{R}$ of resonant sets takes the form of a countable number of countable sets and so

$$\dim\left(\bigcup_{q \in J} R_q \right) = 0$$

and inequality (5.3) trivially holds. Thus, the conditions of Theorem KTV are satisfied and Theorem 2.2 follows. ∎

6. Appendix. We conclude the paper by proving a general result implying Theorems 1.5 & 3.3 as stated in the main body of the paper. The result is an extension of Cassels' inhomogeneous Khintchine-type theorem [3, Chapter VII, Theorem II]. The proof is a modification of Cassels' original argument and also borrows ideas from the work of Gallagher.

THEOREM 6.1. *For any sequence* $\{A_q\}_{q\in\mathbb{N}}$ *of measurable subsets of* $[0,1)^d$ *let* A *denote the set of all pairs* $(\mathbf{x},\boldsymbol{\gamma}) \in [0,1)^d \times [0,1)^d$ *for which there exist infinitely many* $q \in \mathbb{N}$ *and* $\mathbf{p} \in \mathbb{Z}^d$ *such that*

$$(6.1) \qquad\qquad q\mathbf{x} - \boldsymbol{\gamma} - \mathbf{p} \in A_q.$$

Then

$$\mu_{2d}(A) := \begin{cases} 0 & \text{if } \sum_{r=1}^{\infty} \mu_d(A_r) < \infty, \\ 1 & \text{if } \sum_{r=1}^{\infty} \mu_d(A_r) = \infty, \end{cases}$$

where μ_s *denotes* s-*dimensional Lebesgue measure.*

Proof. We begin by considering the case in which the sum $\sum_{r=1}^{\infty} \mu_d(A_r)$ converges. Fix $\boldsymbol{\gamma} \in [0,1)^d$. For each natural number q, a vector $\mathbf{x}$ satisfying (6.1) uniquely determines the integral vector $\mathbf{p}$ in such a way that $|\mathbf{p}| < q$. Therefore, the measure of the set of all $\mathbf{x} \in [0,1)^d$ that satisfy (6.1) for each q is given by

$$\mu_d\left(\bigcup_{\mathbf{p}\in[0,q)^d} \frac{(A_q \oplus \boldsymbol{\gamma}) \oplus \mathbf{p}}{q}\right) = \sum_{\mathbf{p}\in[0,q)^d} \mu_d\left(\frac{(A_q \oplus \boldsymbol{\gamma}) \oplus \mathbf{p}}{q}\right),$$

since the union is disjoint. The dilation property of μ_d implies that this is equivalent to

$$q^{-d}\sum_{\mathbf{p}\in[0,q)^d} \mu_d((A_q \oplus \boldsymbol{\gamma}) \oplus \mathbf{p}) = q^{-d}q^d \cdot \mu_d(A_q \oplus \boldsymbol{\gamma}) = \mu_d(A_q),$$

by the translational invariance of μ_d. Now, if $\sum_{r=1}^{\infty} \mu_d(A_r) < \infty$, then for any $\epsilon > 0$ the set of vectors satisfying (6.1) for any $q \geq Q$ has measure at most $\sum_{q \geq Q} \mu_d(A_q) < \epsilon$ for large enough Q. In particular, the set of $\mathbf{x}$ with infinitely many solutions to (6.1) has measure at most ϵ. This completes the proof of the convergence case.

Let us now assume that the sum $\sum_{r=1}^{\infty} \mu_d(A_r)$ diverges. Define the function $\alpha_q : \mathbb{R}^d \to \mathbb{R}$ for each natural number q as follows. Let

$$\alpha_q(\mathbf{x}) := \begin{cases} 1 & \text{if } \exists \mathbf{p} \in \mathbb{Z}^d, \ \mathbf{x} - \mathbf{p} \in A_q, \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that each α_q is measurable since it is equivalent to the characteristic function of a countable union of measurable sets in $\mathbb{R}^d$. Next, for every

natural number Q define the function $\mathcal{A}_Q : [0,1)^d \times [0,1)^d \to \mathbb{R}$ by

$$\mathcal{A}_Q(\mathbf{x}, \boldsymbol{\gamma}) := \sum_{q \leq Q} \alpha_q(q\mathbf{x} - \boldsymbol{\gamma}).$$

We wish to verify that $\mathcal{A}_Q$ is measurable. To that end, we introduce the following lemma, which is a generalisation of a well-known result in measure theory and follows via simple modification of the classical proof (see for example [22, Chapter 2, Proposition 3.9]).

LEMMA 6.2. *If f is a measurable function on $\mathbb{R}^d$ then the function $F_q(\mathbf{x}, \boldsymbol{\gamma}) := f(q\mathbf{x} - \boldsymbol{\gamma})$ is measurable on $\mathbb{R}^d \times \mathbb{R}^d$ for every natural number q.*

Since α_q is finite-valued (and finite sums of finite-valued measurable functions are measurable functions) Lemma 6.2 implies that $\mathcal{A}_Q$ is indeed measurable on $[0,1)^d \times [0,1)^d$. Furthermore, by construction, it is apparent that $\mathcal{A}_Q(\mathbf{x}, \boldsymbol{\gamma})$ is simply the number of natural q with $q \leq Q$ such that

$$q\mathbf{x} - \boldsymbol{\gamma} - \mathbf{p} \in A_q \quad \text{for some } \mathbf{p} \in \mathbb{Z}^d.$$

Hence, to complete the proof of Theorem 6.1 it suffices to show $\mathcal{A}_Q(\mathbf{x}, \boldsymbol{\gamma}) \to \infty$ almost everywhere as $Q \to \infty$. We will hereafter consider $\mathcal{A}_Q$ as a random variable in a probability space with probability measure μ_d.

For any positive measurable function $f : [0,1)^d \times [0,1)^d \to \mathbb{R}^d_{\geq 0}$ we denote the *expectation* of f by

$$E(f) := \int\limits_{[0,1)^d} \int\limits_{[0,1)^d} f(\mathbf{x}, \boldsymbol{\gamma}) \, d\mathbf{x} \, d\boldsymbol{\gamma}.$$

If the *variance* $V(f) := E(f^2) - E(f)^2$ of f is finite then the famous Paley–Zygmund inequality (see for example [12, ineq. II, p. 8]) states that

$$\mu_d(\{(\mathbf{x}, \boldsymbol{\gamma}) : f(\mathbf{x}, \boldsymbol{\gamma}) \geq \epsilon E(f)\}) \geq (1 - \epsilon)^2 \frac{E(f)^2}{E(f^2)}$$

for any sufficiently small $\epsilon > 0$. We will use this inequality to reach our desired conclusion.

Before applying the Paley–Zygmund inequality to $\mathcal{A}_Q$ we must show that $V(\mathcal{A}_Q)$ is finite. It suffices to show that both $E(\mathcal{A}_Q)$ and $E(\mathcal{A}_Q^2)$ are finite. To do this we require the following lemma [3, Chapter VII, Lemma 3].

LEMMA 6.3 (Cassels). *Let α be a measurable function of period one of the variable $\mathbf{x} \in \mathbb{R}^d$. Then*

$$\int\limits_{[0,1)^d} \alpha(q\mathbf{x} + \boldsymbol{\gamma}) \, d\mathbf{x} = \int\limits_{[0,1)^d} \alpha(\mathbf{x}) \, d\mathbf{x}$$

for any vector $\boldsymbol{\gamma} \in \mathbb{R}^d$ and any integer $q \neq 0$.

We note that α_q is of period one and so

$$
\begin{aligned}
(6.2) \quad E(\mathcal{A}_Q) &= \int_{[0,1)^d} \int_{[0,1)^d} \mathcal{A}_Q(\mathbf{x}, \boldsymbol{\gamma}) \, d\mathbf{x} \, d\boldsymbol{\gamma} \\
&= \sum_{q \leq Q} \int_{[0,1)^d} \int_{[0,1)^d} \alpha_q(q\mathbf{x} - \boldsymbol{\gamma}) \, d\mathbf{x} \, d\boldsymbol{\gamma} \\
&\overset{\text{Lem. } 6.3}{=} \sum_{q \leq Q} \int_{[0,1)^d} \int_{[0,1)^d} \alpha_q(\mathbf{x}) \, d\mathbf{x} \, d\boldsymbol{\gamma} \\
&= \sum_{q \leq Q} \int_{[0,1)^d} \int_{[0,1)^d} \chi_{A_q}(\mathbf{x}) \, d\mathbf{x} \, d\boldsymbol{\gamma} = \sum_{q \leq Q} \mu_d(A_q),
\end{aligned}
$$

which is indeed finite. Further,

$$
\begin{aligned}
E((\mathcal{A}_Q)^2) &= \int_{[0,1)^d} \int_{[0,1)^d} (\mathcal{A}_Q(\mathbf{x}, \boldsymbol{\gamma}))^2 \, d\mathbf{x} \, d\boldsymbol{\gamma} \\
&= \sum_{q,r \leq Q} \int_{[0,1)^d} \int_{[0,1)^d} \alpha_q(q\mathbf{x} - \boldsymbol{\gamma}) \alpha_r(r\mathbf{x} - \boldsymbol{\gamma}) \, d\mathbf{x} \, d\boldsymbol{\gamma} \\
&= \sum_{q,r \leq Q} \int_{[0,1)^d} \int_{[0,1)^d} \alpha_{r-s}(-\boldsymbol{\gamma}') \alpha_r(s\mathbf{x}' - \boldsymbol{\gamma}') \, d\mathbf{x}' \, d\boldsymbol{\gamma}',
\end{aligned}
$$

via the change of variables $\mathbf{x}' := \mathbf{x}$, $\boldsymbol{\gamma}' := \boldsymbol{\gamma} - q\mathbf{x}$ and $s := r - q$. Here, the range of both $\mathbf{x}'$ and $\boldsymbol{\gamma}'$ can be taken as $[0,1)^d$ since the function α_q is periodic. Let

$$
\mathcal{A}^{(r,s)}(\mathbf{x}', \boldsymbol{\gamma}') := \int_{[0,1)^d} \int_{[0,1)^d} \alpha_{r-s}(-\boldsymbol{\gamma}') \alpha_r(s\mathbf{x}' - \boldsymbol{\gamma}') \, d\mathbf{x}' \, d\boldsymbol{\gamma}'.
$$

Then for $r = q$ we have $s = 0$ and

$$
\begin{aligned}
\mathcal{A}^{(r,s)}(\mathbf{x}', \boldsymbol{\gamma}') &= \int_{[0,1)^d} \int_{[0,1)^d} (\alpha_q(-\boldsymbol{\gamma}'))^2 \, d\mathbf{x}' \, d\boldsymbol{\gamma}' \\
&= \int_{[0,1)^d} \int_{[0,1)^d} \alpha_q(-\boldsymbol{\gamma}') \, d\mathbf{x}' \, d\boldsymbol{\gamma}' = \mu_d(A_q).
\end{aligned}
$$

However, if $r \neq q$ then $s \neq 0$ and we get

$$
\begin{aligned}
\mathcal{A}^{(r,s)}(\mathbf{x}', \boldsymbol{\gamma}') &= \int_{[0,1)^d} \alpha_{r-s}(-\boldsymbol{\gamma}') \, d\mathbf{x}' \int_{[0,1)^d} \int_{[0,1)^d} \alpha_r(s\mathbf{x}' - \boldsymbol{\gamma}') \, d\mathbf{x}' \, d\boldsymbol{\gamma}' \\
&\overset{\text{Lem. } 6.3}{=} \mu_d(A_{r-s}) \int_{[0,1)^d} \int_{[0,1)^d} \alpha_r(\mathbf{x}') \, d\mathbf{x}' \, d\boldsymbol{\gamma}' = \mu_d(A_q)\mu_d(A_r).
\end{aligned}
$$

These equivalences yield

$$E((\mathcal{A}_Q)^2) = \sum_{q,r \leq Q} \mathcal{A}^{(r,s)}(\mathbf{x}', \boldsymbol{\gamma}') = \sum_{q \leq Q} \mu_d(A_q) + \sum_{\substack{q,r \leq Q \\ q \neq r}} \mu_d(A_q)\mu_d(A_r)$$

$$\leq \sum_{q \leq Q} \mu_d(A_q) + \Big(\sum_{q \leq Q} \mu_d(A_q)\Big)^2 \leq (1-\epsilon)^{-2}\Big(\sum_{q \leq Q} \mu_d(A_q)\Big)^2$$

$$= (1-\epsilon)^{-2}(E(\mathcal{A}_Q))^2$$

for any sufficiently small $\epsilon > 0$ and large enough Q (since $\sum_{q \leq Q} \mu_d(A_q) \to \infty$ as $Q \to \infty$ by assumption). Note that the final bound is finite as required.

In view of the Paley–Zygmund inequality we have

$$\mu_d\Big(\Big\{(\mathbf{x}, \boldsymbol{\gamma}) : \mathcal{A}_Q(\mathbf{x}, \boldsymbol{\gamma}) \geq \epsilon \sum_{q \leq Q} \mu_d(A_q)\Big\}\Big) \geq (1-\epsilon)^4 \geq 1 - 4\epsilon.$$

Finally, since $\mathcal{A}_Q$ increases monotonically with Q, we have $\mathcal{A}_Q(\mathbf{x}, \boldsymbol{\gamma}) \to \infty$ in $[0,1)^d \times [0,1)^d$ except on a set of measure at most 4ϵ. This completes the proof as the choice of ϵ is arbitrary. ∎

References

[1] D. Badziahin and S. Velani, *Multiplicatively badly approximable numbers and generalised Cantor sets*, arXiv:1007.1848.

[2] Y. Bugeaud, S. Harrap, S. Kristensen and S. Velani, *On shrinking targets for $\mathbb{Z}^m$ actions on tori*, Mathematika 56 (2010), 193–202.

[3] J. W. S. Cassels, *An Introduction to Diophantine Approximation*, Cambridge Tracts in Math. 45, Cambridge Univ. Press, Cambridge, 1957.

[4] J. Chaika, *Shrinking targets for IETs: Extending a theorem of Kurzweil*, arXiv: 0910.2694.

[5] H. Davenport, *A note on Diophantine approximation*, in: Studies in Mathematical Analysis and Related Topics, Stanford Univ. Press, Stanford, CA, 1962, 77–81.

[6] M. Einsiedler, A. Katok and E. Lindenstrauss, *Invariant measures and the set of exceptions to Littlewood's conjecture*, Ann. of Math. 164 (2006), 513–560.

[7] B. Fayad, *Mixing in the absence of the shrinking target property*, Bull. London Math. Soc. 38 (2006), 829–838.

[8] P. Gallagher, *Metric simultaneous diophantine approximation*, J. London Math. Soc. 37 (1962), 387–390.

[9] —, *Metric simultaneous diophantine approximation II*, Mathematika 12 (1965), 123–127.

[10] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 5th ed., Oxford Univ. Press, Oxford, 1979.

[11] L. K. Hua, *Introduction to Number Theory*, Springer, Berlin, 1982.

[12] J.-P. Kahane, *Some Random Series of Functions*, Cambridge Univ. Press, Cambridge, 1985.

[13] A. Khintchine, *Einige Sätze über Kettenbrüche, mit Anwendungen auf die Theorie der Diophantischen Approximationen*, Math. Ann. 92 (1924), 115–125.

[14] D. H. Kim, *The shrinking target property of irrational rotations*, Nonlinearity 20 (2007), 1637–1643.

[15] S. Kristensen, R. Thorn and S. Velani, *Diophantine approximation and badly approximable sets*, Adv. Math. 203 (2006), 132–169.

[16] L. Kronecker, *Näherungsweise ganzzahlige Auflösung linearer Gleichungen*, Monatsber. Königl. Preuss. Akad. Wiss. 1884, Werke, Vol. III, reprint, Chelsea, 1968, 47–109.

[17] J. Kurzweil, *On the metric theory of inhomogeneous diophantine approximations*, Studia Math. 15 (1955), 84–112.

[18] H. Minkowski, *Ueber die Annäherung an eine reelle Grösse durch rationale Zahlen*, Math. Ann. 54 (1901), 91–124.

[19] N. G. Moshchevitin, *A note on badly approximable affine forms and winning sets*, Moscow Math. J. 11 (2011), 129–137.

[20] A. D. Pollington and S. L. Velani, *On a problem in simultaneous Diophantine approximation: Littlewood's conjecture*, Acta Math. 185 (2000), 287–306.

[21] —, —, *On simultaneously badly approximable numbers*, J. London Math. Soc. 66 (2002), 29–40.

[22] E. M. Stein and R. Shakarchi, *Real Analysis. Measure Theory, Integration, and Hilbert Spaces*, Princeton Univ. Press, Princeton, NJ, 2005.

[23] J. Tseng, *Badly approximable affine forms and Schmidt games*, J. Number Theory 129 (2009), 3020–3025.

[24] —, *On circle rotations and the shrinking target properties*, Discrete Contin. Dynam. Systems 20 (2008), 1111–1122.

Stephen Harrap
Department of Mathematics
University of York
Heslington, York, YO10 5DD, United Kingdom
E-mail: sgh111@york.ac.uk