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# On vertical zeros of $\Re\zeta(s)$ and $\Im\zeta(s)$

by

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**1. Introduction.** The following notation will be used:

- $s = \sigma + it$ — complex variable.
- $\zeta(s)$ — the Riemann zeta function.
- $\Gamma(s)$ — the Euler gamma function.
- $\chi(s)$ — the function defined by $\zeta(s) = \chi(s)\zeta(1-s)$. The functional equation of $\zeta(s)$ yields

$$\chi(s) = \frac{\pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)}{\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)} = 2(2\pi)^{s-1}\sin\frac{\pi s}{2}\,\Gamma(1-s).$$

- $T$ — large positive parameter.
- $N(\sigma, T)$ — the number of zeros of $\zeta(s)$ in the region $\Re s \ge \sigma$, $|\Im s| \le T$.

For a fixed real number $\sigma_0$ we consider $\Re\zeta(\sigma_0 + it)$ and $\Im\zeta(\sigma_0 + it)$ as functions of a real variable $t$ and denote them by $A(t) = A(\sigma_0, t)$ and $B(t) = B(\sigma_0, t)$ respectively. Some properties of these functions are described by J. Moser [3]. In particular for $\sigma_0 > 1/2$ he investigated the question of existence of zeros of $A(t) - 1$ and $B(t)$ in short intervals.

Our note is motivated by the suggestion of Professor A. A. Karatsuba to investigate the number of zeros of $A(t)$ and $B(t)$ in the interval $(0, T)$.

Let $N_A(T)$ be the number of zeros of $A(t)$ in $(0, T)$. Analogously define $N_B(T)$. By $N_{A,B}(T)$ we denote any of $N_A(T)$ and $N_B(T)$.

We will use Jensen's inequality in the following form [2, pp. 328–329]:

*Let $f(s)$ be an analytic function in some neighborhood of the disc $|s - s_0| \le R$. Let $0 < r < R$, and suppose $f(s)$ has $n$ zeros* (*counting with multiplicity*) *in the disc $|s - s_0| < r$. Then*

$$\left(\frac{R}{r}\right)^n \le \frac{\max_{|s-s_0|=R}|f(s)|}{|f(s_0)|}.$$

We also need Stirling's formulae [2, pp. 342–344]:

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*Let* $\delta > 0$ *and* $-\pi + \delta < \arg s < \pi - \delta$. *Then*

$$\log \Gamma(s) = (s - 1/2)\log s - s + \log\sqrt{2\pi} + O(|s|^{-1})$$

*where the constant in the* $O$ *symbol depends only on* $\delta$.

Throughout, all constants, including those implicit in the $O$ symbols, may depend only on $\sigma_0$.

Define the functions $F(s) = F(\sigma_0, s)$ and $G(s) = G(\sigma_0, s)$ by

$$F(s) = \zeta(\sigma_0 + s) + \zeta(\sigma_0 - s), \qquad G(s) = \zeta(\sigma_0 + s) - \zeta(\sigma_0 - s).$$

These functions are analytic on the whole complex plane except two simple poles at $s = \sigma_0 - 1$ and $s = 1 - \sigma_0$. The zeros on the imaginary axis of $F(s)$ and $G(s)$ correspond to the zeros of $A(t)$ and $B(t)$ respectively. Therefore, using

$$\max_{|s|=R} |F(s)| \ll R^{cR}, \qquad \max_{|s|=R} |G(s)| \ll R^{cR}$$

and appropriately applying Jensen's inequality we obtain

$$N_{A,B}(T) \ll T\log T.$$

As follows from the definition, the zeros of $F(s)$ as well as of $G(s)$ are symmetric with respect to the real and imaginary axes. Let $N_F(T)$ be the number of zeros of $F(s)$ in the rectangle $-10 < \Re s < 10$, $0 < \Im s < T$. Analogously define $N_G(T)$. Let also $N_{F,G}(T)$ denote either $N_F(T)$ or $N_G(T)$.

THEOREM 1. *For any* $\sigma_0 < 1/2$ *there exists* $c = c(\sigma_0) > 0$ *such that*

$$N_{A,B}(T) > \frac{T}{\pi}\log\frac{T}{2\pi} - \frac{T}{\pi} - N(1-\sigma_0, T) - c\log T \tag{1}$$

*and*

$$N_{F,G}(T) = \frac{T}{\pi}\log\frac{T}{2\pi} - \frac{T}{\pi} + O(\log T). \tag{2}$$

For estimation of $N(1-\sigma_0, T)$ see [2, Chapter V] and [5, pp. 252–253].

COROLLARY. *For any* $\sigma_0 < 1/2$ *there exists* $c_1 = c_1(\sigma_0) > 0$ *such that*

$$N_{A,B}(T) = \frac{T}{\pi}\log\frac{T}{2\pi} - \frac{T}{\pi} + O(T^{1-c_1}).$$

THEOREM 2. *Let* $1/2 < \sigma_0 < 1$. *Then*

$$T \ll N_{A,B}(T) \ll T\log T.$$

**2. Proof of Theorem 1.** We shall first prove (1) with a constant $c$ and the constants implicit in the $O$ symbols being absolute for $0 < \sigma_0 < 1/2$. With these restrictions without loss of generality we may suppose that $\zeta(\sigma_0 + it) \neq 0$ for all $t$, $0 < t \le T$. Really, this holds if $\sigma_0 \le 0$, and we keep $\sigma_0$ unchanged in this case. For $\sigma_0 > 0$ let $N_{A,B}(T) = N_A(T)$ and let $t_1, \ldots, t_n$ be all different zeros of $A(\sigma_0, t)$ with $0 < t < T$. If $t_i$ has multiplicity $k_i$

then the $k_i$th derivative of $A(\sigma_0, t)$ with respect to $t$ does not vanish in a neighborhood of $t_i$. Hence there exist $\delta > 0$, $\varepsilon > 0$ and $\varepsilon_1 > 0$ such that for each index $i$ and for $|\sigma - \sigma_0| < \varepsilon_1, |t - t_i| < \delta$ we have $|d^{k_i} A(\sigma, t)/dt^{k_i}| > \varepsilon$, and for $|\sigma - \sigma_0| < \varepsilon_1$, $|t - t_i| \geq \delta$, $i = 1, \ldots, n$, we have $|A(\sigma, t)| > \varepsilon$. Note that if an interval $(a, b)$ contains $k + 1$ zeros of $A(\sigma, t)$ then there is a $c \in (a, b)$ such that the $k$th derivative of $A(\sigma, t)$ at $t = c$ is zero. It follows that for $|\sigma - \sigma_0| < \varepsilon_1$ the number of zeros of $A(\sigma, t)$ in $(1, T)$ is not greater than the number of zeros of $A(\sigma_0, t)$ in the same interval. Now it remains to choose $\sigma_1 > 0$ with $0 < \sigma_0 - \sigma_1 < \varepsilon_1$ such that $\zeta(\sigma_1 + it) \neq 0$ for all $t$.

We may also suppose that there is no zero of $\zeta(s)$ on the line $\Im s = T$. Then

$$\zeta(\sigma_0 + it) = \zeta(\sigma_0 + i) \exp\left(\int_{\sigma_0+i}^{\sigma_0+it} \frac{\zeta'(s)}{\zeta(s)}\, ds\right).$$

Hence

$$A(t) = |\zeta(\sigma_0 + i)| e^{\phi(t)} \cos\left(\alpha_0 + \Im \int_{\sigma_0+i}^{\sigma_0+it} \frac{\zeta'(s)}{\zeta(s)}\, ds\right),$$

$$B(t) = |\zeta(\sigma_0 + i)| e^{\phi(t)} \sin\left(\alpha_0 + \Im \int_{\sigma_0+i}^{\sigma_0+it} \frac{\zeta'(s)}{\zeta(s)}\, ds\right),$$

where

$$\alpha_0 = \arg \zeta(\sigma_0 + i), \qquad \phi(t) = \Re \int_{\sigma_0+i}^{\sigma_0+it} \frac{\zeta'(s)}{\zeta(s)}\, ds.$$

This yields

$$N_{A,B}(T) > \left| 2\Re \frac{1}{2\pi i} \int_{\sigma_0+i}^{\sigma_0+iT} \frac{\zeta'(s)}{\zeta(s)}\, ds \right| + O(1). \tag{3}$$

Note that

$$\Re \frac{1}{2\pi i} \int_L \frac{\zeta'(s)}{\zeta(s)}\, ds = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} - \frac{1}{2} N(1 - \sigma_0, T) + O(\log T)$$

where $L$ is a positively oriented rectangle with vertices at $\sigma_0 + i, 2 + i, 2 + iT, \sigma_0 + iT$.

Set

$$V(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} - \frac{1}{2} N(1 - \sigma_0, T).$$

Then

$$-\Re \frac{1}{2\pi i} \int_{\sigma_0+i}^{\sigma_0+iT} \frac{\zeta'(s)}{\zeta(s)}\, ds = -\Re \frac{1}{2\pi i} \int_{L_1} \frac{\zeta'(s)}{\zeta(s)}\, ds + V(T) + O(\log T)$$

where $L_1$ is an interval $[2+iT, \sigma_0+iT]$. Hence

$$-\Re\frac{1}{2\pi i}\int\limits_{\sigma_0+i}^{\sigma_0+iT}\frac{\zeta'(s)}{\zeta(s)}\,ds = -\frac{1}{2\pi}\Delta\arg\zeta(s) + V(T) + O(\log T)$$

where $\Delta$ denotes the variation from $2+iT$ to $\sigma_0+iT$. This expression is estimated by $O(\log T)$ according to [5, p. 213]. This proves (1) in view of (3).

We shall now prove (2) following B. C. Berndt [1].

We consider the case $N_{F,G}(T) = N_F(T)$. The other case is analogous.

Let $M$ be a constant such that $M > 10+|\sigma_0|$ and $F(s)\neq 0$ on $\Re s = \pm M$. Let also $T_0$ be a constant sufficiently large with respect to $M$ such that $F(s)\neq 0$ on $\Im s = T_0$. We may also suppose that $F(s)\neq 0$ on $\Im s = T$. Then

$$N_1(T) = \frac{1}{2\pi}\Im\Big\{\int\limits_{L_1} + \int\limits_{L_2} + \int\limits_{L_3} + \int\limits_{L_4}\Big\}\frac{F'(s)}{F(s)}\,ds$$

where $L_1 = [-M+iT_0, M+iT_0], L_2 = [M+iT_0, M+iT], L_3 = [M+iT, -M+iT], L_4 = [-M+iT, -M+iT_0]$ and $N_1(T)$ denotes the number of zeros of $F(s)$ inside the rectangle with sides $L_1, L_2, L_3, L_4$.

The first integral is $O(1)$. From $F(s) = F(-s), F(\overline{s}) = \overline{F(s)}$ we see that the integrals along $L_2$ and $L_4$ are equal. Therefore

$$N_1(T) = \frac{1}{\pi}\Delta_1\arg F(s) + \frac{1}{2\pi}\Delta_2\arg F(s) + O(1)$$

where $\Delta_1$ denotes the variation from $M+iT_0$ to $M+iT$, and $\Delta_2$ denotes the variation from $M+iT$ to $-M+iT$. Since

$$F(M+it) = \zeta(\sigma_0+M+it) + \chi(\sigma_0-M-it)\zeta(1+M-\sigma_0+it)$$

we have

$$\begin{aligned}\Delta_1\arg F(s) &= \Delta_1\arg\{\chi(\sigma_0-M-it)\zeta(1+M-\sigma_0+it)\}\\ &\quad + \Delta_1\arg\left\{1+\frac{\zeta(\sigma_0+M+it)}{\chi(\sigma_0-M-it)\zeta(1+M-\sigma_0+it)}\right\}.\end{aligned}$$

Due to Stirling's formula and the inequality $1/4 < |\zeta(s)| < 10$ for $\Re s > 2$ we can choose $T_0$ such that

$$\left|\frac{\zeta(\sigma_0+M+it)}{\chi(\sigma_0-M-it)\zeta(1+M-\sigma_0+it)}\right| < \frac{1}{2}$$

for any $t > T_0$. Therefore

$$\Delta_1\arg F(s) = \Delta_1\arg\{\chi(\sigma_0-M-it)\zeta(1+M-\sigma_0+it)\} + O(1).$$

Since $\Re\zeta(1+M-\sigma_0+it) > 1/4$, we have

$$\Delta_1\arg F(s) = \Delta_1\arg\chi(\sigma_0-M-it) + O(1).$$

Using Stirling's formula again we easily calculate the above value and obtain the following relation:

$$N_1(T) = \frac{T}{\pi}\log\frac{T}{2\pi} - \frac{T}{\pi} + \frac{1}{2\pi}\Delta_2 \arg F(s) + O(1).$$

Let us now estimate $\Delta_2 \arg F(s)$ using the method of [5, p. 213]. Define

$$F_1(s) = F(s) + F(s - 2iT).$$

Let $D_1, D_2$ be the discs with center $M + iT$ and radii $4M$ and $2M$ respectively. It is easy to see that

$$\max_{s\in D_1} \frac{|F_1(s)|}{|F_1(M+iT)|} \ll T^K$$

for some constant $K = K(M) > 0$. Therefore according to Jensen's inequality there are $O(\log T)$ zeros of $F_1(s)$ inside $D_2$. In particular $F_1(s)$ has $O(\log T)$ zeros in the interval $[M + iT, -M + iT]$. Since

$$F_1(\sigma + iT) = 2\Re F(\sigma + iT),$$

$\Re F(s)$ vanishes $O(\log T)$ times between $M + iT$ and $-M + iT$. Hence

$$\Delta_2 \arg F(s) = O(\log T).$$

Thus

$$N_1(T) = \frac{T}{\pi}\log\frac{T}{2\pi} - \frac{T}{\pi} + O(\log T).$$

If $\sigma_0 \le 0$ then $N(1 - \sigma_0, T) = 0$. Therefore for $\sigma_0 \le 0$ the required estimate of $N_F(T)$ follows from (1) and $N_F(T) \le N_1(T) + O(1)$.

If $0 < \sigma_0 < 1/2$ then we can choose $T_0 = T_0(M)$ so large that $F(s) \ne 0$ in the region $\Re s > 10, \Im s > T_0$, i.e. $N_F(T) = N_1(T) + O(1)$. This completes the proof of Theorem 1.

From the functional equation it follows that

$$F(s) = \zeta(\sigma_0 + s) + 2(2\pi)^{-z}\cos\frac{\pi z}{2}\,\Gamma(z)\zeta(z) \tag{4}$$

where $z = 1 - \sigma_0 + s$. Hence for some $M = M(\sigma_0) > 0$ and $T_0 = T_0(\sigma_0) > 0$ there is no zero of $F(s)$ in the region $\Re s > M$, $\Im s > T_0$. Taking

$$M = 2n_1 - 1 + \sigma_0, \qquad T = 2n_2 - 1 + \sigma_0$$

where $n_1, n_2$ are integers, using (4) and applying an argument of R. Spira [4], we can prove that $F(s)$ and $F(s) - \zeta(\sigma_0 + s)$ have the same number of zeros in the rectangle with vertices at $M \pm iT_0, T \pm iT_0$. Indeed, on the sides of this rectangle due to the $\Gamma$-factor we have

$$|F(s) - (F(s) - \zeta(\sigma_0 + s))| = |\zeta(\sigma_0 + s)| < 1 < \left|2(2\pi)^{-z}\cos\frac{\pi z}{2}\,\Gamma(z)\zeta(z)\right| = |F(s) - \zeta(\sigma_0 + s)|,$$

and thus the assertion follows from Rouché's theorem. For $n$ large enough the function $F(s)$ has different signs at $s = 2n - 1 + \sigma_0$ and $2n + 1 + \sigma_0$, and therefore it has at least one zero in $(2n-1+\sigma_0, 2n+1+\sigma_0)$. Furthermore, the zeros of $|F(s) - \zeta(\sigma_0 + s)|$ inside the considered rectangle are $s_n = 2n + \sigma_0$, and the number of such zeros is equal to the number of zeros of $F(s)$ inside this rectangle. Hence altogether there exists $M_1 = M_1(\sigma_0) > 0$ such that all zeros of $F(s)$ with $|\Re s| > M_1$ are real and exactly one in each interval $(2n-1+\sigma_0, 2n+1+\sigma_0)$. Therefore the following statement is valid:

*For any real numbers $a, b$ with $a+b<1$ almost all zeros of the function $\zeta(a+s)+\zeta(b-s)$ are on the line $\Re s = (b-a)/2$.*

The same remains true for the function $\zeta(a+s)-\zeta(b-s)$.

**3. Concerning Theorem 2.** Theorem 2 can be proved by using the following variant of Voronin's theorem on universality of the Riemann zeta-function, which follows from [2, pp. 241–252]:

*Let $0 < r < 1/4$. Suppose that $f(s)$ is a function which is analytic for $|s| < r$ and continuous for $|s| \le r$. If $f(s) \neq 0$ then for any $\varepsilon > 0$,*

$$\lim_{T\to\infty} \inf \frac{1}{T} \operatorname{meas}\{\tau \in [0,T] : \sup_{|s|\le r} |f(s) - \zeta(s+3/4+i\tau)| < \varepsilon\} > 0.$$

Applying it with $\varepsilon = 1/10$, $|r - 1/4| < \frac{1}{10}(\sigma_0 - 1/2)$ and $f(s) = e^{Cs}$ where $C = C(\sigma_0)$ is a positive number large enough, we see $A(t)$ and $B(t)$ change sign more than $cT$ times in $(1, T)$.

Remark. If $\sigma_0 = 1/2$ then

$$N_{A,B}(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + N_0(T) + O(\log T)$$

where $N_0(T)$ is the number of zeros of $\zeta(1/2+it)$ with $0 < t < T$. This formula follows from

$$\zeta(1/2+it) = e^{-i\theta(t)}Z(t).$$

Here $Z(t), \theta(t)$ are real-valued functions, $\theta(t)$ increases on $(t_0, \infty)$ and

$$\theta(t) = \frac{t}{2}\log\frac{t}{2\pi} - \frac{t}{2} + O(\log t).$$

Problem. *Is it true that for any $\sigma_0$, $1/2 < \sigma_0 < 1$, the inequality*

$$N_{A,B}(T) > T\,\phi(T)$$

*holds for some real-valued function $\phi(t)$ with $\phi(t) \to \infty$ as $t \to \infty$?*

It would be interesting to define $\phi(t)$ explicitly if it exists.

By applying Voronin's method one can extend Theorem 2 to the values of $\sigma_0$ up to $1+\delta$ with some $\delta > 0$. Note that $A(t)$ has no zeros for $\sigma_0 > 2$.

One can study the problem of zeros of $B(t)$ using Hardy's method, by comparing the integrals $\int_T^{T+H} |B(t)|\,dt$ and $|\int_T^{T+H} B(t)\,dt|$. This method yields the following result.

*For any* $\sigma_0 > 1$ *there exists a constant* $H = H(\sigma_0) > 0$ *such that for any* $T > T_0(\sigma_0) > 0$ *the interval* $(T, T + H)$ *contains a zero of* $B(t)$.

Indeed from the simplest approximate functional equation for $\zeta(s)$ we can see that

$$-B(t) = \sum_{n\le T/\pi} n^{-\sigma_0}\sin(t\log n) + O(HT^{-\sigma_0})$$

for $t \in (T, T + H)$. Therefore

$$\int_T^{T+H} B(t)\,dt \ll 1 + H^2T^{-\sigma_0}.$$

On the other hand it can be easily established that

$$\int_T^{T+H} |B(t)|\,dt > \int_T^{T+H} -B(t)\sin(t\log 2)\,dt \ge c_1H - c_2$$

with some positive constants $c_1, c_2$ (which may depend on $\sigma_0$). Hence for $H$ large enough the interval $(T, T + H)$ contains at least one zero of $B(t)$.

In particular for $\sigma_0 > 1$ we have $T \ll N_B(T) \ll T\log T$.

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