ACTA ARITHMETICA
106.1 (2003)

Laurent coefficients of the zeta function of an indefinite quadratic form

by

Makoto Ishibashi (Fukuoka)

1. Introduction. Let K be a real quadratic field of discriminant D, $\zeta(s,A)$ a class zeta function of an ideal class A of K. In [9], Zagier obtained his Kronecker limit formula by using the decomposition

$$D^{s/2}\zeta(s,B^{-1})=\sum_k Z_{Q_k}(s),\qquad \mathrm{Re}(s)>1,$$

of the class zeta function for a narrow ideal class B into the finite sum of zeta functions $Z_{Q_k}(s)$ associated to indefinite quadratic forms Q_k. In view of the above decomposition, getting a Kronecker limit formula amounts to evaluating the constant term of the Laurent expansion of the zeta function

$$Z_Q(s)=\sum_{p=1}^{\infty}\sum_{q=0}^{\infty}\frac{1}{Q(p,q)^s}, \tag{1.1}$$

where $Q(p,q)=ap^2+bpq+cq^2$, with $a,b,c>0$, $b^2-4ac=1$ at $s=1$, after analytic continuation. For this purpose Zagier introduced the function

$$F(x)=\sum_{p=1}^{\infty}\frac{\psi(px)-\log px}{p}=\int_0^{\infty}\left(\frac{1}{1-e^{-t}}-\frac{1}{t}\right)\log(1-e^{-xt})\,dt, \tag{1.2}$$

where $\psi(x)$ denotes the digamma function, i.e. $\psi(x)=(\log\Gamma(x))'$, and obtained the explicit evaluation

$$\lim_{s\to1}\left(Z_Q(s)-\frac{\frac12\log(w/w')}{s-1}\right)=P_0(w,w'),$$

where

$$P_0(x,y)=F(x)-F(y)+\mathrm{Li}_2\left(\frac{y}{x}\right)-\frac{\pi^2}{6}+\log\frac{x}{y}\cdot\left(\gamma-\frac12\log(x-y)+\frac14\log\frac{x}{y}\right),$$

2000 *Mathematics Subject Classification*: 11M41, 11R42.

and $w > w'$ are the roots of the quadratic equation $cw^2 - bw + a = 0$. This, combined with the above-mentioned decomposition, leads to Zagier's formulation of the Kronecker limit formula:

$$\lim_{s\to 1}\left(D^{s/2}\zeta(s,B) - \frac{\log\varepsilon}{s-1}\right) = \sum_k P_0(w_k, w_k'), \tag{1.3}$$

where $\varepsilon > 1$ denotes the smallest unit of K of norm 1. Indeed, the function F was introduced earlier by Herglotz [4] and was used in a similar setting, however, Herglotz's evaluation does not contain the continued fractions but Dedekind sums. Zagier's argument has been simplified considerably by Egami [2] owing to his decomposition (3.3).

Motivated and encouraged by these investigations we shall be concerned in this paper with the evaluation of all the Laurent coefficients of $Z_Q(s)$ at $s = 1$ in closed form. The main result, Theorem 3, provides an explicit formula for the successive kth coefficients $P_k(w, w')$, and in Corollary of Theorem 3, explicit formulas are given for $k = -1, 0, 1$. The formula for $k = 0$ leads immediately to Zagier's formula.

Of course, as in all the formulations of the Kronecker limit formulas, our formulas also undergo the criticism that they are not in final form, involving transcendental integrals, but our method refining those of Zagier and Egami, makes lucid the underlying complexities in evaluating the further Laurent coefficients, and, with some effort, one can in principle compute the higher coefficients P_k, $k = 2, 3, \ldots$, by means of Theorem 3.

The idea is to replace $\psi(x)$ in the definition of the Herglotz–Zagier function $F(x)$ by the derivative of R_k-functions (2.3) to construct a sequence of higher order Herglotz–Zagier functions $\Phi_k(x)$ (viewing $\psi(x) = (\log\Gamma(x))'$ as $R_1'(x)$). After Gut's preliminary investigation on the R_2-function [3], Deninger [1] introduced the R_k-functions as the principal solution of the difference equation (2.1) in the spirit of Artin and extensively developed the theory of the R_2-function for his study on the Chowla–Selberg formula. R_k-functions ($k \ge 2$) were also studied and exploited by Kanemitsu [6] to express the higher order derivatives of Dirichlet L-functions $L(s,\chi)$ at $s = 1$ (see also Ishibashi–Kanemitsu [5]). After expressing the Laurent coefficients $P_k(w, w')$ in terms of Φ_k-functions in Section 3, we go on in Section 4 to study the arithmetical interpretation of the sum $P_k(B) = \sum_B P_k(w, w')$ over all narrow ideal classes, thus giving pre-Chowla–Selberg formulas. We affix "pre" because if we succeed in some way or other in summing $\sum_B P_k(B)$ (as in the case of $k = -1$), then we will get a Chowla–Selberg formula. We hope to study this problem elsewhere.

2. Higher order Herglotz–Zagier functions. We begin with the definition of $R_k(x)$ by Deninger mentioned in Section 1, which in turn is based on Artin's treatment of the gamma function. He introduced the function $R_k(x)$ as the principal solution of the difference equation

$$f(x+1) - f(x) = \log^k x, \qquad f(1) = \lambda, \tag{2.1}$$

which is convex on some interval (A, ∞), $A > 0$, and used mainly $R_2(x)$ in order to calculate $L'(1,\chi)$, the first derivative of the Dirichlet L-function at $s = 1$, thereby proving the Chowla–Selberg formula.

The unique solution $f_k(x)$ of (2.1) with initial condition $f_k(1) = (-1)^{k+1}\zeta^{(k)}(0)$, denoted by $R_k(x)$, admits two equivalent expressions:

$$R_k(x) = \lim_{n\to\infty}\Big((-1)^{k+1}\zeta^{(k)}(0) + x\log^k n - \log^k x - \sum_{\nu=1}^{n-1}(\log^k(x+\nu) - \log^k \nu)\Big), \tag{2.2}$$

$$R_k(x) = (-1)^{k+1}\frac{\partial^k}{\partial s^k}\zeta(0,x), \tag{2.3}$$

where $\zeta(s)$, $\zeta(s,x)$ denote the Riemann zeta function and the Hurwitz zeta function respectively. We note that the Bohr–Mollerup theorem, which characterizes the gamma function by a difference equation and convexity, asserts that $R_1(x) = \log(\Gamma(x)/\sqrt{2\pi})$, making our idea to replace ψ by R_k' legitimate. The analytic continuation of $R_k(x)$ to the whole complex plane except the origin and the negative real axis is accomplished by the Hurwitz zeta-expression (2.3).

Deninger proved many analytic properties of $R_2(x)$ in order to familiarize and assimilate it as one of most commonly used number-theoretic special functions, and among other things, he proved for R_2 an analogue of Plana's formula for $\log\Gamma$:

$$R_2(x) = -\zeta''(0) - 2\int_0^\infty\Big((x-1)e^{-t} + \frac{e^{-xt} - e^{-t}}{1 - e^{-t}}\Big)(\gamma + \log t)\,d\log t, \quad x > 0. \tag{2.4}$$

We find it convenient to have a similar formula for $R_k(x)$:

THEOREM 1. *For $x > 0$,*

$$R_k(x) = (-1)^{k+1}\Big(\zeta^{(k)}(0) + k\int_0^\infty\Big((x-1)e^{-t} + \frac{e^{-xt} - e^{-t}}{1 - e^{-t}}\Big)S_k(t)\,d\log t\Big), \tag{2.5}$$

where $S_k(t) = \sum_{j=0}^{k-1} a_{k,j}\log^j t$, $a_{k,j}$ are defined recursively by

$$a_{k,j} = -\sum_{r=0}^{k-2}\binom{k-1}{r}\Gamma^{(k-r-1)}(1)a_{r+1,j}, \qquad 0 \le j \le k-2,$$

starting from $a_{1,0} = 1$, and $a_{k,k-1} = 1$.

Proof. First we need a formula

$$(2.6)\qquad \int_0^\infty (e^{-\beta t}-e^{-\alpha t})S_k(t)\,d\log t=\frac{(-1)^k}{k}(\log^k\beta-\log^k\alpha)\quad \text{for } \alpha,\beta>0,$$

which is a generalization of formula (2.13) of [1]. From this, we have

$$x\log^k n-\log^k x=(-1)^{k+1}k\int_0^\infty ((x-1)e^{-t}-xe^{-nt}+e^{-xt})S_k(t)\,d\log t,$$

$$\sum_{\nu=1}^{n-1}(\log^k(x+\nu)-\log^k\nu)$$
$$=(-1)^{k+1}k\int_0^\infty\left(\frac{e^{-nt}-e^{-t}}{1-e^{-t}}\,e^{-xt}+\frac{e^{-t}-e^{-nt}}{1-e^{-t}}\right)S_k(t)\,d\log t.$$

Substituting these in (2.2), we conclude that $R_k(x)$ is equal to the right-hand side of (2.5) plus

$$\lim_{n\to\infty}(-1)^k k\int_0^\infty e^{-nt}\left(x-\frac{1-e^{-xt}}{1-e^{-t}}\right)S_k(t)\,d\log t,$$

which is seen to be 0 by Lebesgue's lemma.

It remains to prove (2.6). First we have

$$\int_0^\infty (e^{-\beta t}-e^{-\alpha t})\log^k t\,d\log t=-\int_\alpha^\beta\Big(\int_0^\infty \log^k t\cdot e^{-ut}\,dt\Big)\,du.$$

The inner integral can be evaluated by means of the kth derivative of

$$u^{-x}\Gamma(x)=\int_0^\infty e^{-ut}t^{x-1}\,dt$$

at $x=1$ as

$$\int_0^\infty \log^k t\cdot e^{-ut}\,dt=\sum_{r=0}^k\binom{k}{r}\Gamma^{(k-r)}(1)(-\log u)^r\,\frac{1}{u}.$$

Thus

$$(2.7)\qquad \int_0^\infty (e^{-\beta t}-e^{-\alpha t})\log^k t\,d\log t$$
$$=\sum_{r=0}^k\frac{(-1)^{r+1}}{r+1}\binom{k}{r}\Gamma^{(k-r)}(1)(\log^{r+1}\beta-\log^{r+1}\alpha),$$

which, after inversion, leads to (2.6).

Next, we shall prove an integral representation of $R_k'(x)$, the case $k=1$ of which was used by Zagier to derive an integral representation of $F(x)$.

THEOREM 2. *For $x > 0$,*

$$R'_k(x) - \log^k x = (-1)^k k \int_0^\infty \left(\frac{1}{1-e^{-t}} - \frac{1}{t}\right) S_k(t) e^{-xt}\, dt. \tag{2.8}$$

Proof. Differentiating both sides of (2.5) with respect to x, and substituting $x+1$ for x, we have

$$\begin{aligned} R'_k(x+1) &= (-1)^{k-1} k \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{e^t - 1}\right) S_k(t)\, dt \\ &= (-1)^{k-1} k \int_0^\infty \left[\frac{e^{-t} - e^{-xt}}{t} - \left(\frac{1}{e^t-1} - \frac{1}{t}\right) e^{-xt}\right] S_k(t)\, dt. \end{aligned}$$

Note that the differentiation under the integral sign is justified by Lebesgue's theorem applied to the last expression.

From (2.6), we also have

$$R'_k(x+1) = \log^k x + (-1)^k k \int_0^\infty \left(\frac{1}{e^t-1} - \frac{1}{t}\right) S_k(t) e^{-xt}\, dt. \tag{2.9}$$

On the one hand, (2.1) yields

$$R'_k(x) = R'_k(x+1) - k\,\frac{\log^{k-1} x}{x}, \tag{2.10}$$

and on the other hand, by differentiating (2.6) with $\alpha = 1$ with respect to x $(= \beta)$, we get

$$\int_0^\infty S_k(t) e^{-xt}\, dt = (-1)^{k+1} \frac{\log^{k-1} x}{x}. \tag{2.11}$$

Hence, substituting (2.9) and (2.11) in (2.10), we get the assertion.

We are now ready to introduce the kth order Herglotz–Zagier function $\Phi_k(x)$.

DEFINITION. For $x > 0$,

$$\begin{aligned} \Phi_k(x) &= \sum_{n=1}^\infty \frac{R'_k(nx) - \log^k(nx)}{n} \\ &= (-1)^{k-1} k \int_0^\infty \left(\frac{1}{1-e^{-t}} - \frac{1}{t}\right) S_k(t) \log(1 - e^{-xt})\, dt. \end{aligned} \tag{2.12}$$

We note that $\Phi_1(x)$ coincides with the Herglotz–Zagier function $F(x)$ of Section 1.

3. Laurent expansion of $Z_Q(s)$. It is known by Zagier's theorem that $Z_Q(s)$ has an analytic continuation to the half plane $\mathrm{Re}(s) > 1/2$, with a single pole at $s = 1$. The following two functions will play a fundamental role in the evaluation of the Laurent coefficients of $Z_Q(s)$. For $a, b, c \in \mathbb{N}_0$, let

$$I_{a,b,c}(u,v) = \int_v^u \frac{\log^a x \, \log^b(u-x) \log^c(x-v)}{x} \, dx, \qquad u > v > 0, \tag{3.1}$$

and

$$H(u,v) = \int_0^\infty t^{v-1} \log(1-e^{-ut}) \left(\frac{1}{1-e^{-t}} - \frac{1}{t} \right) dt, \qquad u, v > 0. \tag{3.2}$$

The integral (3.1) can be computed in terms of elementary functions of u and v involving the polylogarithm functions if one of the exponents a, b, c is zero, but in general, it is not the case (see [7], [8]). We hope to study this type of integrals in its full generality elsewhere. Here, we give a few examples for later use.

Lemma 1. *We have*

$$\begin{aligned}
I_{0,0,0} &= \log u - \log v, \\
I_{0,0,1} &= \log v \, \log \frac{u}{v} + \frac{1}{2} \log^2 \frac{v}{u} + \mathrm{Li}_2\left(\frac{v}{u}\right) - \frac{\pi^2}{6}, \\
I_{0,1,0} &= \log u \, \log \frac{u}{v} + \mathrm{Li}_2\left(\frac{v}{u}\right) - \frac{\pi^2}{6}, \\
I_{1,0,0} &= \frac{1}{2} \log^2 u - \frac{1}{2} \log^2 v, \\
I_{2,0,0} &= \frac{1}{3} \log^3 u - \frac{1}{3} \log^3 v, \\
I_{0,0,2} &= -\log^2(u-v) \log \frac{v}{u} - 2 \log(u-v) \, \mathrm{Li}_2\left(1 - \frac{v}{u}\right) + 2 \, \mathrm{Li}_3\left(1 - \frac{u}{v}\right), \\
I_{0,2,0} &= \log^2(u-v) \log \frac{u}{v} + 2 \log(u-v) \, \mathrm{Li}_2\left(1 - \frac{u}{v}\right) - 2 \, \mathrm{Li}_3\left(1 - \frac{u}{v}\right), \\
I_{0,1,1} &= \frac{\pi^2}{6} \log \frac{v}{u} - \frac{\pi^2}{3} \log(u-v) - \log u \, \log v \, \log(u-v) \\
&\quad + \frac{\log^3 v - \log^3 u}{6} + \frac{1}{2} \log^2 u \, \log(v(u-v)^3) + \log^2(u-v) \log \frac{v}{u} \\
&\quad - \frac{1}{2} \log^2 v \, \log(u(u-v)) + 2 \log(u-v) \, \mathrm{Li}_2\left(\frac{v}{u}\right) \\
&\quad + \mathrm{Li}_3\left(1 - \frac{v}{u}\right) - \mathrm{Li}_3\left(1 - \frac{u}{v}\right),
\end{aligned}$$

$$I_{1,0,1} = -\frac{\pi^2}{6}\log v + \frac{\log^3 u - \log^3 v}{3} + \log u \,\mathrm{Li}_2\left(\frac{v}{u}\right) + \mathrm{Li}_3\left(\frac{v}{u}\right) - \zeta(3),$$

$$I_{1,1,0} = -\frac{\pi^2}{6}\log u + \frac{1}{2}\log^3 u - \frac{1}{2}\log^2 v\,\log u + \log v\,\mathrm{Li}_2\left(\frac{v}{u}\right) - \mathrm{Li}_3\left(\frac{v}{u}\right) + \zeta(3),$$

where $\mathrm{Li}_k(z)$ $(k \ge 2)$ *denote the polylogarithm functions*:

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-z)}{z}\,dz, \qquad \mathrm{Li}_k(z) = \int_0^z \frac{\mathrm{Li}_{k-1}(z)}{z}\,dz, \qquad k \ge 3.$$

Proof. See Chapters 1, 6, 8 of [7], especially pp. 5, 154–159, 221, 224, 270, 271.

The following lemma, which follows easily from the definitions (2.12) and (3.2), gives a close relationship between $H(u,v)$ and $\Phi_k(x)$.

LEMMA 2. *We have*

$$\Phi_k(x) = (-1)^{k-1} k \sum_{j=0}^{k-1} a_{k,j}\partial_v^j H(x,1),$$

or conversely

$$\partial_v^k H(x,1) = \sum_{j=0}^{k} b_{k,j+1}\Phi_{j+1}(x),$$

where $b_{k,j+1}$ $(0 \le j \le k-1)$ *are defined recursively by*

$$b_{k,j+1} = -\sum_{r \ge j}^{k-1} a_{k+1,r} b_{r,j+1},$$

starting from $b_{0,1} = 1$, $b_{k,k+1} = (-1)^k (k+1)^{-1}$, *and* $a_{k,j}$ *are as defined in Theorem* 1.

Now we are in a position to state the main result of the paper.

THEOREM 3. *Let* w, w' *be the roots of the quadratic equation* $cw^2 - bw + a = 0$, *labelled so that* $w > w' > 0$. *Then* $Z_Q(s)$ *has a Laurent expansion*

$$Z_Q(s) = \sum_{k=-1}^{\infty} P_k(w,w')(s-1)^k, \qquad P_k(w,w') = \sum_{i+j=k} a_i(b_j + c_j),$$

where

$$a_i = \sum_{l+m=i} \frac{(-1)^l}{l!} \sum_{\substack{s_1+2s_2+\ldots\\+ms_m=m}} \left(\prod_{j=1}^{m} (s_j+1) \left(\frac{-\Gamma^{(j)}(1)}{j!} \right)^{s_j} \right) \log^l(w-w'), \qquad i \geq 0,$$

$$b_j = \frac{2^j}{j!} \sum_{r=0}^{j} b_{j,r+1}(\Phi_{r+1}(w) - \Phi_{r+1}(w')) + \sum_{r=0}^{j-1} \binom{j}{r} \frac{2^r}{j!} \int_{w'}^{w} \log^{j-r}\{(w-u)(u-w')\} \partial_v^r \partial_u H(u,1)\, du, \qquad j \geq 0,$$

$$c_j = \sum_{\substack{l+m+n=j\\ m\geq -1}} \sum_{q+r\leq n} \frac{(-1)^r 2^{l+m+r} \gamma_m \Gamma^{(l)}(1)}{l!n!} \binom{n}{r} \binom{n-r}{q} I_{r,q,n-q-r}(w,w'), \qquad j \geq -1,$$

and γ_m *denotes the generalized Euler constant, i.e. the* m*th Laurent coefficient of the Riemann zeta function.*

Proof. The evaluation of Laurent coefficients in the form given above is based on the following decomposition obtained by Egami [2]:

$$Z_Q(s) = A(s)(B(s) + C(s)), \tag{3.3}$$

where

$$A(s) = \frac{(w-w')^{1-s}}{\Gamma^2(s)},$$

$$B(s) = \int_0^\infty t^{2s-1} \left(\frac{1}{1-e^{-t}} - \frac{1}{t} \right) \int_{w'}^{w} \{(w-u)(u-w')\}^{s-1} \frac{e^{-ut}}{1-e^{-ut}}\, du\, dt,$$

$$C(s) = \Gamma(2s-1)\zeta(2s-1) \int_{w'}^{w} u^{1-2s} \{(w-u)(u-w')\}^{s-1}\, du.$$

The formula for a_i, the ith Taylor coefficient of $A(s)$, follows by noting that

$$\frac{d^k}{dx^k} \left(\frac{1}{\Gamma^2(x)} \right)_{|x=1} = k! \sum_{s_1+2s_2+\ldots+ks_k=k} \prod_{j=1}^{k} (s_j+1) \left(\frac{-\Gamma^{(j)}(1)}{j!} \right)^{s_j}.$$

We need the function $H(u,v)$ for the evaluation of b_j. The following formula enables us to express b_j in terms of higher order Herglotz–Zagier functions (with some additional integrals):

$$B(s) = \int_{w'}^{w} \{(w-u)(u-w')\}^{s-1} \partial_u H(u, 2s-1)\, du.$$

Finally, the formula for c_j, when viewed as the Cauchy product of the Laurent coefficients of the first two factors $\Gamma(2s-1)$, $\zeta(2s-1)$, and of the Taylor coefficients of the last integral $C_1(s)$, say, of $C(s)$, follows from the remark that

$$C_1^{(k)}(1) = \sum_{r=0}^{k} \sum_{j=0}^{k-r} (-2)^r \binom{k}{r} \binom{k-r}{j} I_{r,j,k-r-j}(w, w'),$$

which completes the proof.

Corollary. *For the first three coefficients*, *we have*

$$\begin{aligned}
P_{-1}(w,w') &= \frac{1}{2}\log\frac{w}{w'},\\
P_0(w,w') &= \Phi_1(w) - \Phi_1(w') + \mathrm{Li}_2\left(\frac{w'}{w}\right) - \frac{\pi^2}{6}\\
&\quad + \log\frac{w'}{w}\cdot\left(\gamma_0 - \frac{1}{2}\log(w-w') + \frac{1}{4}\log\frac{w'}{w}\right),\\
P_1(w,w') &= (\Phi_2(w)-\Phi_2(w')) - (\gamma - \log(w-w'))(\Phi_1(w)-\Phi_1(w')) - \frac{\gamma}{3}\pi^2\\
&\quad + \left(2\gamma - \log(w-w') + \log\frac{a}{(w-w')^2}\right)\mathrm{Li}_2\left(\frac{w'}{w}\right) - \mathrm{Li}_3\left(1-\frac{w}{w'}\right)\\
&\quad + \mathrm{Li}_3\left(1-\frac{w'}{w}\right) + \frac{4\gamma_1+\zeta(2)}{2}\log\frac{w}{w'} + \frac{\pi^2}{6}\log(w-w')\\
&\quad + \frac{1}{4}\log a\,\log w\,\log w' - \gamma\log\frac{w}{w'}\log(w-w')\\
&\quad + \frac{\log^3 w - 7\log^3 w'}{12} + \frac{\pi^2}{12}\log\frac{ww'^3}{(w-w')^4}\\
&\quad + \frac{1}{4}\log w\log(w-w')\log\frac{w^5}{w'^3(w-w')^3}\\
&\quad + \frac{1}{4}\log w'\,\log(w-w')\log\frac{w(w-w')^3}{w'^3}\\
&\quad + \int_{w'}^{w}\log\{(w-u)(u-w')\}\partial_u H(u,1)\,du.
\end{aligned}$$

Proof. The formulas for P_{-1} and P_0 are essentially due to Zagier and Egami. Even for these, our method is more transparent and gives immediately

$$P_{-1}(w,w') = \sum_{i+j=-1} a_i(b_j+c_j) = a_0 c_{-1} = \tfrac{1}{2} I_{0,0,0},$$

and similarly

$$\begin{aligned} P_0(w,w') &= \sum_{i+j=0} a_i(b_j+c_j) \\ &= H(w,1) - H(w',1) + \tfrac{1}{2}(I_{0,0,1} + I_{0,1,0} - 2I_{1,0,0}) \\ &\quad + \bigl(\gamma - \tfrac{1}{2}\log(w-w')\bigr) I_{0,0,0}, \end{aligned}$$

whence the result follows from Lemmas 1 and 2.

For P_1, we have

$$\begin{aligned} P_1(w,w') &= \sum_{i+j=1} a_i(b_j+c_j) \\ &= 2(\partial_v H(w,1) - \partial_v H(w',1)) \\ &\quad + \int_{w'}^{w} \log\{(w-u)(u-w')\}\partial_u H(u,1)\,du \\ &\quad + (2\gamma_0 - \log(w-w'))\bigl\{(H(w,1) - H(w',1)) \\ &\qquad\qquad + \tfrac{1}{2}(I_{0,0,1} + I_{0,1,0} - 2I_{1,0,0})\bigr\} \\ &\quad + \tfrac{1}{4}(I_{0,0,2} + 2I_{0,1,1} + I_{0,2,0} - 4I_{1,0,1} - 4I_{1,1,0} + 4I_{2,0,0}) \\ &\quad + (\Gamma^{(2)}(1) + 2\Gamma'(1)\gamma_0 + 2\gamma_1) I_{0,0,0}. \end{aligned}$$

Using Lemma 1, Lemma 2 and the functional equation for polylogarithms ([7, Chapters 1, 6]), we obtain the final form of P_1.

4. Pre-Chowla–Selberg formula. Now coming back to the setting from which we started in Section 1, we consider the class zeta function $\zeta(s,B)$ of a narrow ideal class B of the real quadratic field K of discriminant D. According to Zagier, we have the decomposition

$$D^{s/2}\zeta(s,B^{-1}) = \sum_{l=1}^{l(B)} Z_{Q_l}(s), \tag{4.1}$$

where $l(B)$ denotes the length of the period of the continued fraction expansion of the number $w \in K$ such that $\{1,w\}$ constitutes a basis of some fractional ideal of B (not depending on the choice of w) and $w > 1 > w' > 0$ ($'$ means the conjugate) and Q_l is an indefinite quadratic form with positive real coefficients given by

$$Q_l(x,y) = \frac{1}{w_l - w_l'}\,(y + xw_l)(y + xw_l'),$$

w_l $(1 \le l \le l(B))$ being the elements of K obtained by the cyclic permutation of the cycle of continued fraction expansion of w (see Zagier [9] for all of this).

It is well known that the Dedekind zeta function $\zeta_K(s)$ of K admits two representations, one as the sum of class zeta function $\zeta(s,B)$, and the other as the product of $\zeta(s)$ and the Dirichlet L-function:

$$\zeta_K(s) = \sum_B \zeta(s,B) = \zeta(s)L(s,\chi_D), \tag{4.2}$$

where B runs over the narrow ideal class group of K and χ_D is a primitive Dirichlet character mod D attached to K.

We shall compute the Laurent coefficients of $D^{s/2}\zeta_K(s)$ in two ways based on (4.2) to obtain

THEOREM 4. *Let* $P_i(B) = \sum_{k=1}^{l(B)} P_i(w_k, w_k')$, $i = -1, 0, 1$. *Then*

(1) $\displaystyle\sum_B P_{-1}(B) = 2h_K \log \varepsilon_0,$

(2) $\displaystyle\sum_B P_0(B) = 2\left(2\gamma_0 + \log 2\pi + \frac{\log D}{2}\right) h_K \log\varepsilon_0 - \sum_{\nu=1}^{D-1} \chi_D(\nu) R_2\left(\frac{\nu}{D}\right),$

(3) $\displaystyle\sum_B P_1(B)$

$$\begin{aligned}
&= 2\left\{\frac{3\log 2\pi}{2}\gamma_0 + \frac{5}{4}\gamma_0^2 + \gamma_1 + \frac{\log^2 D}{8}\right. \\
&\qquad \left. + \frac{\log D\cdot(\log 2\pi + 2\gamma_0)}{2} + \frac{\zeta(2) + \log^2 2\pi}{4} - \frac{\pi^2}{16}\right\} h_K \log\varepsilon_0 \\
&\qquad - \frac{\log D + 2\gamma_0}{2} \sum_{\nu=1}^{D-1} \chi_D(\nu) R_2\left(\frac{\nu}{D}\right) \\
&\qquad + \frac{\gamma_0 + \log 2\pi}{2} \sum_{\nu=1}^{D-1} \chi_D(\nu) \log R_2\left(\frac{\nu}{D}\right) + \frac{1}{6}\sum_{\nu=1}^{D-1} \chi_D(\nu) R_3\left(\frac{\nu}{D}\right),
\end{aligned}$$

where B runs over the narrow ideal class group of K, h_K denotes the class number of K, and ε_0 the positive fundamental unit.

Proof. Substituting (4.1) with B^{-1} replaced by B (as we may, since $\zeta(s,B^{-1}) = \zeta(s,B)$) into (4.2), we have

$$\sum_B \left(\sum_{l=1}^{l(B)} Z_{Q_l}(s)\right) = D^{s/2}\zeta_K(s).$$

Then, comparing the Laurent expansions of both sides of the above equality gives

$$\sum_B P_{-1}(B) = \sqrt{D} A_{-1},$$

$$\sum_B P_0(B) = \sqrt{D}\left(A_0 + \frac{\log D}{2}\, A_{-1}\right),$$

$$\sum_B P_1(B) = \sqrt{D}\left(A_1 + \frac{\log D}{2}\, A_0 + \frac{\log^2 D}{8}\, A_{-1}\right),$$

where the A_i denote the Laurent coefficients of $\zeta_K(s)$ at $s = 1$, i.e.

$$\zeta_K(s) = \frac{A_{-1}}{s-1} + A_0 + A_1(s-1) + \dots$$

On the other hand, we can compute the coefficients A_i from the second representation for $\zeta_K(s)$ of (4.2) as

$$\begin{gathered} A_{-1} = L(1,\chi_D), \qquad A_0 = \gamma_0 L(1,\chi_D) + L'(1,\chi_D), \\ A_1 = \gamma_1 L(1,\chi_D) + \gamma_0 L'(1,\chi_D) + \tfrac{1}{2} L''(1,\chi_D), \end{gathered}$$

where we used the expansions

$$\zeta(s) = \sum_{m\ge -1} \gamma_m (s-1)^m, \qquad L(s,\chi_D) = \sum_{m\ge 0} \frac{L^{(m)}(1,\chi_D)}{m!}\,(s-1)^m.$$

Now the results follow from the next lemma.

Lemma 3 ([1], [5]). *For a Dirichlet character χ_D attached to a real quadratic field, we have*

(1) $$L(1,\chi_D) = -\frac{1}{\sqrt{D}} \sum_{\nu=1}^{D-1} \chi_D(\nu) \log 2\, \sin\frac{\pi\nu}{D} = \frac{2h_K \log\varepsilon_0}{\sqrt{D}},$$

(2) $$L'(1,\chi_D) = -\frac{\gamma + \log 2\pi}{\sqrt{D}} \sum_{\nu=1}^{D-1} \chi_D(\nu) \log 2\, \sin\frac{\pi\nu}{D} - \frac{1}{\sqrt{D}} \sum_{\nu=1}^{D-1} \chi_D(\nu) R_2\left(\frac{\nu}{D}\right),$$

(3) $$\begin{aligned} L''(1,\chi_D) &= \frac{1}{\sqrt{D}}\left[\frac{1}{3}\sum_{\nu=1}^{D-1} \chi_D(\nu) R_3\left(\frac{\nu}{D}\right) + (\gamma + \log 2\pi) \sum_{\nu=1}^{D-1} \chi_D(\nu) \log R_2\left(\frac{\nu}{D}\right)\right. \\ &\quad \left. - \left(\frac{\zeta(2) + \gamma^2 + \log^2 2\pi + 2\gamma\log 2\pi}{2} - \frac{\pi^2}{8}\right) \sum_{\nu=1}^{D-1} \chi_D(\nu) \log 2\, \sin\frac{\pi\nu}{D}\right]. \end{aligned}$$

Proof. This is the special case of $\chi = \chi_D$ in [1], [5]. We notice that χ_D is even and since we treat real quadratic fields, the Gauss sum attached to χ_D is equal to $\sqrt{D}$.

Remark. Formula (1) can be proved directly from

$$P_{-1}(B) = \sum_{l=1}^{l(B)} P_{-1}(w_l, w_l') = \log \varepsilon = \begin{cases} \log \varepsilon_0 & \text{if } h_K^* = 2h_K, \\ 2\log \varepsilon_0 & \text{if } h_K^* = h_K, \end{cases}$$

where h_K^* denotes the class number of K in the narrow sense, ε being the smallest totally positive unit > 1 (see [9]).

Acknowledgements. I would like to thank Professor S. Kanemitsu and Professor M. Katsurada for their many suggestions and useful advice. Especially, Prof. S. Kanemitsu read the manuscript carefully and made the obscure parts clear and encouraged the author to write Section 4 which did not exist in the first version of this paper. I also would like to thank the referee for his useful comments.

References

[1] C. Deninger, *On the analogue of the formula of Chowla and Selberg for real quadratic fields*, J. Reine Angew. Math. 351 (1984), 172–191.

[2] S. Egami, *A note on the Kronecker limit formula for real quadratic fields*, Mathematika 33 (1986), 239–243.

[3] M. Gut, *Die Zetafunktion, die Klassenzahl und die Kroneckersche Grenzformel eines beliebigen Kreiskörpers*, Comment. Math. Helv. 1 (1929), 160–226.

[4] G. Herglotz, *Über die Kroneckersche Grenzformel für reelle quadratische Körper I, II*, Ber. Verhandl. Sächsischen Akad. Wiss. Leipzig 75 (1923), 3–14, 31–37.

[5] M. Ishibashi and S. Kanemitsu, *Dirichlet series with periodic coefficients*, Results Math. 35 (1999), 70–88.

[6] S. Kanemitsu, *On evaluation of certain limits in closed form*, in: Théorie des nombres, J.-M. Koninck (ed.), de Gruyter, 1989, 459–474.

[7] L. Lewin, *Polylogarithms and Associated Functions*, Elsevier and North-Holland, 1981.

[8] G. Wechsung, *Functional equations of hyperlogarithms*, in: Structural Properties of Polylogarithms, L. Lewin (ed.), Math. Surveys Monographs 37, Amer. Math. Soc., 1991, 171–184.

[9] D. Zagier, *A Kronecker limit formula for real quadratic fields*, Math. Ann. 213 (1975), 153–184.

Inst. Math. Sci. & Bio.
S.J. Ampio 302
Sasaoka 1-13-24
Chuo-ku, Fukuoka 810-0034, Japan
E-mail: m-isibasi@nifty.ne.jp

Received on 24.4.2001
and in revised form on 20.2.2002 (4019)