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On a Brun–Titchmarsh inequality for multiplicative functions

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0. Introduction. We investigate the distribution of non-negative multiplicative functions on arithmetic progressions. More precisely, we are motivated by the following problem. When does the inequality

$$\sum_{\substack{n\le x\\ n\equiv a\,(\mathrm{mod}\,q)}} f(n) \ll \frac{1}{\varphi(q)} \sum_{\substack{n\le x\\ (n,q)=1}} f(n)$$

hold, where f is a non-negative multiplicative function and φ is Euler's totient function? Note that such an inequality is an analogue of the Brun–Titchmarsh inequality in the theory of the distribution of prime numbers. The following conjecture appears to have been first formulated in [Ba1]. Here and throughout the paper we use the letter p to denote a prime.

CONJECTURE. *Fix ε, $0<\varepsilon\le 1/2$. Let $x\ge 3$ and a and q be integers* (*not necessarily coprime*) *with $1\le q\le x^{1-\varepsilon}$. Then*

$$\sum_{\substack{n\le x\\ n\equiv a\,(\mathrm{mod}\,q)}} f(n) \ll_\varepsilon \frac{1}{\varphi(q)} \sum_{\substack{n\le x\\ (n,q)=1}} f(n), \tag{0.1}$$

uniformly for all multiplicative functions f satisfying $0\le f\le 1$ and

$$\sum_{p\le x}\frac{f(p)}{p} \ge \varepsilon\log\log x. \tag{0.2}$$

Our new results represent only a modest step in this direction but they do provide some evidence in support of this conjecture. In particular, we obtain a logarithmic form of (0.1) (Theorem 1) and show that, in a certain sense, the conjecture is true on average (Theorem 2). Furthermore, there is at least one interesting application (please see below) where our results can be used successfully in place of (0.1).

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Before stating our new results we give a brief discussion of what is known on this topic (we refer the reader to [Ba1] for a more thorough discussion). Asymptotic results of the form

$$\sum_{\substack{n\le x\\ n\equiv a\,(\mathrm{mod}\,q)}} g(n) \sim \frac{1}{\varphi(q)} \sum_{\substack{n\le x\\ (n,q)=1}} g(n),$$

where $(a,q)=1$ and g is a complex-valued multiplicative function satisfying $|g|\le 1$, have been obtained by A. Hildebrand [Hi] and in a series of papers by P. D. T. A. Elliott. The best such result is due to Elliott and is found in the latest paper of this series [El]. As one might expect, this is a very difficult problem and such asymptotic formulae are known to hold in rather limited ranges. For the sake of brevity we refrain from giving the exact statement of Elliott's result and simply mention that it does not imply (0.1) unless the function f and modulus q satisfy

$$\sum_{n\le x} f(n) \gg x\left(\frac{\log\log x+\log q}{\log x}\right)^{1/8}.$$

Given a non-negative multiplicative function f set

$$S(x)=S(f;x)=\sum_{p\le x}\frac{f(p)}{p}, \tag{0.3}$$

$$S_q(x)=\sum_{\substack{p\le x\\ (p,q)=1}}\frac{f(p)}{p}. \tag{0.4}$$

Furthermore, let $\mathcal{F}$ denote the class of multiplicative functions f satisfying $0\le f\le 1$. The best-known result in the direction of (0.1) is due to P. Shiu [Sh], a special case of which is the following estimate (see Lemma 2 for the full strength of what was proved in [Sh]). Given ε, $0<\varepsilon\le 1/2$, we have

$$\sum_{\substack{x-y<n\le x\\ n\equiv a\,(\mathrm{mod}\,q)}} f(n) \ll_\varepsilon \frac{y}{\varphi(q)\log x}\, e^{S_q(x)}, \tag{0.5}$$

uniformly for all $f\in\mathcal{F}$ and all $x\ge 3$, y, a and q satisfying

$$(a,q)=1,\qquad 1\le q\le y^{1-\varepsilon},\qquad x^\varepsilon\le y\le x.$$

This estimate is sharp in the sense that there are functions $f\in\mathcal{F}$ for which (0.5) holds with $\ll$ replaced by $\gg$. Furthermore, (0.5) implies (0.1) for those functions $f\in\mathcal{F}$ which satisfy

$$\sum_{\substack{n\le x\\ (n,q)=1}} f(n) \asymp \frac{x}{\log x}\, e^{S_q(x)}, \tag{0.6}$$

in view of an estimate of R. R. Hall [Ha] which shows that (0.6) with $\asymp$ replaced by $\ll$ is always true.

It is worth emphasizing that while (0.5) holds uniformly for all $f \in \mathcal{F}$, the assumption (0.2) is certainly necessary for the validity of (0.1). To see this, take f to be a completely multiplicative function whose value on primes is given by $f(p) = 1$ if $p \equiv 1 \pmod q$, and 0 otherwise, and take $a = 1$ on the left-hand side of (0.1). We further remark that finding a useful general criterion which implies (0.6) is itself an interesting problem. Assuming that f is supported on a positive proportion of primes in the sense of (0.2) is certainly not sufficient. To see this take $f = f_y$ to be a characteristic function of y-smooth numbers, i.e., a completely multiplicative function with values on primes given by $f(p) = 1$ or 0, as $p \le y$ or $> y$. In fact, we believe that this function is a good test function for our conjecture. It is of note that even though this function has been extensively investigated (please see the excellent survey [HT]), it is not even known if the claim of the conjecture is true with this choice of the function f (see discussion in [Ba1], and [Gr] for the best-available estimate in this special case).

1. Statement of results. In view of the somewhat technical nature of our main result Theorem 0, we begin with some of its applications. A logarithmic form of (0.1) was obtained by the author in [Ba1]. A special case of that result gives the bound

$$\sum_{\substack{n\le x\\ n\equiv a\,(\mathrm{mod}\,q)}} \frac{f(n)}{n} \ll_\varepsilon \frac{1}{\varphi(q)} \sum_{\substack{n\le x\\ (n,q)=1}} \frac{f(n)}{n}, \tag{1.1}$$

uniformly for all $f \in \mathcal{F}$ satisfying (0.2) and $(a, q) = 1$ with

$$q \le \sum_{n\le x} \frac{f(n)}{n}. \tag{1.2}$$

This result is best possible since (1.1) may not hold unless both (0.2) and (1.2) are satisfied. The example of the previous section we used to show that (0.1) may fail if (0.2) is not satisfied also shows that (1.1) may fail in this case. The severe restriction on the size of q (1.2) is seen to be necessary simply because otherwise the first term on the left-hand side of (1.1) could already be larger than the entire right-hand side of (1.1). Thus for large q one should consider instead a more interesting problem of obtaining bounds of the form

$$\sum_{\substack{x_0\le n\le x\\ n\equiv a\,(\mathrm{mod}\,q)}} \frac{f(n)}{n} \ll \frac{1}{\varphi(q)} \sum_{\substack{n\le x\\ (n,q)=1}} \frac{f(n)}{n}, \tag{1.3}$$

with $x_0 \ge q$. Our first result gives such an estimate with $x_0 = q^{1+\varepsilon}$. In fact, this and the following results are applicable to a larger class of functions than the class $\mathcal{F}$, a class which we now define. For any real numbers $A \ge 1$ and $\gamma \ge 0$, let $\mathcal{F}_{A,\gamma}$ be the class of non-negative multiplicative functions f satisfying

$$\begin{cases} f(p^\nu) \le A^\nu & \text{for all primes } p \text{ and } \nu = 1, 2, \dots, \\ f(n) \le A n^\gamma & \text{for all natural numbers } n. \end{cases} \tag{1.4}$$

THEOREM 1. *Fix* ε, $0 < \varepsilon \le 1/2$. *Then*

$$\sum_{\substack{q^{1+\varepsilon} \le n \le x \\ n \equiv a \,(\mathrm{mod}\, q)}} \frac{f(n)}{n} \ll_\varepsilon \frac{1}{\varphi(q)} \sum_{\substack{n \le x \\ (n,q)=1}} \frac{f(n)}{n},$$

uniformly for all $x \ge q^{1+\varepsilon}$, *arbitrary* a, *and all functions* $f \in \mathcal{F}$ *satisfying*

$$S(x) - S(q) = \sum_{q < p \le x} \frac{f(p)}{p} \ge \varepsilon \sum_{q < p \le x} \frac{1}{p}. \tag{1.5}$$

Furthermore, if the residue a *is assumed to be coprime with* q, *then there exists a constant* $\gamma_\varepsilon > 0$, *depending only on* ε, *such that the same inequality holds, with* $\ll_\varepsilon$ *replaced by* $\ll_{A,\varepsilon}$, *uniformly for all* $f \in \mathcal{F}_{A,\gamma_\varepsilon}$.

Our second result shows that (0.1) is true on average. To this end let us introduce the following notation. Given a function f and a modulus q we set

$$M(y) = \frac{1}{y} \sum_{\substack{n \le y \\ (n,q)=1}} f(n), \tag{1.6}$$

$$M_q(y) = \max_{a \,(\mathrm{mod}\, q)} \frac{1}{y} \sum_{\substack{n \le y \\ n \equiv a \,(\mathrm{mod}\, q)}} f(n). \tag{1.7}$$

According to the conjecture we expect that if $y \ge q^{1+\varepsilon}$ and if f is supported on the positive proportion of primes up to y, then $M_q(y) \ll M(y)/\varphi(q)$. We will prove

THEOREM 2. *Fix* ε, $0 < \varepsilon \le 1/2$. *Then*

$$\sum_{q^{1+\varepsilon} \le 2^i \le x} M_q(2^i) \ll_\varepsilon \frac{1}{\varphi(q)} \sum_{2^i \le x} M(2^i), \tag{1.8}$$

uniformly for all $x \ge q^{1+\varepsilon}$ *and all functions* $f \in \mathcal{F}$ *satisfying* (1.5). *Furthermore, if we restrict the definition of* $M_q(y)$ *by taking the maximum in* (1.7) *only over* $(a, q) = 1$, *then there exists a constant* $\gamma_\varepsilon > 0$, *depending only on* ε, *such that the same inequality holds, with* $\ll_\varepsilon$ *replaced by* $\ll_{A,\varepsilon}$, *uniformly for all* $f \in \mathcal{F}_{A,\gamma_\varepsilon}$.

Theorems 1 and 2 provide some evidence in support of the conjecture, although they obviously fall well short of the mark. Even the logarithmic form of (0.1) is not entirely settled. In particular, it seems that the estimate

$$\sum_{\substack{q<n\le q^2\\ n\equiv a\,(\mathrm{mod}\,q)}} \frac{f(n)}{n} \ll \frac{1}{\varphi(q)} \sum_{\substack{n\le q^2\\ (n,q)=1}} \frac{f(n)}{n}$$

ought to be true for those functions $f\in\mathcal{F}$ satisfying $S(q)\gg\log\log q$, but this remains an open problem.

We now turn to our main result of which the first two theorems are simple corollaries.

Theorem 0. *Let an arbitrary natural number q and a triple of real numbers $\tau\ge 3$, $b>1$ and $\varepsilon>0$ be given. Then for any two sequences of real numbers x_i and y_i, $1\le i\le\kappa$, satisfying the conditions*

$$y_1\ge(q\tau)^{1+\varepsilon},\qquad y_{i+1}\ge by_i,\qquad \text{and}\qquad y_i\le x_i\le\tau y_i, \tag{1.9}$$

and an arbitrary sequence of integers a_i, we have

$$\begin{aligned}\sum_{1\le i\le\kappa}\frac{1}{y_i}\sum_{\substack{x_i-y_i<n\le x_i\\ n\equiv a_i\,(\mathrm{mod}\,q)}} f(n)\\ \ll_{\varepsilon,b}\min\Bigl(1,\sum_{1\le i\le\kappa}\frac{1}{\log x_i}\Bigr)\frac{1}{\varphi(q)}\Bigl(e^{S_q(x_1)}\Bigl(\frac{\log x_\kappa}{\log x_1}\Bigr)^{\varepsilon}+e^{S_q(x_\kappa)}\Bigr),\end{aligned} \tag{1.10}$$

uniformly for all $f\in\mathcal{F}$. Furthermore, if the sequence a_i is assumed to be coprime with q, then there exists a constant $\gamma_\varepsilon>0$, depending only on ε, such that the same inequality holds, with $\ll_{\varepsilon,b}$ replaced by $\ll_{A,\varepsilon,b}$, uniformly for all $f\in\mathcal{F}_{A,\gamma_\varepsilon}$.

For the subclasses of $\mathcal{F}$ and $\mathcal{F}_{A,\gamma}$ consisting of functions f supported on the positive proportion of primes in the interval $(x_1,x_\kappa]$ we immediately obtain the following corollary. Here and below we use $\log_2$ as an abbreviation for $\log\log$.

Corollary 3. *If all the hypotheses of Theorem 0 hold, then*

$$\sum_{1\le i\le\kappa}\frac{1}{y_i}\sum_{\substack{x_i-y_i<n\le x_i\\ n\equiv a_i\,(\mathrm{mod}\,q)}} f(n)\ll_{\varepsilon,b}\min\Bigl(1,\sum_{1\le i\le\kappa}\frac{1}{\log x_i}\Bigr)\frac{1}{\varphi(q)}\,e^{S_q(x_\kappa)}, \tag{1.11}$$

uniformly for all the functions $f\in\mathcal{F}$ satisfying

$$\begin{aligned}S(x_\kappa)-S(x_1)&=S_q(x_\kappa)-S_q(x_1)\\ &=\sum_{x_1<p\le x_\kappa}\frac{f(p)}{p}\ge\varepsilon(\log_2 x_\kappa-\log_2 x_1).\end{aligned} \tag{1.12}$$

Furthermore, if the sequence a_i is assumed to be coprime with q, then there exists a constant $\gamma_\varepsilon > 0$, depending only on ε, such that the same inequality holds, with $\ll_{\varepsilon,b}$ replaced by $\ll_{A,\varepsilon,b}$, uniformly for all $f \in \mathcal{F}_{A,\gamma_\varepsilon}$.

Next we make several remarks focusing on the cleaner estimate of the corollary. Basically, Corollary 3 is a more general form of Theorem 2 showing that the conjecture is true on average. In fact, Theorem 2 is equivalent to Corollary 3 with $b = 2$ and $y_i = x_i = 2^{i_0+i}$, $1 \le i \le \kappa$, where $i_0 = \lfloor \log q^{1+\varepsilon}/\log 2 \rfloor$ and $\kappa \gg \log q$, as is readily apparent from the proof of that theorem. Since it is obvious that there are examples for which (1.8) holds with $\ll$ replaced by $\gg$, it is then immediate that the same is true for (1.11). In fact, one can give such examples with functions f satisfying $S(x) = \lambda \log_2 x + O(1)$, for any $0 < \lambda \le 1$.

To note some quantitative aspects of (1.11) let us now assume, for simplicity, that $f \in \mathcal{F}$. Then estimating sums over n on the left-hand side of (1.11) by y_i/q yields the trivial bound κ/q. Furthermore, by (1.9), each sum over n on the left-hand side of (1.11) can be estimated by Shiu's bound (0.5) yielding

$$(1.13)\qquad \sum_{1\le i\le\kappa} \frac{1}{y_i} \sum_{\substack{x_i - y_i < n \le x_i \\ n \equiv a_i \,(\mathrm{mod}\, q)}} f(n) \ll_\varepsilon \frac{1}{\varphi(q)} \sum_{1\le i\le\kappa} \frac{1}{\log x_i}\, e^{S_q(x_i)}$$
$$\ll \frac{1}{\varphi(q)}\, e^{S_q(x_\kappa)} \sum_{1\le i\le\kappa} \frac{1}{\log x_i},$$

by (0.4) and (1.9). Thus, if $\log x_\kappa \ll \log x_1$, then we obtain in this way the bound

$$(1.14)\qquad \sum_{1\le i\le\kappa} \frac{1}{y_i} \sum_{\substack{x_i - y_i < n \le x_i \\ n \equiv a_i \,(\mathrm{mod}\, q)}} f(n) \ll_\varepsilon \frac{1}{\varphi(q)} \cdot \frac{\kappa}{\log x_\kappa}\, e^{S_q(x_\kappa)}$$
$$\ll \frac{\kappa}{q} \exp\Big(- \sum_{\substack{p \le x_\kappa \\ (p,q)=1}} \frac{1 - f(p)}{p} \Big),$$

by (0.4) and Mertens's formula. In this form the savings over the trivial bound are immediately apparent and in general this bound is optimal.

The condition $\log x_\kappa \ll \log x_1$ translates into the sum over i being "short" and "dense". For longer sums (1.13) gives only the estimate which is by a factor $(\log_2 x_\kappa - \log_2 x_1)$ worse than (1.11). In view of the fact that $S_q(x_\kappa) \le \log_2 x_\kappa + O(1)$, this loss, which could be of size $\log_2 x_\kappa$, is quite substantial. An obvious strategy for additional savings is to take advantage of the fact that for typical $i < j$ we have $S_q(x_i) < S_q(x_j)$. This, however, appears to be a non-trivial matter and the argument of the present paper exploiting this fact is somewhat lengthy.

If $\kappa \asymp_b \log x_\kappa$, i.e., if summation over i is "long" and "dense", then (1.11) is equivalent to (1.14). In this case too our estimate is in general optimal, as we already observed by relating it to Theorem 2.

Thus we have seen that if the ratios y_{i+1}/y_i or, equivalently, x_{i+1}/x_i are, on average, not too large, i.e., the sum over i is "dense" (short or long), then our estimate is equivalent to (1.14).

It is then natural to ask if (1.14) holds in general. The answer is no. For example, let us take $\kappa = 2$ and $\log x_2 = (\log x_1)^2$. One can then readily construct examples of functions f for which (1.13) holds with $\ll$ replaced by $\asymp$. It now follows that, in this case, (1.14) fails. This construction easily generalizes to arbitrary κ and sufficiently sparse sequences x_i.

Finally, we note that aside from providing evidence in support of the conjecture these results can be used in place of it for the purpose of estimating certain exponential sums. More precisely, set

$$G(x,\alpha) = \sum_{n\le x} g(n)e^{2\pi i\alpha n},$$

where α is real and g is a complex-valued multiplicative function.

The problem of providing estimates for $G(x,\alpha)$ valid uniformly for large classes of functions g, e.g., $|g| \le 1$, has been initiated by H. Daboussi and H. Delange [Da], [DD1] and [DD2]. The best-known result here is due to H. L. Montgomery and R. C. Vaughan [MV], a special case of which is as follows. Suppose that $|\alpha - s/r| \le 1/r^2$ and $2 \le R \le r \le x/R$ for some coprime integers s and r. Then we have

$$G(x,\alpha) \ll \frac{x}{\log x} + \frac{x}{\sqrt{R}}(\log R)^{3/2},$$

uniformly for all $|g| \le 1$. Using Corollary 3 we can now strengthen this estimate to

$$G(x,\alpha) \ll_\lambda \frac{x}{\log x} + \frac{x}{\sqrt{R}}\exp\biggl(-\sum_{p\le x}\frac{1-|g(p)|}{p}\biggr)(\log R)^{1/2}(\log_2 R)^{3/2},$$

valid uniformly for all $|g| \le 1$ satisfying

$$\sum_{p\le x}\frac{|g(p)|}{p} \ge \lambda \log_2 x \qquad (\lambda > 0).$$

We refer the reader to [Ba2] for a more thorough discussion of this topic as well as for several other variants of the latter result. The proofs of our exponential sums estimates will be contained in the forthcoming paper [Ba3].

2. Preliminaries. In this section we collect various lemmas needed for the proofs of our main results.

Lemma 1. *We have*

$$\sum_{\substack{n\le x\\(n,q)=1}} \frac{f(n)}{n} \asymp_A e^{S_q(x)},$$

uniformly for all $f\in\mathcal{F}_{A,1/3}$, $x\ge 3$, *and all natural numbers* q.

Proof. A related result was obtained in [Ba1, Lemma 1] where it was shown that the claim of this lemma holds true for all multiplicative functions h satisfying the Wirsing condition

$$0\le h(p^\nu)\le \lambda_1\lambda_2^{\nu-1}\qquad (\nu=1,2,\ldots), \tag{2.1}$$

where $\lambda_1\ge 1$ and $1\le\lambda_2<2$, with the implied constant depending only on λ_1 and λ_2. We now show how our lemma follows from that result.

Let h_1 and h_2 be multiplicative functions defined by

$$h_1(p^\nu)=\begin{cases} f(p) & \text{if } \nu=1,\\ 0 & \text{if } \nu\ge 2,\end{cases}\qquad h_2(p^\nu)=\begin{cases} 0 & \text{if } \nu=1,\\ f(p^\nu) & \text{if } \nu\ge 2.\end{cases}$$

Then

$$h_1(n)\le f(n)\le h_1*h_2\,(n). \tag{2.2}$$

By definition (1.4) of $\mathcal{F}_{A,1/3}$,

$$\sum_{n=1}^{\infty}\frac{h_2(n)}{n}=\prod_p\left(1+\frac{h_2(p^2)}{p^2}+\frac{h_2(p^3)}{p^3}+\ldots\right)=\prod_p\left(1+O_A\left(\frac{1}{p^{4/3}}\right)\right)\ll_A 1.$$

Therefore (2.2) yields

$$\sum_{\substack{n\le x\\(n,q)=1}} \frac{f(n)}{n} \asymp_A \sum_{\substack{n\le x\\(n,q)=1}} \frac{h_1(n)}{n}.$$

But h_1 satisfies (2.1) with $\lambda_1=A$ and $\lambda_2=1$ and

$$S_q(x)=\sum_{\substack{p\le x\\(p,q)=1}}\frac{f(p)}{p}=\sum_{\substack{p\le x\\(p,q)=1}}\frac{h_1(p)}{p}.$$

Hence, the desired estimate follows from the afore-mentioned result [Ba1, Lemma 1]. ∎

Our next lemma, a special case of which was mentioned in the introduction, is due to Shiu [Sh].

Lemma 2. *Let* ε, $0<\varepsilon\le 1/2$, *be fixed. Then the inequality*

$$\sum_{\substack{x-y<n\le x\\ n\equiv a\,(\mathrm{mod}\,q)}} f(n)\ll_\varepsilon \frac{y}{\varphi(q)\log x}\,e^{S_q(x)}$$

holds uniformly for all $f \in \mathcal{F}$, $x \geq 3$, $x^\varepsilon \leq y \leq x$, $1 \leq q \leq y^{1-\varepsilon}$, *and all integers* a. *Furthermore*, *if we assume in addition that* $(a,q) = 1$, *then there exists a constant* $\gamma_\varepsilon > 0$, *depending only on* ε, *such that the same inequality holds*, *with* $\ll_\varepsilon$ *replaced by* $\ll_{A,\varepsilon}$, *uniformly for all* $f \in \mathcal{F}_{A,\gamma_\varepsilon}$.

Proof. Let us first consider the second claim of the lemma. Although this result is stronger than what is stated by Shiu in [Sh], one readily verifies that his argument certainly applies to give the result stated here with $\gamma_\varepsilon = \varepsilon^2/100$, say.

Shiu considered only the case $(a,q) = 1$, but the general case can be deduced from this. In particular, the first claim of our lemma was derived in such a way in [Ba4, Lemma 3.2]. We remark that the latter result was stated with $S(x)$ in place of $S_q(x)$, but the argument given there does give the estimate stated here. ∎

We also need the following extension of Lemma 2.

LEMMA 3. *Let real numbers* ε, $0 < \varepsilon \leq 1/2$, *and* $C \geq 1$ *be fixed. Then there exists a constant* $z_{\varepsilon,C}$ *such that for all* $z \geq z_{\varepsilon,C}$ *the inequality*

$$\sum_{\substack{x-y<n\leq x \\ n\equiv a \,(\mathrm{mod}\, q),\, p\mid n \Rightarrow p\leq z}} f(n) \ll_{\varepsilon,C} \frac{y}{\varphi(q)\log x} \exp\left(S_q(z) - C\,\frac{\log x}{\log z}\right)$$

holds uniformly for all $f \in \mathcal{F}$, $x \geq 3$, $x^\varepsilon \leq y \leq x$, $1 \leq q \leq y^{1-\varepsilon}$, *and all integers* a. *Furthermore*, *if we assume in addition that* $(a,q) = 1$, *then there also exists a constant* $\gamma_\varepsilon > 0$, *depending only on* ε, *such that the same inequality holds*, *with* $\ll_{\varepsilon,C}$ *replaced by* $\ll_{A,\varepsilon,C}$, *uniformly for all* $f \in \mathcal{F}_{A,\gamma_\varepsilon}$.

Proof. This estimate is contained in a recent work of M. Nair and G. Tenenbaum [NT]. Arguments of that paper are intended to prove much more general results, of which Lemma 3 is a special case, and are rather involved. Alternatively, a much simpler argument for this estimate may be constructed as follows.

Analogously to the situation in Lemma 2, the first claim of this lemma is readily seen to follow from the second claim by the argument given in [Ba4, Proof of Lemma 3.2]. The second claim, with $\gamma_\varepsilon = \varepsilon^2/200$ and $z_{\varepsilon,C} = e^{2C/\gamma^2}$, say, is readily deduced from Lemma 2 by the Rankin–Tenenbaum method (see [Te, III.5.1, Proof of Theorem 1]). ∎

3. Proof of Theorems 1 and 2. In this and the remaining two sections we will find it convenient to suppress the dependence of various estimates on the quantities A, ε and b, e.g., we will write $\ll$ in place of $\ll_{A,\varepsilon,b}$.

We begin by observing that both theorems follow from the estimate

$$\sum_{q^{1+\varepsilon}\le 2^i\le x} M_q(2^i) \ll \frac{1}{\varphi(q)}\, e^{S_q(x)}. \tag{3.1}$$

Indeed, $S_q(x)=S_q(x/2)+O(1)$, by Lemma 1 and definition (1.6) we have

$$\begin{aligned} e^{S_q(x)} &\ll e^{S_q(x/2)} \ll \sum_{\substack{n\le x/2\\(n,q)=1}} \frac{f(n)}{n} \\ &\le \sum_{1\le 2^i\le x}\ \sum_{\substack{2^{i-1}<n\le 2^i\\(n,q)=1}} \frac{f(n)}{n} \ll \sum_{2^i\le x} M(2^i), \end{aligned} \tag{3.2}$$

and Theorem 2 is proved. For Theorem 1 we write

$$\sum_{\substack{q^{1+\varepsilon}\le n\le x\\ n\equiv a\,(\mathrm{mod}\,q)}} \frac{f(n)}{n} \le \sum_{q^{1+\varepsilon}\le 2^i\le 2x}\ \sum_{\substack{2^{i-1}<n\le 2^i\\ n\equiv a\,(\mathrm{mod}\,q)}} \frac{f(n)}{n} \ll \sum_{q^{1+\varepsilon}\le 2^i\le 2x} M_q(2^i),$$

by definition (1.7), and the desired estimate follows from (3.1) and (3.2) (with x replaced by $2x$).

By Mertens's formula there is a constant $c=c_\varepsilon\ge 2$ depending only on ε such that if $x\ge q^c$, then the validity of (1.5) implies the validity of (1.12) with $x_\kappa=x$, x_1 equal to the smallest 2^i occurring in the sum (3.1) and with ε replaced by $\varepsilon/2$ (we assume, as we may, that $q\ge 2$). Therefore, if $x\ge q^c$, then (3.1) follows from Corollary 3. If, on the other hand, $x<q^c$, then (3.1) follows by applying Lemma 2 to each of the terms $M_q(2^i)$ in (3.1). This completes the proof of Theorems 1 and 2.

4. Proof of Theorem 0; reductions, principal estimate. We will prove in detail only the first claim of the theorem. The second claim follows easily by making obvious modifications to the argument below (the resulting argument is in fact slightly simpler than the one given here).

Our first two reductions are trivial consequences of Lemma 2. As we already observed in Section 1, by (1.9) each sum over n on the left-hand side of (1.10) can be estimated by Lemma 2 yielding the bound (1.13). Thus it only remains to obtain (1.10) with $\min(1,\sum_{1\le i\le\kappa} 1/\log x_i)$ replaced by 1. Furthermore, replacing the range $1\le i\le\kappa$ in (1.13) by the range $i:(q\tau)^{1+\varepsilon}\le y_i<(q\tau)^2$, shows, by (1.9), that the sum over the latter range can be estimated by $\ll e^{S_q(x_\kappa)}/\varphi(q)$. Whence we may assume that $y_1\ge(q\tau)^2$.

Next we introduce some conventions. Let us assume, as we may, that $q\ge 2$ and that $1<b\le 2$, and let the function $f\in\mathcal{F}$ be given. Furthermore,

let i_- be the smallest integer satisfying

$$b^{i_-} \geq (q\tau)^2, \tag{4.1}$$

and define sequences x_i, y_i and a_i, $i_- \leq i \leq i_+$, in terms of the function f as follows. For each i we set

$$\frac{1}{y_i} \sum_{\substack{x_i - y_i < n \leq x_i \\ n \equiv a_i \,(\mathrm{mod}\, q)}} f(n) = \max_{(x,y,a)} \frac{1}{y} \sum_{\substack{x-y<n\leq x \\ n\equiv a\,(\mathrm{mod}\, q)}} f(n), \tag{4.2}$$

where the maximum is taken over x, y and a satisfying

$$b^{i-1} < y \leq b^i, \qquad y \leq x \leq \tau y \tag{4.3}$$

and a arbitrary (not necessarily coprime with q). It is readily seen that it suffices to prove the theorem with this choice of the sequences x_i, y_i and a_i (defined over the range $i_- \leq i \leq i_+$ in place of the range $1 \leq i \leq \kappa$ as in the statement of the theorem). Moreover, set

$$F_i = \frac{1}{y_i} \sum_{\substack{x_i - y_i < n \leq x_i \\ n \equiv a_i \,(\mathrm{mod}\, q)}} f(n), \tag{4.4}$$

$$S_i = \sum_{\substack{p \leq y_i \\ (p,q)=1}} \frac{f(p)}{p}, \tag{4.5}$$

for $i_- \leq i \leq i_+$. Observe that, with these conventions, the statement of the theorem follows from the bound

$$\sum_{i_- \leq i \leq i_+} F_i \ll \frac{1}{\varphi(q)} \left(e^{S_{i_-}} \left(\frac{i_+}{i_-} \right)^{\varepsilon} + e^{S_{i_+}} \right), \tag{4.6}$$

since by Mertens's formula, (4.3) and (4.1), we have

$$\sum_{\substack{p \leq x_i \\ (p,q)=1}} \frac{f(p)}{p} - \sum_{\substack{p \leq y_i \\ (p,q)=1}} \frac{f(p)}{p} \leq \sum_{y_i < p \leq x_i} \frac{1}{p} \ll 1. \tag{4.7}$$

Since our proof of (4.6) is somewhat involved, we now give, for the convenience of the reader, a brief sketch highlighting the main points of our argument. We begin by noting that if the range of summation $i_- \leq i \leq i_+$ is sufficiently large with respect to i_-, let us say $i_+ \geq 2i_-$, then we regard the estimate

$$\sum_{i_- \leq i \leq i_+} F_i \ll \frac{1}{\varphi(q)}\, e^{S_{i_+}} \tag{S.1}$$

as the expected bound. This estimate is certainly best possible, and it is not at all difficult to construct functions $f \in \mathcal{F}$ (with $S(x) = \lambda \log_2 x + O(1)$, $0 < \lambda \leq 1$) for which (S.1) holds with $\ll$ replaced by $\gg$. On the other hand, estimating each F_i by Lemma 2 apparently leads to the "extra" $\log i_+$ term (corresponding to $\log_2 x_\kappa$ in the original notation) in the event that $\log(i_+/i_-) \asymp \log i_+$, viz.

$$\sum_{i_- \le i \le i_+} F_i \ll \frac{1}{\varphi(q)} \sum_{i_- \le i \le i_+} \frac{1}{i} e^{S_i} \ll \frac{1}{\varphi(q)} e^{S_{i_+}} \log \frac{i_+}{i_-},$$

as we already noted in Section 1. Our new idea that enables us to get (S.1) for those functions $f \in \mathcal{F}$ which are supported on the positive proportion of primes in the interval $(x_-, x_+]$, i.e., f satisfying

$$S_{i_+} - S_{i_-} \gg \log \frac{i_+}{i_-}, \tag{S.2}$$

consists of three parts:

(i) Fix a positive parameter δ. The assumption (S.2) implies that the entire range of summation $i_- \le i \le i_+$ contains "good sub-ranges" $i' < i \le i''$, i.e., ranges over which both

$$\sum_{i' < i \le i''} F_i \ll \frac{1}{\varphi(q)} \sum_{i' < i \le i''} \frac{1}{i} e^{S_i} \ll \frac{1}{\varphi(q)} e^{S_{i''}} \tag{S.3}$$

and

$$S_{i''} - S_{i'} \ge \delta. \tag{S.4}$$

(ii) Take i'' of part (i) as large as possible. Instead of directly estimating F_i by Lemma 2 (as in (S.3)) in the "unfavorable range" $i'' < i \le i_+$, estimate the sum over this range by the sum over $i_- \le i \le i''$ to get the bound

$$\sum_{i'' < i \le i_+} F_i \le (ce^{S_{i_+} - S_{i''}} - 1) \sum_{i_- \le i \le i''} F_i + \text{Acceptable Error Term}, \tag{S.5}$$

for some parameter c.

(iii) By applying (i) and (ii), reduce proving (S.1) to proving that (S.1) holds with i' in place of i_+. Now proceed inductively.

To illustrate this method let us perform two iterations. By (S.5),

$$\sum_{i_- \le i \le i_+} F_i = \sum_{i_- \le i \le i''} F_i + \sum_{i'' < i \le i_+} F_i \le ce^{S_{i_+} - S_{i''}} \sum_{i_- \le i \le i''} F_i,$$

where, for the sake of simplicity, we ignore the "Acceptable Error Term". This, (S.3) and (S.4) yield

$$\begin{aligned} \sum_{i_- \le i \le i_+} F_i &\le ce^{S_{i_+} - S_{i''}} \Bigg(\sum_{i_- \le i \le i'} F_i + O\bigg(\frac{1}{\varphi(q)} e^{S_{i''}} \bigg) \Bigg) \\ &\le ce^{-\delta} e^{S_{i_+} - S_{i'}} \sum_{i_- \le i \le i'} F_i + O\bigg(\frac{c}{\varphi(q)} e^{S_{i_+}} \bigg). \end{aligned}$$

Iterating this argument the second time we obtain the bound

$$\sum_{i_-\le i\le i_+} F_i \le (ce^{-\delta})^2 e^{S_{i_+}-S_{i'}} \sum_{i_-\le i\le i'} F_i + O\biggl(\frac{c}{\varphi(q)}\, e^{S_{i_+}}(1+ce^{-\delta})\biggr),$$

where the value of i' in the last display is different and strictly smaller than the value of i' in the penultimate display. It is now plain that further iterations of this argument yield, essentially,

$$\sum_{i_-\le i\le i_+} F_i \ll \frac{c}{\varphi(q)}\, e^{S_{i_+}} \sum_{j=0}^{\infty}(ce^{-\delta})^j \ll \frac{1}{\varphi(q)}\, e^{S_{i_+}},$$

as desired, by a suitable choice of parameters c and δ.

In the remainder of this section we will obtain two estimates for the sum

$$\sum_{i'<i\le i''} F_i \qquad (i_- - 1 \le i' < i'' \le i_+), \tag{4.8}$$

which will then be used in the next section to deduce (4.6). We give these estimates in two lemmas below.

Lemma 4. *We have*

$$\sum_{i'<i\le i''} F_i \ll \frac{1}{\varphi(q)}\, e^{S_{i''}} \log\frac{i''}{i'}.$$

Proof. We simply observe that, by (4.4), (4.3) and (4.1), each term F_i can be estimated by Lemma 2. This yields the bound

$$\sum_{i'<i\le i''} F_i \ll \frac{1}{\varphi(q)} \sum_{i'<i\le i''} \frac{1}{\log x_i}\, e^{S_q(x_i)} \ll \frac{1}{\varphi(q)}\, e^{S_q(x_{i''})} \log\frac{i''}{i'},$$

and the lemma follows by (4.5) and (4.7). ∎

Our second estimate for (4.8) is more complicated. It takes a recursive form of providing an estimate for this sum in terms of itself, but over the preceding range $i_- \le i \le i'$.

Lemma 5. *We have*

$$\sum_{i'<i\le i''} F_i \le (ce^{S_{i''}-S_{i'}} - 1) \sum_{i_-\le i\le i'} F_i + O\biggl(\frac{1}{\varphi(q)}\, e^{S_{i''}}\Bigl(1 + e^{S_{i_-}-S_{i'}} \log\frac{i''}{i'}\Bigr)\biggr),$$

for some absolute constant $c \ge 2$.

Proof. Let N be an arbitrary natural number. Then for $i' \le N$ we have, by (4.4), (4.3) and (4.1),

$$\sum_{i'<i\le N} F_i \ll_N \frac{1}{q}.$$

Furthermore, if $i' \le 3i_-$, then for any $j \le 3i_- + 1$ we have, by Lemma 4,

$$\sum_{i'<i\le j} F_i \ll \frac{1}{\varphi(q)}\, e^{S_j}.$$

This shows that we may assume in what follows that i', and hence $y_{i'}$, is sufficiently large and that, in particular, it satisfies the condition

$$i' > 3i_-. \tag{4.9}$$

For the rest of the proof of this lemma we adopt the following notation.

We let m denote natural numbers free of prime factors greater than $y_{i'}$ and we let l denote natural numbers all of whose prime factors lie in the interval $(y_{i'}, x_{i''}]$, i.e.,

$$p \,|\, m \ \Rightarrow\ p \le y_{i'} \quad \text{and} \quad p \,|\, l \ \Rightarrow\ y_{i'} < p \le x_{i''}. \tag{4.10}$$

Observe that any natural number $n \le x_{i''}$ can be written in a unique way in the form $n = ml$. This permits us to split the sum (4.8) as follows, we write

$$\begin{aligned} \sum_{i'<i\le i''} F_i &= \sum_{i'<i\le i''} \frac{1}{y_i} \sum_{\substack{x_i - y_i < ml \le x_i \\ ml \equiv a_i \,(\mathrm{mod}\, q)}} f(ml) \\ &= \Big(\sum_{m\le y_{i_-}} + \sum_{y_{i_-}<m\le y_{i'}} + \sum_{y_{i'}<m\le x_{i''}} \Big) f(m) \sum_{i'<i\le i''} \frac{1}{y_i} \sum_{\substack{(x_i-y_i)/m < l \le x_i/m \\ ml\equiv a_i \,(\mathrm{mod}\, q)}} f(l) \\ &= \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3, \end{aligned} \tag{4.11}$$

say. We will proceed somewhat differently in order to estimate each of the three sums on the right-hand side of (4.11), but each of these estimates will ultimately rest on an application of a Shiu-type bound, namely Lemmas 2 and 3.

We estimate $\sum_1$ by applying Lemma 2 to its innermost sum. To this end we first observe that by (4.1), (4.3) and (4.9) the conditions of Lemma 2 are satisfied. Indeed, we have

$$\frac{y_i}{m} \ge \frac{y_i}{y_{i_-}} > b^{i_-} > q^2 \quad \text{and} \quad \tau m \le \tau y_{i_-} < b^{2i_-} < y_i,$$

from which it follows that

$$\frac{y_i}{m} = \sqrt{\frac{y_i^2}{m^2}} > \sqrt{\frac{y_i \tau}{m}} \ge \sqrt{\frac{x_i}{m}}.$$

Furthermore, we note that by definition of l (see (4.10)) and (4.1) we have

$(l,q)=1$. We thus obtain

$$\sum\nolimits_1 = \sum_{d|q} \sum_{\substack{m\le y_{i\_} \\ (m,q)=d}} f(m) \sum_{\substack{i'<i\le i'' \\ (a_i,q)=d}} \frac{1}{y_i} \sum_{\substack{(x_i-y_i)/m<l\le x_i/m \\ ml\equiv a_i \,(\mathrm{mod}\, q)}} f(l)$$

$$\ll \sum_{d|q} \frac{1}{\varphi(q/d)} \sum_{\substack{m\le y_{i\_} \\ (m,q)=d}} \frac{f(m)}{m} \sum_{\substack{i'<i\le i'' \\ (a_i,q)=d}} \frac{1}{\log(x_i/m)}\, e^{S_q(x_i/m)-S_q(y_{i'})},$$

by (4.10). Letting d_0 denote the value of d which maximizes the quantity

$$\frac{1}{\varphi(q/d)} \sum_{\substack{m\le y_{i\_} \\ (m,q)=d}} \frac{f(m)}{m},$$

we get, by (4.5), (4.7), (4.3) and (4.9),

$$\sum\nolimits_1 \ll \frac{1}{\varphi(q/d_0)} \sum_{\substack{m\le y_{i\_} \\ (m,q)=d_0}} \frac{f(m)}{m} \sum_{i'<i\le i''} \frac{1}{i}\, e^{S_i-S_{i'}} \tag{4.12}$$

$$\ll \frac{1}{\varphi(q/d_0)}\, e^{S_{i''}-S_{i'}} \log\frac{i''}{i'} \sum_{\substack{m\le y_{i\_} \\ (m,q)=d_0}} \frac{f(m)}{m}.$$

Now set

$$d_0 = d_1 d_2 \quad \text{and} \quad q = d_1 D_2 q'$$

where

$$(d_0, q') = 1, \qquad \left(d_1, \frac{q}{d_1}\right) = 1 \quad \text{and} \quad p \,|\, d_2 \;\Rightarrow\; p \,\Big|\, \frac{D_2}{d_2}.$$

With this notation and by definition of m (see (4.10)) the sum on the right-hand side of (4.12) is certainly bounded by

$$\sum_{\substack{m\le y_{i\_} \\ (m,q)=d_0}} \frac{f(m)}{m} \le \frac{f(d_2)}{d_2} \sum_{\substack{1\le k\le\infty \\ p|k\Rightarrow p|d_1}} \frac{f(d_1 k)}{d_1 k} \sum_{\substack{m\le y_{i\_} \\ (m,q)=1}} \frac{f(m)}{m}.$$

Estimating the sum over k trivially and the last sum over m by Lemma 1, we obtain

$$\sum_{\substack{m\le y_{i\_} \\ (m,q)=d_0}} \frac{f(m)}{m} \ll \frac{1}{d_2}\cdot\frac{1}{d_1}\prod_{p|d_1}\left(1-\frac{1}{p}\right)^{-1} e^{S_{i\_}} = \frac{1}{d_2}\cdot\frac{1}{\varphi(d_1)}\, e^{S_{i\_}}.$$

Substituting this into (4.12) finally gives

$$\sum\nolimits_1 \ll \frac{1}{\varphi(q)}\, e^{S_{i''}} e^{S_{i\_}-S_{i'}} \log\frac{i''}{i'}. \tag{4.13}$$

Next we estimate $\sum_3$. Let us rewrite this sum in the form

$$\sum\nolimits_3 \le \sum_{l \le x_{i''}} f(l) \sum_{\substack{i' < i \le i'' \\ x_i > y_{i'} l}} \frac{1}{y_i} \sum_{\substack{(x_i - y_i)/l < m \le x_i/l \\ ml \equiv a_i \,(\mathrm{mod}\, q)}} f(m). \tag{4.14}$$

Observe that, by (4.1), (4.3) and (4.9), the parameters of the innermost sum on the right-hand side of (4.14) satisfy the inequalities

$$\tau l < \tau \frac{x_i}{y_{i'}} \le y_i \frac{\tau^2}{y_{i'}} < y_i,$$

and hence

$$\frac{y_i}{l} > \sqrt{\frac{\tau y_i}{l}} \ge \sqrt{\frac{x_i}{l}},$$

as well as

$$\frac{y_i}{l} > \frac{y_i y_{i'}}{x_i} \ge \frac{y_{i'}}{\tau} > q^2.$$

These and the definition of m (see (4.10)) shows that we may apply Lemma 3, with $\varepsilon = 1/2$, $C = 1$ and $z = y_{i'}$, to the innermost sum on the right-hand side of (4.14), provided only that $y_{i'}$ is larger than some absolute constant, as we may assume. Recalling also that $(l, q) = 1$, we thus obtain

$$\sum\nolimits_3 \ll \frac{1}{\varphi(q)} \sum_{l \le x_{i''}} \frac{f(l)}{l} \sum_{\substack{i' < i \le i'' \\ x_i > y_{i'} l}} \frac{1}{\log(x_i/l)} \exp\Big(S_{i'} - \frac{\log(x_i/l)}{\log y_{i'}} \Big). \tag{4.15}$$

But, by (4.3) and (4.1), we have

$$\begin{aligned} \sum_{\substack{i' < i \le i'' \\ x_i > y_{i'} l}} \frac{1}{\log(x_i/l)} \exp\Big(-\frac{\log(x_i/l)}{\log y_{i'}} \Big) \\ < \frac{1}{\log y_{i'}} \Big(\sum_{y_{i'} l/\tau < y_i \le y_{i'} l} 1 + \sum_{y_i > y_{i'} l} \exp\Big(-\frac{\log(y_i/l)}{\log y_{i'}} \Big) \Big) \ll 1, \end{aligned} \tag{4.16}$$

while, by Lemma 1, (4.10) and (4.7),

$$\sum_{l \le x_{i''}} \frac{f(l)}{l} \ll e^{S_{i''} - S_{i'}}. \tag{4.17}$$

Therefore, by (4.15)–(4.17) we finally obtain the bound

$$\sum\nolimits_3 \ll \frac{1}{\varphi(q)} e^{S_{i''}}. \tag{4.18}$$

We now turn to $\sum_2$. We can provide a satisfactory estimate for this term only by means of an iterative bound. To this end we now split this sum further and write

$$\sum\nolimits_2 = \sum_{l \le x_{i''}} f(l) \times \Bigg(\sum_{\substack{i'<i\le i'' \\ y_i \le l y_{i\_}}} + \sum_{\substack{i'<i\le i'' \\ l y_{i\_} < y_i \le l y_{i'}}} + \sum_{\substack{i'<i\le i'' \\ y_i > l y_{i'}}} \Bigg) \frac{1}{y_i} \sum_{\substack{(x_i - y_i)/l < m \le x_i/l \\ ml \equiv a_i \,(\mathrm{mod}\, q) \\ y_{i\_} < m \le y_{i'}}} f(m) = \sum\nolimits_{21} + \sum\nolimits_{22} + \sum\nolimits_{23}, \tag{4.19}$$

say. The main difficulty here is in estimating $\sum_{22}$. So let us dispose of the other two terms first.

Our method for estimating $\sum_{21}$ is analogous to our method for estimating $\sum_3$. In the first place we have, by (4.3),

$$\sum\nolimits_{21} \le \sum_{l \le x_{i''}} f(l) \sum_{\substack{i'<i\le i'' \\ l y_{i\_}/\tau < y_i \le l y_{i\_}}} \frac{1}{y_i} \sum_{\substack{(x_i - y_i)/l < m \le x_i/l \\ ml \equiv a_i \,(\mathrm{mod}\, q)}} f(m).$$

Computations analogous to the ones performed prior to estimating the innermost sum on the right-hand side of (4.14) show that we can apply Lemma 2 to the last sum over m. This yields the following analogue of (4.15)

$$\sum\nolimits_{21} \ll \frac{1}{\varphi(q)} \sum_{l \le x_{i''}} \frac{f(l)}{l} \sum_{\substack{i'<i\le i'' \\ l y_{i\_}/\tau < y_i \le l y_{i\_}}} \frac{e^{S_{i'}}}{\log(x_i/l)} \tag{4.20}$$

(for the present estimate it suffices to appeal to Lemma 2 whereas we needed Lemma 3 to estimate $\sum_3$). But, by (4.3) and (4.1), we have

$$\sum_{l y_{i\_}/\tau < y_i \le l y_{i\_}} \frac{1}{\log(x_i/l)} < \frac{1}{\log(y_{i\_}/\tau)} \sum_{l y_{i\_}/\tau < y_i \le l y_{i\_}} 1 \ll 1. \tag{4.21}$$

Thus, by (4.20), (4.21) and (4.17), we obtain

$$\sum\nolimits_{21} \ll \frac{1}{\varphi(q)} e^{S_{i''}}. \tag{4.22}$$

To estimate $\sum_{23}$ we first write

$$\sum\nolimits_{23} \le \sum_{l \le x_{i''}} f(l) \sum_{\substack{i'<i\le i'' \\ y_i > l y_{i'}}} \frac{1}{y_i} \sum_{\substack{m \le y_{i'} \\ ml \equiv a_i \,(\mathrm{mod}\, q)}} f(m).$$

Familiar-by-now computations resting on Lemma 2, (4.3) and (4.17) now yield

$$\sum\nolimits_{23} \ll \frac{1}{\varphi(q)} \cdot \frac{y_{i'}}{\log y_{i'}}\, e^{S_{i'}} \sum_{l \le x_{i''}} f(l) \sum_{\substack{i' < i \le i'' \\ y_i > l y_{i'}}} \frac{1}{y_i} \tag{4.23}$$

$$\ll \frac{1}{\varphi(q)}\, e^{S_{i'}} \sum_{l \le x_{i''}} \frac{f(l)}{l} \ll \frac{1}{\varphi(q)}\, e^{S_{i''}}.$$

Finally, to estimate $\sum_{22}$ we write

$$\sum\nolimits_{22} \le \sum_{1 < l \le x_{i''}} \frac{f(l)}{l} \sum_{\substack{i' < i \le i'' \\ l y_{i_-} < y_i \le l y_{i'}}} \frac{1}{y_i/l} \sum_{\substack{(x_i - y_i)/l < m \le x_i/l \\ lm \equiv a_i \, (\mathrm{mod}\, q)}} f(m). \tag{4.24}$$

Recalling that $(l, q) = 1$, we observe that for each l the summation over i on the right-hand side of (4.24) is bounded by $2\sum_{i_- \le i \le i'} F_i$, by definitions of F_i and the sequences x_i, y_i and a_i (see (4.2)–(4.4)). This and (4.17) yield the inequality

$$\sum\nolimits_{22} \le (c e^{S_{i''} - S_{i'}} - 1) \sum_{i_- \le i \le i'} F_i, \tag{4.25}$$

with $c = 2c'$, where $c' \ge 1$ is the absolute constant implicit in (4.17).

Therefore, by (4.19), (4.22), (4.23) and (4.25), we get the estimate

$$\sum\nolimits_{2} \le (c e^{S_{i''} - S_{i'}} - 1) \sum_{i_- \le i \le i'} F_i + O\left(\frac{1}{\varphi(q)}\, e^{S_{i''}}\right). \tag{4.26}$$

Finally, combining (4.11), (4.13), (4.18) and (4.26) gives the desired estimate and completes the proof of the lemma. ∎

5. Proof of Theorem 0; conclusion. Recall that the proof of the theorem was reduced to establishing (4.6).

Our first task here is to dispose of two easy cases, cases when either the quantity i_+/i_- or the quantity $S_{i_+} - S_{i_-}$ are small. In both of these cases (4.6) follows immediately from Lemma 4. Indeed, if $i_+/i_- \le 10$, say, then Lemma 4 yields

$$\sum_{i_- \le i \le i_+} F_i \ll \frac{1}{\varphi(q)}\, e^{S_{i_+}},$$

establishing (4.6) in this case. As for the second case, we first fix our notion of what we mean by saying that the quantity $S_{i_+} - S_{i_-}$ is small so as to be convenient for our later considerations. To this end, let us now fix a sufficiently large (in an absolute sense) real number δ satisfying

$$\delta \ge 2 \log c, \tag{5.1}$$

where c is the absolute constant from Lemma 5, and assume that $S_{i_+} - S_{i_-} \leq 2\delta$. Then Lemma 4 yields

$$\sum_{i_- \leq i \leq i_+} F_i \ll \frac{1}{\varphi(q)}\, e^{S_{i_+}} \log \frac{i_+}{i_- - 1} \ll \frac{1}{\varphi(q)}\, e^{S_{i_-}} \log\biggl(2\, \frac{i_+}{i_-}\biggr),$$

establishing (4.6) in this case as well. Thus it only remains to consider the case when

$$\frac{i_+}{i_-} > 10 \quad \text{and} \quad S_{i_+} - S_{i_-} > 2\delta,$$

which is what we assume from now on.

Next, let us introduce the sequence i_k, $0 \leq k \leq K$, defined as follows. We set $i_0 = 5i_-$, and then proceed recursively by letting i_k, $1 \leq k \leq K$, be the smallest integer i for which the inequality

$$S_i - S_{i_{k-1}} \geq \delta \tag{5.2}$$

holds, where, in addition, we let $i_K \leq i_+$ be the unique integer satisfying

$$S_{i_+} - S_{i_K} < \delta. \tag{5.3}$$

Observe that the sequence i_k is well defined, provided δ is chosen sufficiently large, for by Mertens's formula, (4.3) and (4.1) we have

$$\begin{aligned} S_{i_0} - S_{i_-} &\leq \sum_{y_{i_-} < p \leq y_{i_0}} \frac{1}{p} = \log_2 y_{i_0} - \log_2 y_{i_-} + O\biggl(\frac{1}{\log y_{i_-}}\biggr) \\ &\leq \log \frac{i_0}{i_- - 1} + O(1) < \delta, \end{aligned} \tag{5.4}$$

since the constant implicit by the O symbol in (5.4) is absolute. As will be seen shortly, our argument will consider two different subcases depending on the growth conditions of the sequence i_k. But before we get into these considerations, let us establish a fact, equation (5.5) below, which will be used in both subcases. We show that the relation

$$S_{i_+} = S_{i_-} + K\delta + O(1) \tag{5.5}$$

holds. To this end we first observe that the sequence i_k increases very rapidly and that, in particular, it certainly satisfies the growth condition

$$i_k > e^{\delta/2} i_{k-1}, \tag{5.6}$$

provided δ is sufficiently large. Indeed, (5.6) follows readily from definition (5.2) of i_k which gives

$$\begin{aligned} \delta \leq S_{i_k} - S_{i_{k-1}} &\leq \sum_{y_{i_{k-1}} < p \leq y_{i_k}} \frac{1}{p} \\ &= \log_2 y_{i_k} - \log_2 y_{i_{k-1}} + O\biggl(\frac{1}{\log y_{i_{k-1}}}\biggr) = \log \frac{i_k}{i_{k-1}} + O\biggl(\frac{1}{\log y_{i_{k-1}}}\biggr), \end{aligned}$$

by Mertens's formula and (4.3). Secondly, we note that the relation

$$S_{i_k} - S_{i_{k-1}} = \delta + O\left(\frac{1}{i_k}\right) \tag{5.7}$$

holds. To see this we, once again, appeal to (5.2), Mertens's formula and (4.3) to get

$$\delta \le S_{i_k} - S_{i_{k-1}} < \delta + S_{i_k} - S_{i_k - 1} \le \delta + \sum_{y_{i_k-1} < p \le y_{i_k}} \frac{1}{p}$$
$$= \delta + \log_2 y_{i_k} - \log_2 y_{i_k-1} + O\left(\frac{1}{\log y_{i_k-1}}\right) = \delta + O\left(\frac{1}{i_k}\right).$$

Now, by (5.3), (5.4) and (5.7), we have

$$S_{i_+} - S_{i_-} = S_{i_0} - S_{i_-} + \sum_{k=1}^{K} (S_{i_k} - S_{i_{k-1}}) + S_{i_+} - S_{i_K} = K\delta + O\left(1 + \sum_{k=1}^{K} \frac{1}{i_k}\right).$$

From this (5.5) follows by (5.6).

As we have already indicated, we will split the case presently under consideration into two different subcases depending on the growth conditions of the sequence i_k. To this end let us now assume, as we may, that the parameter ε appearing in the statement of the theorem satisfies $0 < \varepsilon < 1$, and introduce another parameter

$$\Delta = e^{2\delta/\varepsilon}. \tag{5.8}$$

We consider two subcases as follows:

(i) $$\frac{i_k}{i_{k-1}} > \Delta \quad \text{for all } k,$$

(ii) $$\frac{i_k}{i_{k-1}} \le \Delta \quad \text{for some } k.$$

Our argument in subcase (i) is much simpler, resting entirely on an application of Lemma 4, so we dispose of this case first.

We have, by Lemma 4 and (5.5),

$$\sum_{i_- \le i \le i_+} F_i \ll \frac{1}{\varphi(q)}\, e^{S_{i_-} + K\delta} \log \frac{i_+}{i_-}. \tag{5.9}$$

But, in subcase (i),

$$K < \frac{\log(i_+/i_-)}{\log \Delta}, \tag{5.10}$$

since

$$\frac{i_+}{i_-} > \frac{i_K}{i_0} > \Delta^K.$$

Using this in (5.9) together with (5.8) yields

$$\sum_{i_- \le i \le i_+} F_i \ll \frac{1}{\varphi(q)}\, e^{S_{i_-}} \left(\frac{i_+}{i_-}\right)^{\varepsilon/2} \log \frac{i_+}{i_-} \ll \frac{1}{\varphi(q)}\, e^{S_{i_-}} \left(\frac{i_+}{i_-}\right)^{\varepsilon}.$$

We have thus established the validity of (4.6) for subcase (i), and it only remains to consider subcase (ii).

We start our argument for (ii) by introducing a subsequence of all those members of the sequence i_k for which (ii) holds. Let us temporarily use the natural notation i_{k_t}, $1 \le t \le T$, to denote this subsequence, i.e., i_{k_t} is the tth member of the sequence i_k for which $i_{k_t}/i_{k_t-1} \le \Delta$, and $T \ge 1$ denotes the number of such elements of the sequence i_k. This notation clearly suffers from the problem of "mounting subscripts", and we only use it as an aid for introducing simpler notation. We now let j_t and j'_t, $1 \le t \le T$, be two sequences defined by

$$j_t = i_{k_t} \quad \text{and} \quad j'_t = i_{k_t-1}.$$

Thus we have

$$j_t/j'_t \le \Delta \tag{5.11}$$

and, by (5.2),

$$S_{j_t} - S_{j'_t} \ge \delta. \tag{5.12}$$

Furthermore, since $j_t \ge i_t$, we also have

$$S_{j_t} \ge S_{i_t} \ge S_{i_-} + t\delta, \tag{5.13}$$

by (5.2).

We are now ready for the final assault. We begin by appealing to Lemma 5 to get

$$\sum_{j_T < i \le i_+} F_i \le (c e^{S_{i_+} - S_{j_T}} - 1) \sum_{i_- \le i \le j_T} F_i + O\bigg(\frac{1}{\varphi(q)}\, e^{S_{i_+}} \bigg(1 + e^{S_{i_-} - S_{j_T}} \log \frac{i_+}{j_T}\bigg)\bigg). \tag{5.14}$$

This yields the bound

$$\sum_{i_- \le i \le i_+} F_i \le c e^{S_{i_+} - S_{j_T}} \sum_{i_- \le i \le j_T} F_i + O\bigg(\frac{1}{\varphi(q)}\, e^{S_{i_+}} \bigg(1 + e^{S_{i_-} - S_{j_T}} \log \frac{i_+}{j_T}\bigg)\bigg). \tag{5.15}$$

But by Lemma 4, (5.11) and (5.8) we have

$$\sum_{j'_T < i \le j_T} F_i \ll \frac{1}{\varphi(q)}\, e^{S_{j_T}} \log \frac{j_T}{j'_T} \ll \frac{1}{\varphi(q)}\, e^{S_{j_T}}, \tag{5.16}$$

since the constants c and δ are absolute. Using (5.16) in (5.15) gives

$$\sum_{i_-\le i\le i_+} F_i \le c e^{S_{i_+}-S_{j_T}} \sum_{i_-\le i\le j_T'} F_i + O\biggl(\frac{1}{\varphi(q)}\, e^{S_{i_+}}\Bigl(1+e^{S_{i_-}-S_{j_T}}\log\frac{i_+}{j_T}\Bigr)\biggr). \tag{5.17}$$

Furthermore, by (5.12), (5.1) and (5.13), we now obtain

$$\sum_{i_-\le i\le i_+} F_i \le e^{-\delta/2} e^{S_{i_+}-S_{j_T'}} \sum_{i_-\le i\le j_T'} F_i + O\biggl(\frac{1}{\varphi(q)}\, e^{S_{i_+}}\Bigl(1+e^{-T\delta}\log\frac{i_+}{j_T}\Bigr)\biggr). \tag{5.18}$$

Next we repeat estimates (5.14)–(5.18) with i_+, j_T and j_T' replaced by j_T', j_{T-1} and j_{T-1}' respectively. This gives the bound

$$\sum_{i_-\le i\le j_T'} F_i \le e^{-\delta/2} e^{S_{j_T'}-S_{j_{T-1}'}} \sum_{i_-\le i\le j_{T-1}'} F_i + O\biggl(\frac{1}{\varphi(q)}\, e^{S_{j_T'}}\Bigl(1+e^{-(T-1)\delta}\log\frac{j_T'}{j_{T-1}}\Bigr)\biggr).$$

We remark that our estimates make perfect sense in all possible cases, including the case when $i_+ = j_T$ or $j_T' = j_{T-1}$. Substituting our last estimate into (5.18) we obtain

$$\begin{aligned}\sum_{i_-\le i\le i_+} F_i &\le (e^{-\delta/2})^2 e^{S_{i_+}-S_{j_{T-1}'}} \sum_{i_-\le i\le j_{T-1}'} F_i \\ &\quad + O\biggl\{\frac{1}{\varphi(q)}\, e^{S_{i_+}}\Bigl(1+e^{-T\delta}\log\frac{i_+}{j_T} \\ &\quad + e^{-\delta/2}\Bigl(1+e^{-(T-1)\delta}\log\frac{j_T'}{j_{T-1}}\Bigr)\Bigr)\biggr\} \\ &= (e^{-\delta/2})^2 e^{S_{i_+}-S_{j_{T-1}'}} \sum_{i_-\le i\le j_{T-1}'} F_i \\ &\quad + O\biggl\{\frac{1}{\varphi(q)}\, e^{S_{i_+}}\Bigl((1+e^{-\delta/2}) \\ &\quad + e^{-T\delta/2}\Bigl(e^{-T\delta/2}\log\frac{i_+}{j_T}+e^{-(T-1)\delta/2}\log\frac{j_T'}{j_{T-1}}\Bigr)\Bigr)\biggr\}.\end{aligned} \tag{5.19}$$

We continue estimating $\sum_{i_-\le i\le i_+} F_i$ inductively with each step being the appropriate analogue of going from (5.18) to (5.19). Omitting the intermediate steps we proceed directly to the final outcome. To this end, let E_1 and

E_2 denote the quantities given by

$$E_1 = \sum_{t=0}^{T-1} e^{-t\delta/2} \tag{5.20}$$

and

$$E_2 = e^{-T\delta/2}\biggl(e^{-T\delta/2}\log\frac{i_+}{j_T} + \sum_{t=1}^{T-1} e^{-(T-t)\delta/2}\log\frac{j'_{T-t+1}}{j_{T-t}} + \log\frac{j'_1}{i_-}\biggr). \tag{5.21}$$

One then readily verifies that the final outcome of our inductive process is the estimate

$$\begin{aligned}\sum_{i_-\le i\le i_+} F_i \le e^{-T\delta/2} e^{S_{i_+}-S_{j'_1}} \sum_{i_-\le i\le j'_1} F_i \\ + O\biggl\{\frac{1}{\varphi(q)}\, e^{S_{i_+}}\biggl(E_1 + E_2 - e^{-T\delta/2}\log\frac{j'_1}{i_-}\biggr)\biggr\}.\end{aligned} \tag{5.22}$$

Furthermore, estimating the sum over i on the right-hand side of (5.22) by Lemma 4 gives

$$\sum_{i_-\le i\le i_+} F_i \ll \frac{1}{\varphi(q)}\, e^{S_{i_+}}(E_1+E_2). \tag{5.23}$$

But, by (5.20), $E_1 \ll 1$, while, by (5.21) and the fact that $j_t > j'_t$,

$$E_2 < e^{-T\delta/2}\biggl(\log\frac{i_+}{j_T} + \sum_{t=1}^{T-1}\log\frac{j'_{T-t+1}}{j_{T-t}} + \log\frac{j'_1}{i_-}\biggr) < e^{-T\delta/2}\log\frac{i_+}{i_-}.$$

Using these estimates on the right-hand side of (5.23) we finally obtain

$$\sum_{i_-\le i\le i_+} F_i \ll \frac{1}{\varphi(q)}\, e^{S_{i_+}}\biggl(1 + e^{-T\delta/2}\log\frac{i_+}{i_-}\biggr). \tag{5.24}$$

We complete our estimate of $\sum_{i_-\le i\le i_+} F_i$ in the present case by considering two possibilities. First, (5.24) immediately gives the bound

$$\sum_{i_-\le i\le i_+} F_i \ll \frac{1}{\varphi(q)}\, e^{S_{i_+}}, \tag{5.25}$$

unless T satisfies

$$T\delta < 2\log_2\frac{i_+}{i_-}. \tag{5.26}$$

In the latter case, by (5.5), we get

$$S_{i_+} = S_{i_-} + K\delta + O(1) < S_{i_-} + (K-T)\delta + 2\log_2\frac{i_+}{i_-} + O(1). \tag{5.27}$$

We now recall that by definition of the sequences i_k and j_t and the quantities K and T there are exactly $K-T$ members of the sequence i_k, with $1 \le k \le K$, which satisfy the inequality

$$\frac{i_k}{i_{k-1}} > \Delta.$$

This yields, analogously to (5.10), the bound

$$K - T < \frac{\log(i_+/i_-)}{\log \Delta}. \tag{5.28}$$

Hence, by (5.24), (5.27), (5.28) and (5.8), the assumption (5.26) leads to the estimate

$$\begin{aligned} \sum_{i_- \le i \le i_+} F_i &\ll \frac{1}{\varphi(q)}\, e^{S_{i_+}} \log \frac{i_+}{i_-} \\ &\ll \frac{1}{\varphi(q)}\, e^{S_{i_-}} \left(\frac{i_+}{i_-}\right)^{\varepsilon/2} \left(\log \frac{i_+}{i_-}\right)^3 \ll \frac{1}{\varphi(q)}\, e^{S_{i_-}} \left(\frac{i_+}{i_-}\right)^{\varepsilon}. \end{aligned} \tag{5.29}$$

Combining (5.25) and (5.29) establishes (4.6) in the present case and thus completes the proof of the theorem.

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