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Greither's maximal independent system of units in global function fields

by

JAEHYUN AHN and HWANYUP JUNG (Daejon)

1. Introduction and notations. In the number field case, many authors (Ramachandra [7], Levesque [6], Greither [2] and Kučera [5]) studied a certain maximal independent system of units. In the function field case, Feng and Yin [1] gave results analogous to those of Ramachandra and Levesque when the base field is a rational function field, and their results are generalized by Xu and Zhao [11] to any subfield of cyclotomic function field over a global function field. In [4], we gave results analogous to those of Greither and Kučera when the base field is a rational function field. In this paper, we extend our previous results to the global function field case.

We introduce some basic notations and facts which are needed later. Let k be a global function field with constant field $\mathbb{F}_q$ of q elements, and let ∞ be a fixed place of k with degree d_∞. Let k_∞ be the completion of k at ∞, and Ω be the completion of an algebraic closure of k_∞. Let $\mathbb{A}$ be the Dedekind ring of functions in k which are holomorphic away from ∞. Let $\mathbb{F}_\infty$ ($\simeq \mathbb{F}_{q^{d_\infty}}$) be the residue field at ∞ and $W_\infty = |\mathbb{F}_\infty^*| = q^{d_\infty} - 1$. Throughout the paper we fix a sign-function $\mathrm{sgn} : k_\infty^* \to \mathbb{F}_\infty^*$ (cf. [3, Section 12]). An element z of k_∞^* is called *positive* if $\mathrm{sgn}(z) = 1$. For each integral ideal $\mathfrak{m}$ of $\mathbb{A}$ one uses a sgn-normalized Drinfeld module of rank one to construct a field extension $K_\mathfrak{m}$, called the $\mathfrak{m}$*th cyclotomic function field*, and its maximal real subfield $K_\mathfrak{m}^+$. For more details we refer to Hayes's article [3, Part II]. Let $\xi(\mathfrak{m}) \in \Omega$ be an invariant associated to the ideal $\mathfrak{m}$, which is characterized by the condition that the lattice $\xi(\mathfrak{m})\mathfrak{m}$ corresponds to some sgn-normalized Drinfeld module of rank one, say ϱ. Let

$$\Lambda_\mathfrak{m}^\varrho = \{\alpha \in \Omega : \varrho_x(\alpha) = 0 \text{ for } x \in \mathfrak{m}\}$$

be the set of $\mathfrak{m}$-torsion points associated to ϱ, which is an $\mathbb{A}$-module via ϱ isomorphic to $\mathbb{A}/\mathfrak{m}$. In fact, $\xi(\mathfrak{m})$ is determined up to a factor in $\mathbb{F}_\infty^*$ ([3, Proposition 13.1]). Thus we should fix the ξ-invariants as in [12, Section 2].

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Let $e_{\mathfrak{m}}(z)$ be the exponential function associated to the lattice $\mathfrak{m}$, i.e.,

$$e_{\mathfrak{m}}(z) = z \prod_{a \in \mathfrak{m},\, a \neq 0} (1 - z/a).$$

Let $\lambda_{\mathfrak{m}} = \xi(\mathfrak{m}) e_{\mathfrak{m}}(1) \in K_{\mathfrak{m}}$. Then $\lambda_{\mathfrak{m}}$ is a generator of $\Lambda_{\mathfrak{m}}^{\varrho}$.

Let F/k be a finite abelian extension which is contained in a cyclotomic function field. Let $\mathbb{F}_F$ be the constant field of F with $W_F = |\mathbb{F}_F^*|$, the order of nonzero elements of $\mathbb{F}_F$. By the *conductor* $\mathfrak{m}$ of F, we mean the integral ideal $\mathfrak{m}$ of $\mathbb{A}$ such that $K_{\mathfrak{m}}$ is the smallest cyclotomic function field which contains F. If $\mathfrak{m} = \mathfrak{e}$, then F is unramified at every finite place $\mathfrak{p} \neq \infty$. Let F^+ be the maximal real subfield of F in which ∞ splits completely. We say that F/k is a *real extension* if $F = F^+$. Let $G_F = \mathrm{Gal}(F/k)$ and $J_F = \mathrm{Gal}(F/F^+)$ with $\delta_F = |J_F|$, its order. For any integral ideal $\mathfrak{f}$ of $\mathbb{A}$, let $F_{\mathfrak{f}} = F \cap K_{\mathfrak{f}}$ and $F_{\mathfrak{f}}^+ = F \cap K_{\mathfrak{f}}^+$. Let $\widehat{G}_F$ be the character group of G_F with values in $\mathbb{C}$. A character χ is called *real* if $\chi(J_F) = 1$ and *nonreal* otherwise. We denote by $\widehat{G}_F^+$ the set of all real characters of G_F and $\widehat{G}_F^- = \widehat{G}_F \backslash \widehat{G}_F^+$. We also denote the conductor of a character $\chi \in \widehat{G}_F$ by $\mathfrak{f}_\chi$, which is an integral ideal of $\mathbb{A}$. For $\chi \in \widehat{G}_F$ and an ideal $\mathfrak{a}$ of $\mathbb{A}$, we define $\chi(\mathfrak{a})$ as follows. If $(\mathfrak{a}, \mathfrak{f}_\chi) = \mathfrak{e}$, we let $\chi(\mathfrak{a}) = \chi(\sigma_{\mathfrak{a}})$, where $\sigma_{\mathfrak{a}} = (\mathfrak{a}, F_{\mathfrak{f}_\chi}/k)$ is the Artin symbol. If $(\mathfrak{a}, \mathfrak{f}_\chi) \neq \mathfrak{e}$, we put $\chi(\mathfrak{a}) = 0$. Let $h(F)$ and $h(F^+)$ be the divisor class number of F and F^+, respectively. We have the following analytic class number formulas (see [10, Chapter VII, §6, Theorem 4]):

$$h(F) = \frac{W_F[\mathbb{F}_F : \mathbb{F}_q]}{q-1} h(k) \prod_{1 \neq \chi \in \widehat{G}_F} L_k(0, \overline{\chi}),$$

$$h(F^+) = \frac{W_{F^+}[\mathbb{F}_{F^+} : \mathbb{F}_q]}{q-1} h(k) \prod_{1 \neq \chi \in \widehat{G}_F^+} L_k(0, \overline{\chi}), \tag{1.1}$$

where $L_k(s, \chi)$ is the Artin L-function associated to the character χ.

Let $\mathcal{O}_F$ be the integral closure of $\mathbb{A}$ in F and $\mathcal{O}_F^*$ be the unit group of $\mathcal{O}_F$. Let $h(\mathcal{O}_F)$ and $h(\mathcal{O}_{F^+})$ be the ideal class number of $\mathcal{O}_F$ and $\mathcal{O}_{F^+}$, respectively. Then by [8, Lemma 4.1 and its Corollary], we have

$$d_\infty h(F) = R(F) h(\mathcal{O}_F), \qquad d_\infty h(F^+) = R(F^+) h(\mathcal{O}_{F^+}),$$

where $R(F)$ and $R(F^+)$ are the regulator of F and F^+, respectively. Let $Q_0 = [\mathcal{O}_F^* : \mathcal{O}_{F^+}^*]$ be the index of units.

LEMMA 1.1. $R(F) = \delta_F^{[F^+:k]-1} R(F^+)/Q_0$.

Proof. Following the proof of [12, Corollary 1.6], we get the result. We note that "κQ_0" in [12] corresponds to Q_0 in our notation. ∎

We recall the logarithm map l_F of F, which is defined by

$$l_F : F^* \to \mathbb{Q}[G_F], \quad x \mapsto l_F(x) = \sum_{\sigma\in G} v_\infty(x^\sigma)\sigma^{-1},$$

where v_∞ is the extension to Ω of the normalized valuation of k_∞ at ∞. We also write $l_F^* = (1-e_1)l_F$. Here e_1 is the idempotent element associated to the trivial character. Let $e^+ = s(J_F)/\delta_F$. Let R_0 be the augmentation ideal of $R = \mathbb{Z}[G_F]$.

LEMMA 1.2. $(e^+R_0 : l_F(\mathcal{O}_F^*)) = R(F)$.

Proof. As in [12, §4, (4.1)], this follows from the definition of the logarithm map l_F and the regulator $R(F)$. ■

2. Maximal independent system of units. In this section, we fix a finite abelian extension F/k with conductor $\mathfrak{m} = \prod_{i=1}^{s} \mathfrak{p}_i^{e_i}$ and let $G = G_F, R = \mathbb{Z}[G]$ for simplicity. For any ideal $\mathfrak{f} \neq \mathfrak{e}$, we define

$$\lambda_{\mathfrak{f},F} = N_{K_\mathfrak{f}/F_\mathfrak{f}}(\lambda_\mathfrak{f}).$$

For any prime ideal $\mathfrak{p}$ of $\mathbb{A}$, let $T_\mathfrak{p}$ and $D_\mathfrak{p}$ be the inertia group and decomposition group of $\mathfrak{p}$ in G. Let $\mathcal{F}_\mathfrak{p} \in D_\mathfrak{p}$ be a Frobenius automorphism associated to $\mathfrak{p}$, which is determined modulo $T_\mathfrak{p}$. We set $\overline{\sigma}_\mathfrak{p} = \mathcal{F}_\mathfrak{p}^{-1}s(T_\mathfrak{p})/|T_\mathfrak{p}|$, which is the unique element of $\mathbb{C}[G]$ satisfying $\chi(\overline{\sigma}_\mathfrak{p}) = \overline{\chi}(\mathfrak{p})$ for any $\chi \in \widehat{G}$. For any subset T of G, we write $s(T) = \sum_{\sigma\in T}\sigma \in \mathbb{Z}[G]$. We also define

$$\omega_F = W_\infty \sum_{\chi\neq 1,\,\text{real}} L_k(0,\overline{\chi})e_\chi,$$

where $e_\chi = (1/|G|)\sum_{\sigma\in G}\chi(\sigma)\sigma^{-1}$ is the idempotent element associated to the character χ.

LEMMA 2.1. *For any integral ideal* $\mathfrak{f}$ *of* $\mathbb{A}$, *let* $I_\mathfrak{f} = \mathrm{Gal}(F/F_\mathfrak{f})$. *Then*

$$l_F^*(\lambda_{\mathfrak{f},F}) = \omega_F s(I_\mathfrak{f}) \prod_{\mathfrak{p}|\mathfrak{f}} (1-\overline{\sigma}_\mathfrak{p}). \tag{2.1}$$

Moreover, *for* $\chi \neq 1$ *real*,

$$\chi(l_F(\lambda_{\mathfrak{f},F})) = W_\infty L_k(0,\overline{\chi})\chi(s(I_\mathfrak{f})) \prod_{\mathfrak{p}|\mathfrak{f}} (1-\overline{\chi}(\mathfrak{p})).$$

Proof. From [12, Proposition 3.1], we have $l_{K_\mathfrak{f}}^*(\lambda_\mathfrak{f}) = \omega_{K_\mathfrak{f}}\prod_{\mathfrak{p}|\mathfrak{f}}(1-\overline{\sigma}_\mathfrak{p}) \in \mathbb{Q}[G_{K_\mathfrak{f}}]$. Applying the restriction map $\mathrm{res}_{K_\mathfrak{f}/F_\mathfrak{f}} : \mathbb{Q}[G_{K_\mathfrak{f}}] \to \mathbb{Q}[G_{F_\mathfrak{f}}]$ and the corestriction map $\mathrm{cor}_{F_\mathfrak{f}/F} : \mathbb{Q}[G_{F_\mathfrak{f}}] \to \mathbb{Q}[G]$, we get (2.1). ■

For any integral ideal $\mathfrak{n} \neq \mathfrak{e}$ of $\mathbb{A}$, let $N_\mathfrak{n}$ be the subgroup of $G_{F_\mathfrak{e}^+}$ generated by the Artin symbols $\tau_\mathfrak{p} = (\mathfrak{p}, F_\mathfrak{e}^+/k)$ for all primes $\mathfrak{p} \mid \mathfrak{n}$. We choose an ideal $\mathfrak{m}'$ which is coprime to $\mathfrak{m}$ and $N_{\mathfrak{m}\mathfrak{m}'} = G_{F_\mathfrak{e}^+}$. Let $\overline{\mathfrak{m}} = \mathfrak{m}\mathfrak{m}' = \prod_{i=1}^{s+t}\mathfrak{p}_i^{e_i}$. Let $S = \{1,\ldots,s+t\}$ and $\mathbb{P}_S$ be the set of all proper subsets of S. For each $i \in S$, we write $T_i = T_{\mathfrak{p}_i}$, $D_i = D_{\mathfrak{p}_i}$ and $\mathcal{F}_i = \mathcal{F}_{\mathfrak{p}_i}$ for simplicity. Let $t_i = |T_i|, f_i = |D_i|/|T_i|$ and $g_i = |G|/|D_i|$ be the ramification degree, inertia degree and decomposition degree of $\mathfrak{p}_i$ in F, respectively. We also set $\nu_i = \sum_{j=1}^{f_i} \mathcal{F}_i^j \in R$. For each subset I of S, we also introduce the following notations: $\overline{\mathfrak{m}}_I = \prod_{i\in I} \mathfrak{p}_i^{e_i}$, $T_I = \prod_{i\in I} T_i$, $D_I = \prod_{i\in I} D_i$, $\nu_I = \prod_{i\in I} \nu_i$ and $n_I = (\prod_{i\in I} t_i)/|T_I|$. For each $I \in \mathbb{P}_S$, we put

$$\lambda_I = N_{K_{\overline{\mathfrak{m}}/\overline{\mathfrak{m}}_I}/F_{\overline{\mathfrak{m}}/\overline{\mathfrak{m}}_I}}(\lambda_{\overline{\mathfrak{m}}/\overline{\mathfrak{m}}_I}) = \lambda_{\overline{\mathfrak{m}}/\overline{\mathfrak{m}}_I,F}.$$

For any given function $\beta : \mathbb{P}_S \to R$, we define

$$\lambda(\beta) = \prod_{I\in\mathbb{P}_S} \lambda_I^{n_I\beta(I)}.$$

Since $\lambda_I^{\sigma-1} \in \mathcal{O}_F^*$ for any $\sigma \in G$, we have $\lambda(\beta)^{\sigma-1} \in \mathcal{O}_F^*$. Let $\mathcal{R}$ be a system of representatives for G/J_F containing 1 and $\mathcal{R}^* = \mathcal{R}\backslash\{1\}$. Let C_β be the subgroup of $\mathcal{O}_F^*$ generated by $\mathbb{F}_F^*$ and $\{\lambda(\beta)^{\sigma-1} : \sigma \in \mathcal{R}^*\}$. Let $r = [F^+ : k] - 1$.

THEOREM 2.2. *For any function $\beta : \mathbb{P}_S \to R$, we have*

$$[\mathcal{O}_F^* : C_\beta] = \frac{Q_0(q-1)}{W_F[\mathbb{F}_F : \mathbb{F}_q]} \left(\frac{W_\infty}{\delta_F}\right)^r \frac{h(\mathcal{O}_{F^+})}{h(\mathbb{A})}\, i_\beta,$$

where

$$i_\beta = \Big| \prod_{\substack{\chi\neq 1\\ real}} \sum_{\substack{I\in\mathbb{P}_S\\ (\mathfrak{f}_\chi,\overline{\mathfrak{m}}_I)=\mathfrak{e}}} n_I|T_I|\chi(\beta(I)) \prod_{i\notin I}(1-\overline{\chi}(\mathfrak{p}_i))\Big|.$$

Furthermore if $i_\beta = 0$, then the index of C_β in $\mathcal{O}_F^*$ is infinite.

Proof. Since $\ker l_F \cap \mathcal{O}_F^* = \ker l_F \cap C_\beta = \mathbb{F}_F^*$, by Lemmas 1.1 and 1.2, we have

$$\begin{aligned}(2.2)\quad [\mathcal{O}_F^* : C_\beta] &= [l_F(\mathcal{O}_F^*) : l_F(C_\beta)] = (l_F(\mathcal{O}_F^*) : e^+R_0)(e^+R_0 : l_F(C_\beta)) \\ &= \frac{\delta_F^{1-[F^+:k]}Q_0}{R(F^+)}\,(e^+R_0 : l_F(C_\beta)).\end{aligned}$$

Now we consider the transition matrix of the generators $\{l_F(\lambda(\beta)^{\sigma-1}) : \sigma \in \mathcal{R}^*\}$ of $l(C_\beta)$ with respect to the basis $\{e^+(\sigma^{-1}-1) : \sigma \in \mathcal{R}^*\}$ of e^+R_0. Since J_F is the inertia group of ∞ and $\lambda(\beta)^{\sigma-1}$ is a unit, we have

$$\begin{aligned}l_F(\lambda(\beta)^{\sigma-1}) &= \sum_{\tau\in G} v_\infty(\lambda(\beta)^{(\sigma-1)\tau})\tau^{-1} = \sum_{\tau\in\mathcal{R}^*} \delta_F v_\infty(\lambda(\beta)^{(\sigma-1)\tau})e^+(\tau^{-1}-1) \\ &= \sum_{\tau\in\mathcal{R}^*} \delta_F(v_\infty(\lambda(\beta)^{\sigma\tau}) - v_\infty(\lambda(\beta)^{\tau}))e^+(\tau^{-1}-1).\end{aligned}$$

From the Dedekind determinant formula (cf. [9, Lemma 5.26]), we get

$$(2.3)\qquad (e^+R_0 : l_F(C_\beta)) = |\det(\delta_F(v_\infty(\lambda(\beta)^{\sigma\tau}) - v_\infty(\lambda(\beta)^\tau)) : \sigma,\tau\in\mathcal{R}^*)|$$
$$= \Big|\prod_{\substack{\chi\neq 1\\ \text{real}}}\sum_{\sigma\in\mathcal{R}}\overline{\chi}(\sigma)\delta_F v_\infty(\lambda(\beta)^\sigma)\Big|$$
$$= \Big|\prod_{\substack{\chi\neq 1\\ \text{real}}}\sum_{I\in\mathbb{P}_S}\sum_{\sigma\in G}\overline{\chi}(\sigma)v_\infty(\lambda_I^{\sigma n_I\beta(I)})\Big|.$$

Fix $\chi\neq 1$ real and $I\in\mathbb{P}_S$. Since $\mathrm{Gal}(F/F_{\overline{\mathfrak{m}}/\overline{\mathfrak{m}}_I}) = T_I$, Lemma 2.1 yields

$$(2.4)\qquad \sum_{\sigma\in G}\overline{\chi}(\sigma)v_\infty(\lambda_I^{\sigma n_I\beta(I)}) = \chi(\beta(I))\chi(l_F(\lambda^{n_I}_{\overline{\mathfrak{m}}/\overline{\mathfrak{m}}_I,F}))$$
$$= \chi(\beta(I))n_I W_\infty L_k(0,\overline{\chi})\chi(s(T_I))\prod_{i\notin I}(1-\overline{\chi}(\mathfrak{p}_i)).$$

Note that if $\mathfrak{f}_\chi\nmid\overline{\mathfrak{m}}/\overline{\mathfrak{m}}_I$, then $\chi(s(T_I)) = 0$. Thus by combining (2.2)–(2.4), we get

$$(e^+R_0 : l_F(C_\beta))$$
$$= \Big|\prod_{\substack{\chi\neq 1\\ \text{real}}}W_\infty L_k(0,\overline{\chi})\Big|\cdot\Big|\prod_{\substack{\chi\neq 1\\ \text{real}}}\sum_{\substack{I\in\mathbb{P}_S\\ (\mathfrak{f}_\chi,\overline{\mathfrak{m}}_I)=\mathfrak{e}}} n_I|T_I|\chi(\beta(I))\prod_{i\notin I}(1-\overline{\chi}(\mathfrak{p}_i))\Big|$$
$$= W_\infty^r\frac{(q-1)h(F^+)}{W_F[\mathbb{F}_F:\mathbb{F}_q]h(k)}\,i_\beta,$$

where the second equality comes from the class number formula (1.1). Since $h(F^+) = R(F^+)h(\mathcal{O}_{F^+})/d_\infty$ and $h(k) = h(\mathbb{A})/d_\infty$, we have completed the proof of the theorem. ∎

A function $\beta:\mathbb{P}_S\to R$ is called *multiplicative* if $\beta(\emptyset)=1$ and $\beta(I\cup J) = \beta(I)\beta(J)$ whenever both sides are defined and the intersection $I\cap J$ is empty. Clearly, a multiplicative function β is determined by the values $\beta(\{i\})$ and these can be assigned arbitrarily. We denote $\beta(\{i\})$ by $\beta(i)$ for simplicity.

PROPOSITION 2.3. *For a multiplicative function $\beta:\mathbb{P}_S\to R$, we have*

$$i_\beta = \Big|\prod_{\substack{\chi\neq 1\ \mathit{real}\\ \mathfrak{f}_\chi\neq\mathfrak{e}}}\prod_{\mathfrak{p}_i\nmid\mathfrak{f}_\chi}(t_i\chi(\beta(i)) + 1 - \overline{\chi}(\mathfrak{p}_i))$$
$$\times\prod_{\substack{\chi\neq 1\ \mathit{real}\\ \mathfrak{f}_\chi=\mathfrak{e}}}\Big(\prod_{i=1}^{s+t}(t_i\chi(\beta(i)) + 1 - \overline{\chi}(\mathfrak{p}_i)) - \prod_{i=1}^{s+t}t_i\chi(\beta(i))\Big)\Big|.$$

Proof. For any nontrivial real character χ, we consider the factor

$$T_\chi = \sum_{\substack{I\in\mathbb{P}_S\\(\mathfrak{f}_\chi,\overline{\mathfrak{m}}_I)=\mathfrak{e}}} n_I|T_I|\chi(\beta(I))\prod_{i\notin I}(1-\overline{\chi}(\mathfrak{p}_i))$$

of i_β in Theorem 2.2. We also consider $U_\chi = \prod_{\mathfrak{p}_i\nmid\mathfrak{f}_\chi}(t_i\chi(\beta(i))+1-\overline{\chi}(\mathfrak{p}_i))$. Since β is multiplicative, $n_I|T_I| = \prod_{i\in I} t_i$ and

$$\prod_{i\notin I}(1-\overline{\chi}(\mathfrak{p}_i)) = \prod_{\substack{i\notin I\\ \mathfrak{p}_i\nmid\mathfrak{f}_\chi}}(1-\overline{\chi}(\mathfrak{p}_i)),$$

we have

$$T_\chi = \sum_{\substack{I\in\mathbb{P}_S\\(\mathfrak{f}_\chi,\overline{\mathfrak{m}}_I)=\mathfrak{e}}} \prod_{i\in I} t_i\chi(\beta(i)) \prod_{\substack{i\notin I\\ \mathfrak{p}_i\nmid\mathfrak{f}_\chi}}(1-\overline{\chi}(\mathfrak{p}_i)).$$

Let $S_\chi = \{i\in S : \mathfrak{p}_i\nmid\mathfrak{f}_\chi\}$. Then, if $\mathfrak{f}_\chi\neq\mathfrak{e}$, the I which occur in the summation for T_χ are exactly the subsets of S_χ. By expanding the product U_χ, we get $T_\chi = U_\chi$. If $\mathfrak{f}_\chi=\mathfrak{e}$, then S_χ becomes S and so $\prod_{i\in S} t_i\chi(\beta(i))$ occurs in the expansion of U_χ. Therefore, we have $T_\chi = U_\chi - \prod_{i=1}^{s+t} t_i\chi(\beta(i))$ in the case $\mathfrak{f}_\chi=\mathfrak{e}$. From this, the proposition follows. ■

Now we choose a multiplicative function $\beta : \mathbb{P}_S \to R$ with $\beta(i)=\nu_i$ for each $i\in S$. Since $\lambda_I \in F_{\overline{\mathfrak{m}}/\overline{\mathfrak{m}}_I}$ and $\beta(I)$ is uniquely determined modulo $T_I = \mathrm{Gal}(F/F_{\overline{\mathfrak{m}}/\overline{\mathfrak{m}}_I})$, C_β is independent of the choice of $\mathcal{F}_i$. Then as in the rational function field case [4, Theorem 4.1], we have the following result.

Proposition 2.4. *Let β be as above. Let $z_i = |(J_F\cap D_i)/(J_F\cap T_i)|$. Then*

$$i_\beta = \prod_{i=1}^{s+t} t_i^{[G:J_FD_i]-1} f_i^{2[G:J_FD_i]-1} z_i^{-[G:J_FD_i]}.$$

In particular, if F/k is real, then $i_\beta = \prod_{i=1}^{s+t} t_i^{g_i-1} f_i^{2g_i-1}$.

Proof. Any unramified nontrivial character χ may be viewed as a nontrivial character of $G_{F_\mathfrak{e}^+}$. Since $N_{\overline{\mathfrak{m}}} = G_{F_\mathfrak{e}^+}$, we have $\chi(\mathfrak{p}_i)\neq 1$ for some $i\in S$. Thus $\chi(\nu_i)=0$ for such $i\in S$ and so $\prod_{i=1}^{s+t}\chi(\beta(i))=0$. We follow the argument in the proof of [4, Theorem 4.1] to get the result. ■

Suppose F/k is a real extension. For any divisor $\mathfrak{n}$ of $\overline{\mathfrak{m}}$, let $\overline{K}_\mathfrak{n}^+ = K_\mathfrak{n}^+\cdot K_\mathfrak{e}$, the compositum of $K_\mathfrak{n}^+$ and $K_\mathfrak{e}$. Then $\lambda_\mathfrak{n}^{\sigma-1}\in\overline{K}_\mathfrak{n}^+$ as in [12, Section 2] and

$$\begin{aligned} N_{K_\mathfrak{n}/F_\mathfrak{n}}(\lambda_\mathfrak{n})^{\sigma-1} &= N_{\overline{K}_\mathfrak{n}^+/F_\mathfrak{n}}(N_{K_\mathfrak{n}/\overline{K}_\mathfrak{n}^+}(\lambda^{\sigma-1}))\\ &= N_{\overline{K}_\mathfrak{n}^+/F_\mathfrak{n}}((\lambda_\mathfrak{n}^{\sigma-1})^{q-1}) = (N_{\overline{K}_\mathfrak{n}^+/F_\mathfrak{n}}(\lambda_\mathfrak{n}^{\sigma-1}))^{q-1}. \end{aligned}$$

Thus for $\sigma\in G$, there exists $\varepsilon_\sigma\in\mathcal{O}_F^*$ such that $\varepsilon_\sigma^{q-1} = \lambda(\beta)^{\sigma-1}$ and one can construct ε_σ explicitly from the above relation and the definition of $\lambda(\beta)$. We define C'_β as the subgroup of $\mathcal{O}_F^*$ generated by $\mathbb{F}_F^* \cup \{\varepsilon_\sigma : \sigma \in G, \sigma \neq 1\}$. Then it is easy to see that both C_β and C'_β are isomorphic to the augmentation ideal R_0 of R as R-module.

COROLLARY 2.5. *When F/k is a real extension, we have*

$$[\mathcal{O}_F^* : C'_\beta] = \frac{q-1}{W_F[\mathbb{F}_F : \mathbb{F}_q]} \left(\frac{W_\infty}{q-1}\right)^{[F:k]-1} \frac{h(\mathcal{O}_F)}{h(\mathbb{A})} \prod_{i=1}^{s+t} t_i^{g_i-1} f_i^{2g_i-1}.$$

3. Comparison of indices. To compare our index with Xu–Zhao's [11, Theorem 1], we change the notations in [11] to ours. Then the index in [11, Theorem 1] reads

$$[\mathcal{O}_{F^+}^* : \mathfrak{C}(\mathfrak{n}, \mathcal{D})] = \frac{q-1}{W_F} \left(\frac{W_\infty}{\delta_F}\right)^r \frac{h(\mathcal{O}_{F^+})}{h(\mathbb{A})} i(\mathfrak{n}, \mathcal{D}),$$

with $\mathfrak{m} \mid \mathfrak{n}$. We also note that the constant field of k in [11] is enlarged so that $[\mathbb{F}_F : \mathbb{F}_q] = 1$ in the Xu–Zhao's index. Thus it suffices to compare i_β in our index with $i(\mathfrak{n}, \mathcal{D})$. Note that our choice of $\overline{\mathfrak{m}}$ satisfies the condition $i(\overline{\mathfrak{m}}, \mathcal{D}) \neq 0$ in [11, Theorem 2]. Define $T_0 = \{i \in S : \chi(\mathfrak{p}_i) = 1$ for some nontrivial $\chi \in \widehat{G}_{F^+}\}$. For $T_0 \subseteq T \subseteq S$, we let $\mathcal{D} = \mathcal{D}(T) = \{\overline{\mathfrak{m}}/\overline{\mathfrak{m}}_I \neq \mathfrak{e} : I \subseteq T\}$. For any integral ideal $\mathfrak{a}$ of $\mathbb{A}$, let $\Phi(\mathfrak{a}) = |(\mathbb{A}/\mathfrak{a})^*|$. Then $i(\overline{\mathfrak{m}}, \mathcal{D})$ in [11, Theorem 1] can be written as

$$\begin{aligned} i(\overline{\mathfrak{m}}, \mathcal{D}) &= \Big| \prod_{\substack{\chi \neq 1 \\ \text{real}}} \sum_{\substack{I \subseteq T, I \neq S \\ (\overline{\mathfrak{m}}_I, \mathfrak{f}_\chi) = \mathfrak{e}}} \Phi(\overline{\mathfrak{m}}_I) \prod_{i \notin I} (1 - \chi(\mathfrak{p}_i)) \Big| \\ &= \Big| \prod_{\substack{\chi \neq 1 \text{ real} \\ \mathfrak{f}_\chi \neq \mathfrak{e}}} \prod_{\substack{i \in T \\ \mathfrak{p}_i \nmid \mathfrak{f}_\chi}} (\Phi(\mathfrak{p}_i^{e_i}) + 1 - \chi(\mathfrak{p}_i)) \prod_{i \notin T} (1 - \chi(\mathfrak{p}_i)) \Big| \\ &\quad \times \Big| \prod_{\substack{\chi \neq 1 \text{ real} \\ \mathfrak{f}_\chi = \mathfrak{e}}} \Big(\prod_{i \in T} (\Phi(\mathfrak{p}_i^{e_i}) + 1 - \chi(\mathfrak{p}_i)) - \delta_{T,S} \Phi(\overline{\mathfrak{m}}) \Big) \prod_{i \notin T} (1 - \chi(\mathfrak{p}_i)) \Big| \end{aligned}$$

where $\delta_{T,S} = 1$ if $T = S$ and 0 otherwise. If $T = S$, the above index is difficult to compute. Thus we assume that $T \subsetneq S$. For simplicity, we also assume that F is real. Note that T_0 is just the set of all $i \in I$ with $g_i > 1$ (cf. [9, Theorem 3.7]). Thus for $i \notin T$, both factors in i_β and $i(\overline{\mathfrak{m}}, \mathcal{D})$ are equal to f_i. For $i \in T$, as in the proof of Proposition 2.4, we see that

$$\prod_{\chi \neq 1, \mathfrak{p}_i \nmid \mathfrak{f}_\chi} (\Phi(\mathfrak{p}_i^{e_i}) + 1 - \chi(\mathfrak{p}_i)) = \frac{((\Phi(\mathfrak{p}_i^{e_i}) + 1)^{f_i} - 1)^{g_i}}{\Phi(\mathfrak{p}_i^{e_i})}.$$

Note that $t_i \leq \Phi(\mathfrak{p}_i^{e_i})$ and $t_i = 1$ for $i > s$. It is easy to see that this factor is greater than our factor $t_i^{g_i-1} f_i^{2g_i-1}$ as in the rational function field case.

So we have $i_\beta \le i(\overline{\mathfrak{m}}, \mathcal{D}(T))$. We also note that both i_β and $i(\overline{\mathfrak{m}}, \mathcal{D})$ depend on the choice of $\overline{\mathfrak{m}}$.

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Department of Mathematics
KAIST
Daejon, South Korea 305-701
E-mail: jaehyun@mathx.kaist.ac.kr
hyjung@mathx.kaist.ac.kr

Current address of H. Jung:
Department of Mathematical Education
Chungbuk National University
Cheongju, Chungbuk, South Korea 361-763

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