ACTA ARITHMETICA
108.3 (2003)

On the Waring–Goldbach problem with small non-integer exponent

by

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1. Introduction. I. I. Piatetski-Shapiro [6] considered the following variant of the Waring–Goldbach problem. Let $c > 1$ be non-integer and denote by $H(c)$ the least k such that the inequality

$$|p_1^c + p_2^c + \ldots + p_k^c - N| < \varepsilon$$

has a solution in prime numbers $p_1, p_2, \ldots, p_k$ for every $\varepsilon > 0$ and $N > N_0(c, \varepsilon)$. It is proved in [6] that

$$\limsup_{c\to\infty} \frac{H(c)}{c\log c} \le 4.$$

Piatetski-Shapiro also proved that if $1 < c < 3/2$ then $H(c) \le 5$.

In this paper we sharpen the last result and prove the following theorem:

Theorem. *If* $1 < c \le (\sqrt{5}+1)/2$ *then* $H(c) \le 5$.

It should be pointed out that if c is near to unity then $H(c) \le 3$. More precisely D. I. Tolev [7] showed that if $1 < c < 15/14$ then $H(c) \le 3$. Afterwards several authors sharpened Tolev's result improving on the range for c (see [1], [3], [5]). The best improvement to date is due to A. Kumchev [3]; he proved that $H(c) \le 3$ for $1 < c < 61/55$. Note that from the result of A. Kumchev and M. B. S. Laporta [4] it follows that $H(c) \le 4$ for $1 < c < 6/5$ and for almost all $c \in (1,2)$ (in the sense of Lebesgue measure). We also refer the readers to [8].

2. Auxiliary lemmas. Set $A = \{n \in \mathbb{Z} : P^{c-1}/2 \le n \le P^{c-1}\}$, $B = \{n \in \mathbb{Z} : P/10 \le n \le P/5\}$.

Lemma 1. *Let* $1 < c \le (\sqrt{5}+1)/2$. *Then for any real* N_1 *the number of solutions of the inequality*

2000 *Mathematics Subject Classification*: 11P05, 11P32.

$$(1) \qquad |x^c + y^c - u^c - v^c - N_1| < 1, \qquad x, u \in A,\ y, v \in B,$$

is $O(P^c \log P)$.

Proof. For any given integer k the number I_k of solutions of the equation

$$[x^c] + [y^c] = [u^c] + [v^c] + k, \qquad x, u \in A,\ y, v \in B,$$

satisfies

$$I_k = \int_0^1 \Big| \sum_{x \in A} \sum_{y \in B} e^{2\pi i \alpha([x^c]+[y^c])} \Big|^2 e^{-2\pi i \alpha k}\, d\alpha.$$

Hence $I_k \le I_0$. Furthermore, for any solution x, y, u, v of (1), one has

$$[x^c] + [y^c] = [u^c] + [v^c] + N_1 + 4\theta,$$

where $|\theta| \le 1$, and $N_1 + 4\theta$ is an integer. Therefore, if we prove that the number J of solutions of the equation

$$[x^c] + [y^c] = [u^c] + [v^c], \qquad x, u \in A,\ y, v \in B,$$

is $O(P^c \log P)$ then we are done. To do that we follow our work [2].

Obviously

$$(2) \qquad J < P^c + 2 \sum_{1 \le l < P^{c-1}} J_l,$$

where J_l denotes the number of solutions of the equation

$$[x^c] + [y^c] = [(x+l)^c] + [z^c], \qquad x, x+l \in A,\ y, z \in B.$$

In order to estimate J_l we fix $x = x_0 = x_0(l)$ such that $J_l < P^{c-1} J_l'$ where J_l' denotes the number of solutions of the equation

$$[x_0^c] + [y^c] = [(x_0+l)^c] + [z^c], \qquad y, z \in B,$$

in variables y, z. It then follows that

$$y^c - z^c < (x_0 + l)^c - x_0^c + 2.$$

Since $x_0 + l \in A, z \in B$, we have

$$0 < (y-z)P^{c-1} < c_1 l P^{(c-1)^2}, \quad \text{i.e.} \quad 0 < y - z < c_1 l P^{c^2-3c+2}.$$

Therefore

$$(3) \qquad J_l < P^{c-1} \sum_{m < c_1 l P^{c^2-3c+2}} J_l'(m)$$

where $J_l'(m)$ denotes the number of solutions in z of the equation

$$(4) \qquad [(z+m)^c] - [z^c] = a, \qquad z, z+m \in B,$$

where $a = a(m, P)$ is some fixed integer.

Suppose that z_0 is the smallest solution of (4). Then for any other solution z of (4),

$$(z+m)^c - z^c < (z_0+m)^c - z_0^c + 4.$$

This inequality can be written as

$$c(c-1)\int\limits_{z_0}^{z}\int\limits_{0}^{m}(\phi+\psi)^{c-2}\,d\phi\,d\psi<4.$$

Hence

$$0\le c_2(z-z_0)mP^{c-2}<4,\quad \text{i.e.}\quad J_l'(m)\le c_3(1+P^{2-c}m^{-1}).$$

In view of (3) we obtain

$$J_l\ll_c lP^{c^2-2c+1}+P\log P$$

and therefore in view of (2),

$$J\ll_c P^{c^2-1}+P^c\log P.$$

Since $c\le(\sqrt{5}+1)/2$, the result follows.

For the proof of the following three lemmas see [6].

Lemma 2. *Let s be a positive integer*,

$$\delta>0,\qquad \varDelta>s\delta,\qquad \varPhi_s(\xi)=\frac{\sin(\varDelta\xi)\sin^s(\delta\xi)}{\delta^s\xi^{s+1}}$$

and

$$\phi_s(y)=\frac{1}{\pi}\int\limits_{-\infty}^{\infty}\varPhi_s(\xi)e^{i\xi y}\,d\xi.$$

Then

$$\phi_s(y)=\begin{cases}0 & \text{if } |y|\ge\varDelta+s\delta,\\ 1 & \text{if } |y|\le\varDelta-s\delta,\end{cases}$$

and also

$$0<\phi_s<1\qquad \text{if } \varDelta-s\delta<|y|<\varDelta+s\delta.$$

Furthermore, *if $k\ge 0$ is an integer and $s\ge k+2$, then*

$$g(y)=\int\limits_{-\infty}^{\infty}|\varPhi_s(\xi)\xi^k|e^{-i\xi y}\,d\xi=O(y^{-2})\qquad \text{as } y\to\infty.$$

Define

$$S(\xi)=\sum_{P/2<p\le 2P}e^{i\xi p^c}.$$

Lemma 3. *Let $H=e^{\sqrt{\log P}}$, $P^c/2<N_1<P^c$. Then under the assumptions of Lemma 2 we have*

$$\frac{1}{\pi}\int\limits_{|\xi|<HP^{-c}}\varPhi_s(\xi)S(\xi)e^{-i\xi N_1}\,d\xi\gg\frac{P^{1-c}}{\log P}.$$

LEMMA 4. *For any fixed m,*

$$\max_{HP^{-c}<\xi<\log P} |S(\xi)| \ll \frac{P}{\log^m P}.$$

3. Proof of the Theorem. The proof is similar to one in [6]. Let $N > N_0(c,\varepsilon)$ be a large enough real number, $P = N^{1/c}$. We shall prove that the inequality

$$|p^c + p_1^c + p_2^c + p_3^c + p_4^c - N| < \varepsilon \tag{5}$$

is solvable in prime numbers subject to

$$P/2 < p < 2P, \qquad p_1, p_3 \in A,\ p_2, p_4 \in B.$$

Set $\Delta = \varepsilon/2$, $\delta = \varepsilon/50$, $s = 20$ and $\Phi(\xi) = \Phi_{20}(\xi)$. From Lemma 2 it follows that the number J of solutions of inequality (5) satisfies

$$J \ge \frac{1}{\pi} \int_{-\infty}^{\infty} \Phi(\xi) S_A^2(\xi) S_B^2(\xi) S(\xi) e^{-i\xi N}\, d\xi$$

where

$$S_A(\xi) = \sum_{p\in A} e^{i\xi p^c}, \qquad S_B(\xi) = \sum_{p\in B} e^{i\xi p^c}.$$

Therefore

$$\pi J > J_1 + J_2 + J_3 \tag{6}$$

where

$$J_1 = \int_{|\xi|<HP^{-c}} \Phi(\xi) S_A^2(\xi) S_B^2(\xi) S(\xi) e^{-i\xi N}\, d\xi,$$

$$J_2 = \int_{HP^{-c}<|\xi|<\log P} \Phi(\xi) S_A^2(\xi) S_B^2(\xi) S(\xi) e^{-i\xi N}\, d\xi,$$

$$J_3 = \int_{|\xi|>\log P} \Phi(\xi) S_A^2(\xi) S_B^2(\xi) S(\xi) e^{-i\xi N}\, d\xi.$$

From now on, all constants implicit in the Vinogradov symbols $\ll$ and $\gg$ may depend on c and ε.

We estimate J_1 from below using Lemma 3. If $N_1 = N - p_1^c - \ldots - p_4^c$ then $P^c/2 < N_1 < P^c$. Hence

$$J_1 = \sum_{p_1,p_3\in A} \sum_{p_2,p_4\in B} \int_{|\xi|<HP^{-c}} \Phi(\xi) S(\xi) e^{-i\xi N_1}\, d\xi \gg \left(\frac{P^{c-1}}{\log P}\right)^2 \left(\frac{P}{\log P}\right)^2 \frac{P^{1-c}}{\log P},$$

i.e.

$$J_1 \gg \frac{P^{c+1}}{\log^5 P}. \tag{7}$$

Now we shall obtain upper bounds for J_2, J_3. Since

$$\sum_{p_1,p_3\in A}\sum_{p_2,p_4\in B} e^{i\xi(p_1^c+p_2^c-p_3^c-p_4^c)}$$

is a non-negative real number for any real ξ, Lemma 4 (with $m=10$) yields

$$J_2 \ll \frac{P}{\log^{10} P}\sum_{p_1,p_3\in A}\sum_{p_2,p_4\in B}\int_{-\infty}^{\infty}|\Phi(\xi)|e^{i\xi(p_1^c+p_2^c-p_3^c-p_4^c)}\,d\xi.$$

Now we make use of Lemma 1. For any integer j there exists $t_j\in(j-1,j+1)$ such that

$$J_2 \ll \frac{P}{\log^{10} P}P^c\log P\sum_j\Big|\int_{-\infty}^{\infty}|\Phi(\xi)|e^{i\xi t_j}\,d\xi\Big|.$$

Therefore from the second part of Lemma 2 we derive

$$J_2 \ll \frac{P^{c+1}}{\log^9 P}. \tag{8}$$

For J_3 we use the trivial estimation of $S(\xi)$:

$$\begin{aligned} J_3 &\le P\int_{|\xi|>\log P}|\Phi(\xi)S_A^2(\xi)S_B^2(\xi)|\,d\xi \\ &\le \frac{P}{\log^{10} P}\int_{|\xi|>\log P}|\xi^{10}\Phi(\xi)S_A^2(\xi)S_B^2(\xi)|\,d\xi, \end{aligned}$$

i.e.

$$J_3 \le \frac{P}{\log^{10} P}\sum_{p_1,p_3\in A}\sum_{p_2,p_4\in B}\int_{-\infty}^{\infty}|\xi^{10}\Phi(\xi)|e^{i\xi(p_1^c+p_2^c-p_3^c-p_4^c)}\,d\xi.$$

Now as in the case of J_2 we obtain

$$J_3 \ll \frac{P^{c+1}}{\log^9 P}. \tag{9}$$

Thus from (6)–(9) we obtain $J \gg P^{c+1}/\log^5 P$.

The Theorem is proved.

Acknowledgements. The author would like to thank the referee for several corrections.

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Received on 13.3.2002
and in revised form on 29.10.2002 (4247)