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The distribution of Fourier coefficients of cusp forms over sparse sequences

by

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1. Introduction and main results. According to the Langlands program, the "most general" L-function should be a product of L-functions of automorphic cuspidal representations of $\mathrm{GL}_m/\mathbb{Q}$. Therefore these automorphic L-functions do deserve deep investigation. The Hecke L-function is an important automorphic L-function.

Let $S_k(\Gamma)$ be the space of holomorphic cusp forms of even integral weight k for the full modular group $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. Suppose that $f(z)$ is an eigenfunction of all the Hecke operators belonging to $S_k(\Gamma)$. Then the Hecke eigenform $f(z)$ has the following Fourier expansion at the cusp ∞:

$$f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i n z},$$

where we normalize $f(z)$ so that $a_f(1) = 1$. Instead of $a_f(n)$, one often considers the normalized Fourier coefficient

$$\lambda_f(n) = \frac{a_f(n)}{n^{(k-1)/2}}.$$

It is well-known that $\lambda_f(n)$ is real and has the multiplicative property

$$\lambda_f(m)\lambda_f(n) = \sum_{d\mid(m,n)} \lambda_f(mn/d^2), \tag{1.1}$$

where $m, n \geq 1$ are any integers. The Fourier coefficients of cusp forms are interesting objects. In 1974, P. Deligne [2] proved the Ramanujan–Petersson conjecture

$$|\lambda_f(n)| \leq d(n), \tag{1.2}$$

where $d(n)$ is the divisor function.

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The *Hecke L-function* attached to $f \in S_k(\Gamma)$ is defined, for $\mathrm{Re}(s) > 1$, by

$$L(f,s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s}.$$

For the sum of the normalized Fourier coefficients over natural numbers, Rankin [20] proved that

$$S(x) = \sum_{n \le x} \lambda_f(n) \ll x^{1/3} (\log x)^{-\delta},$$

where $0 < \delta < 0.06$.

In 2001, Ivić [6] studied the sum of the normalized Fourier coefficients over squares, i.e.

$$S_2(x) = \sum_{n \le x} \lambda_f(n^2).$$

By using (1.1), the Rankin–Selberg method, and the zero-free region of Riemann zeta function, he gave a nontrivial estimate

$$S_2(x) \ll_f x \exp\bigl(-A (\log x)^{3/5} (\log\log x)^{-1/5}\bigr),$$

where A is a suitable positive constant.

Later Fomenko [3] observed that

$$S_2(x) \ll_f x^{1/2} (\log x)^3.$$

Recently Sankaranarayanan [22] showed that

$$S_2(x) \ll x^{3/4} (\log x)^{19/2} \log\log x$$

uniformly for any holomorphic cusp form of even integral weight k for the full modular group satisfying $k \ll x^{1/3} (\log x)^{22/3}$.

Subsequently by using the properties of symmetric power L-functions, Lü [16] proved that for any $\varepsilon > 0$,

$$S_3(x) = \sum_{n \le x} \lambda_f(n^3) \ll_{f,\varepsilon} x^{3/4+\varepsilon}, \qquad S_4(x) = \sum_{n \le x} \lambda_f(n^4) \ll_{f,\varepsilon} x^{7/9+\varepsilon}.$$

On the other hand, Rankin [19] and Selberg [23] studied the average behavior of $\lambda_f^2(n)$ over natural numbers and showed that

$$\sum_{n \le x} \lambda_f^2(n) = c_1 x + O_f(x^{3/5}),$$

where c_1 is a positive constant depending on f. Recently we studied the asymptotic formula for the sum

$$\sum_{n \le x} \lambda_f^2(n^j), \qquad j = 2, 3, 4.$$

By using the properties of symmetric power L-functions and their Rankin–Selberg L-functions (which have been established in [4], [7], [9], [10], [11], [14], and [24]), in [12] we proved that for any $\varepsilon > 0$, we have

$$\sum_{n\le x} \lambda_f^2(n^j) = c_j x + O_{f,\varepsilon}\big(x^{1-\frac{2}{(j+1)^2+2}+\varepsilon}\big), \qquad j = 2, 3, 4, \tag{1.3}$$

where c_j are suitable constants depending on f.

In this paper we first improve these results by applying the convolution method arguments and a classical lemma of Landau.

THEOREM 1.1. *Let $f(z) \in S_k(\Gamma)$ be a Hecke eigenform of even integral weight k for the full modular group, and let $\lambda_f(n)$ denote its nth normalized Fourier coefficient. Then*

$$\sum_{n\le x} \lambda_f^2(n^j) = c_j x + O_f\big(x^{1-\frac{2}{(j+1)^2+1}}\big), \qquad j = 2, 3, 4.$$

Furthermore by applying an identity among automorphic L-functions and some techniques of analytic number theory, we can still improve Theorem 1.1 for $j = 2$. More precisely, we prove:

THEOREM 1.2. *Let $f(z) \in S_k(\Gamma)$ be a Hecke eigenform of even integral weight k for the full modular group. Then for any $\varepsilon > 0$,*

$$\sum_{n\le x} \lambda_f^2(n^2) = c_2 x + O_{f,\varepsilon}(x^{53/69+\varepsilon}).$$

For comparison, we have $9/11 = 0.818\ldots$ (for $j = 2$ in (1.3)), $4/5 = 0.8$ (for $j = 2$ by Theorem 1.1) and $53/69 = 0.768\ldots$.

2. Some lemmas. According to Deligne [2], for any prime number p there are $\alpha_f(p)$ and $\beta_f(p)$ such that

$$\lambda_f(p) = \alpha_f(p) + \beta_f(p) \quad \text{and} \quad |\alpha_f(p)| = \alpha_f(p)\beta_f(p) = 1. \tag{2.1}$$

The jth symmetric power L-function attached to $f \in S_k(\Gamma)$ is defined as

$$L(\mathrm{sym}^j f, s) := \prod_p \prod_{m=0}^{j} (1 - \alpha_f(p)^{j-m}\beta_f(p)^m p^{-s})^{-1} \tag{2.2}$$

for $\mathrm{Re}(s) > 1$. In particular,

$$L(\mathrm{sym}^0 f, s) = \zeta(s), \quad L(\mathrm{sym}^1 f, s) = L(f, s).$$

In the half-plane $\mathrm{Re}(s) > 1$, we can write $L(\mathrm{sym}^j f, s)$ as a Dirichlet series

$$L(\mathrm{sym}^j f, s) = \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^j f}(n)}{n^s}.$$

The *Rankin–Selberg L-function* associated to $\mathrm{sym}^j f \times \mathrm{sym}^j f$ is defined as

$$\text{(2.3)} \qquad L(\mathrm{sym}^j f \times \mathrm{sym}^j f, s) := \prod_p \prod_{m=0}^{j} \prod_{u=0}^{j} (1 - \alpha_f(p)^{j-m}\beta_f(p)^m \alpha_f(p)^{j-u}\beta_f(p)^u p^{-s})^{-1}$$

for $\mathrm{Re}(s) > 1$.

LEMMA 2.1 (Lao and Sankaranarayanan [12, Lemma 2.1]). *Let $f(z) \in S_k(\Gamma)$ be a Hecke eigenform of even integral weight k for the full modular group. For $j = 2, 3, 4$, we introduce*

$$\text{(2.4)} \qquad L_j(s) := \sum_{n=1}^{\infty} \frac{\lambda_f^2(n^j)}{n^s} \quad \text{for } \mathrm{Re}(s) > 1.$$

Then

$$\text{(2.5)} \qquad L_j(s) = L(\mathrm{sym}^j f \times \mathrm{sym}^j f, s) U_j(s) \quad \text{for } \mathrm{Re}(s) > 1,$$

where $U_j(s)$ converges uniformly and absolutely in the half-plane $\mathrm{Re}(s) \ge 1/2 + \varepsilon$ for any $\varepsilon > 0$.

LEMMA 2.2. *For $\mathrm{Re}(s) > 1$, we have*

$$L(\mathrm{sym}^2 f \times \mathrm{sym}^2 f, s) = \zeta(s) L(\mathrm{sym}^2 f, s) L(\mathrm{sym}^4 f, s).$$

Proof. This follows from (2.2) with $j = 0, 2, 4$, and from (2.3) with $j = 3$. ∎

Based on the work of Cogdell and Michel [1], Lau and Wu [14] showed that for $j = 2, 3, 4$, $L(\mathrm{sym}^j f, s)$ and $L(\mathrm{sym}^j f \times \mathrm{sym}^j f, s)$ have meromorphic continuations to the whole complex plane, and satisfy a functional equation.

LEMMA 2.3 (Cogdell and Michel [1, Section 3.2.1]). *Let $f(z) \in S_k(\Gamma)$ be a Hecke eigencuspform of even integral weight k. For $j = 2, 3, 4$, the archimedean local factor of $L(\mathrm{sym}^j f, s)$ is*

$$L_\infty(\mathrm{sym}^j f, s) = \begin{cases} \prod_{v=0}^{n} \Gamma_{\mathbb{C}}(s + (v + 1/2)(k-1)) & \text{if } j = 2n+1, \\ \Gamma_{\mathbb{R}}(s + \delta_{2\nmid n}) \prod_{v=1}^{n} \Gamma_{\mathbb{C}}(s + v(k-1)) & \text{if } j = 2n, \end{cases}$$

where $\Gamma_{\mathbb{R}} = \pi^{-s/2}\Gamma(s/2)$, $\Gamma_{\mathbb{C}} = 2(2\pi)^{-s}\Gamma(s)$, and $\delta_{2\nmid n}$ is 1 when 2 does not divide n, and 0 otherwise.

For $2 \le j \le 4$, the complete L-function

$$\Lambda(\mathrm{sym}^j f, s) = L_\infty(\mathrm{sym}^j f, s) L(\mathrm{sym}^j f, s)$$

is an entire function on $\mathbb{C}$, and satisfies the functional equation

$$\Lambda(\mathrm{sym}^j f, s) = \epsilon_{\mathrm{sym}^j f} \Lambda(\mathrm{sym}^j f, 1-s),$$

where $\epsilon_{\mathrm{sym}^j f} = \pm 1$.

LEMMA 2.4 (Lau and Wu [14, Proposition 2.1]). *Let $f(z) \in S_k(\Gamma)$ be a Hecke eigenform of even integral weight k. For $j = 2, 3, 4$, the archimedean local factor of $L(\mathrm{sym}^j f \times \mathrm{sym}^j f, s)$ is*

$$L_\infty(\mathrm{sym}^j f \times \mathrm{sym}^j f, s) = \Gamma_{\mathbb{R}}(s)^{\delta_{2|j}} \Gamma_{\mathbb{C}}(s)^{[j/2]+\delta_{2\nmid j}} \prod_{v=1}^{j} \Gamma_{\mathbb{C}}(s + v(k-1))^{j-v+1},$$

where $\delta_{2|j} = 1 - \delta_{2\nmid j}$. The complete L-function

$$\Lambda(\mathrm{sym}^j f \times \mathrm{sym}^j f, s) := L_\infty(\mathrm{sym}^j f \times \mathrm{sym}^j f, s) L(\mathrm{sym}^j f \times \mathrm{sym}^j f, s)$$

is entire except possibly for simple poles at $s = 0, 1$ and satisfies the functional equation

$$\Lambda(\mathrm{sym}^j f \times \mathrm{sym}^j f, s) = \epsilon_{\mathrm{sym}^j f \times \mathrm{sym}^j f} \Lambda(\mathrm{sym}^j f \times \mathrm{sym}^j f, 1-s)$$

with $|\epsilon_{\mathrm{sym}^j f \times \mathrm{sym}^j f}| = 1$.

LEMMA 2.5. *For any $\varepsilon > 0$, $\sigma \geq 1/2$, and $|t| \geq 2$, we have*

$$\zeta(\sigma + it) \ll_\varepsilon (1 + |t|)^{\max\{\frac{1}{3}(1-\sigma), 0\}+\varepsilon},$$

$$L(\mathrm{sym}^2 f, \sigma + it) \ll_{f,\varepsilon} (1 + |t|)^{\max\{\frac{11}{8}(1-\sigma), 0\}+\varepsilon},$$

$$L(\mathrm{sym}^j f, \sigma + it) \ll_{f,\varepsilon} (1 + |t|)^{\max\{\frac{j+1}{2}(1-\sigma), 0\}+\varepsilon}, \qquad j = 3, 4.$$

Proof. For any $\varepsilon > 0$, we have (see [18])

$$\zeta(\sigma + it) \ll_\varepsilon (1 + |t|)^{\frac{1}{3}(1-\sigma)+\varepsilon}, \qquad 1/2 \leq \sigma \leq 1,\ |t| \geq 2.$$

The estimate

$$L(\mathrm{sym}^2 f, \sigma + it) \ll_{f,\varepsilon} (1 + |t|)^{\frac{11}{8}(1-\sigma)+\varepsilon}, \qquad 1/2 \leq \sigma \leq 1,\ |t| \geq 2,$$

is due to X. Q. Li [15]. From Lemma 2.3, we have

$$L(\mathrm{sym}^j f, \sigma + it) \ll_{f,\varepsilon} (1 + |t|)^{\frac{j+1}{2}(1-\sigma)+\varepsilon}, \qquad 1/2 \leq \sigma \leq 1,\ |t| \geq 2,\ j = 3, 4.$$

The claim for $\sigma > 1$ holds by the absolute convergence of the Dirichlet series involved, which follows from (1.2). ∎

LEMMA 2.6. *Let $j = 2, 3, 4$. Then for $T \geq T_0$ (where T_0 is sufficiently large),*

$$\int_T^{2T} |L(\mathrm{sym}^j f, 1/2 + \varepsilon + it)|^2\, dt \ll_{f,\varepsilon} T^{\frac{j+1}{2}+\varepsilon},$$

where ε is any positive constant.

Proof. From (2.2), the L-function $L\left(\mathrm{sym}^j f, s\right)$ is of degree $j + 1$. Lemma 2.4 shows that the L-function $L\left(\mathrm{sym}^j f, s\right)$ can be extended as an entire function and also satisfy a nice functional equation of the Riemann zeta type. Thus we can write the functional equation here as

$$L(\mathrm{sym}^j f, s) = \chi(s) L(\mathrm{sym}^j f, 1-s),$$

where

$$|\chi(s)| \asymp |t|^{\frac{j+1}{2}(1-2\sigma)} \quad \text{as } |t| \to \infty$$

uniformly in any fixed strip $a \le \sigma \le b$. Now we follow the arguments of Sankaranarayanan [21, Theorem 4.1(i)]. The only necessary changes are that we need the free parameters Y and Y_1 therein to be $Y = Y_1 = cT^{(j+1)/2}$, where c is a suitable positive constant. This leads to the estimate of this lemma. ∎

LEMMA 2.7 (Heath-Brown [5]). *For $T \ge 1$,*

$$\int_1^T |\zeta(1/2+it)|^{12}\,dt \ll T^{2+\varepsilon}.$$

LEMMA 2.8. *Let $a_n \ge 0$ and set*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

Suppose $f(s)$ is convergent in some half-plane and has an analytic continuation, except for a pole at $s = \alpha$ of order k, to the entire complex plane and it satisfies a functional equation

$$c^s \Delta(s) f(s) = c^{1-s} \Delta(1-s) f(1-s),$$

where c is a positive constant and $\Delta(s) = \prod_{i=1}^{N} \Gamma(\alpha_i s + \beta_i)$ $(\alpha_i > 0)$. Then

$$\sum_{n \le x} a_n = x^{\alpha} P_{k-1}(\log x) + O(x^{\alpha(1-\frac{2}{2A+1})} \log^{k-1} x),$$

where $A = \sum_{i=1}^{N} \alpha_i$ and $P_{k-1}(y)$ is a polynomial in y of degree $k-1$.

Proof. This is one of the many possible versions of a classical lemma of Landau. See for e.g. Murty [17, Lemma 1]. ∎

3. Proof of Theorem 1.1.

The product over primes in (2.3) gives a Dirichlet series representation

$$L(\mathrm{sym}^j f \times \mathrm{sym}^j f, s) = \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^j f \times \mathrm{sym}^j f}(n)}{n^s} \quad \text{for } \mathrm{Re}(s) > 1,$$

where $\lambda_{\mathrm{sym}^j f \times \mathrm{sym}^j f}(n)$ is nonnegative in view of [13, Lemma 3.1(a)]. By Lemma 2.4, $L(\mathrm{sym}^j f \times \mathrm{sym}^j f, s)$ satisfies the conclusion of Lemma 2.8 with $\alpha = 1$, $k = 1$, and $2A = (j+1)^2$. Then we have

$$\sum_{n \le x} \lambda_{\mathrm{sym}^j f \times \mathrm{sym}^j f}(n) = d_j x + O\big(x^{1-\frac{2}{(j+1)^2+1}}\big),$$

where d_j is a suitable constant depending on f. By Lemma 2.1,

$$\lambda_f^2(n^j) = \sum_{n=ml} \lambda_{\mathrm{sym}^j f\times \mathrm{sym}^j f}(m) u_j(l),$$

where

$$\sum_{l\le x} |u_j(l)| l^{-v} \ll 1 \quad \text{for } v \ge 1/2+\varepsilon.$$

Hence

$$\begin{aligned}\sum_{n\le x} \lambda_f^2(n^j) &= \sum_{ml\le x} \lambda_{\mathrm{sym}^j f\times \mathrm{sym}^j f}(m) u_j(l) = \sum_{l\le x} u_j(l) \sum_{m\le x/l} \lambda_{\mathrm{sym}^j f\times \mathrm{sym}^j f}(m)\\ &= \sum_{l\le x} u_j(l)\big\{d_j(x/l) + O\big((x/l)^{1-\frac{2}{(j+1)^2+1}}\big)\big\}\\ &=: c_j x + O\big(x^{1-\frac{2}{(j+1)^2+1}}\big).\end{aligned}$$

This completes the proof of Theorem 1.1.

4. Proof of Theorem 1.2.

Recall that

$$L_2(s) = \sum_{n=1}^{\infty} \frac{\lambda_f^2(n^2)}{n^s} \quad \text{for } \mathrm{Re}(s) > 1.$$

From Lemmas 2.1 and 2.2, we observe that

$$L_2(s) = L(\mathrm{sym}^2 f\times \mathrm{sym}^2 f, s) U_2(s) = \zeta(s) L(\mathrm{sym}^2 f, s) L(\mathrm{sym}^4 f, s) U_2(s)$$

can be meromorphically continued to the half-plane $\mathrm{Re}(s) > 1/2$. In this region, $L_2(s)$ has only a simple pole at $s=1$.

Now, we begin to prove Theorem 1.2. By Perron's formula (see [8, Proposition 5.54]), we have

$$\sum_{n\le x} \lambda_f^2(n^2) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} L_2(s) \frac{x^s}{s}\, ds + O(x^{1+\varepsilon}/T),$$

where $b = 1+\varepsilon$ and $1\le T\le x$ is a parameter to be chosen later. Here we have used (1.2).

Next we move the integration to the parallel segment with $\mathrm{Re}(s) = 1/2+\varepsilon$. By Cauchy's residue theorem, we have

$$\begin{aligned}\sum_{n\le x} \lambda_f^2(n^2) &= \frac{1}{2\pi i}\Big\{ \int_{1/2+\varepsilon-iT}^{1/2+\varepsilon+iT} + \int_{1/2+\varepsilon+iT}^{b+iT} + \int_{b-iT}^{1/2+\varepsilon-iT} \Big\} L_2(s) \frac{x^s}{s}\, ds\\ &\quad + \mathrm{Res}_{s=1}(L_2(s)x^s/s) + O(x^{1+\varepsilon}/T)\\ &=: I_1 + I_2 + I_3 + c_2 x + O(x^{1+\varepsilon}/T).\end{aligned} \tag{4.1}$$

For I_1, by Lemma 2.1,

$$\begin{aligned} I_1 &\ll x^{1/2+\varepsilon} \\ &\quad + x^{1/2+\varepsilon} \int_1^T |L(\mathrm{sym}^2 f \times \mathrm{sym}^2 f, 1/2+\varepsilon+it) U_j(1/2+\varepsilon+it)| t^{-1}\, dt \\ &\ll x^{1/2+\varepsilon} + x^{1/2+\varepsilon} \int_1^T |L(\mathrm{sym}^2 f \times \mathrm{sym}^2 f, 1/2+\varepsilon+it)| t^{-1}\, dt. \end{aligned}$$

Therefore

$$\begin{aligned} I_1 &\ll x^{1/2+\varepsilon} \\ &\quad + x^{1/2+\varepsilon} \sum_{1\le j\le [\frac{\log T}{\log 2}]+1} \int_{T/2^j}^{T/2^{j-1}} |L(\mathrm{sym}^2 f \times \mathrm{sym}^2 f, 1/2+\varepsilon+it)| t^{-1}\, dt \\ &\ll x^{1/2+\varepsilon} \\ &\quad + x^{1/2+\varepsilon} \log T \max_{T_1\le T} \bigg\{ \frac{1}{T_1} \int_{T_1/2}^{T_1} |L(\mathrm{sym}^2 f \times \mathrm{sym}^2 f, 1/2+\varepsilon+it)|\, dt \bigg\}. \end{aligned}$$

Using the decomposition in Lemma 2.2, by Hölder's inequality, we have

$$\begin{aligned} I_1 \ll x^{1/2+\varepsilon} + x^{1/2+\varepsilon} \log T \max_{T_1\le T} \bigg\{ & \frac{1}{T_1} \Big(\int_{T_1/2}^{T_1} |\zeta(1/2+\varepsilon+it)|^{12}\, dt \Big)^{1/12} \\ & \times \Big(\int_{T_1/2}^{T_1} |L(\mathrm{sym}^2 f, 1/2+\varepsilon+it)|^{12/5}\, dt \Big)^{5/12} \\ & \times \Big(\int_{T_1/2}^{T_1} |L(\mathrm{sym}^4 f, 1/2+\varepsilon+it)|^{2}\, dt \Big)^{1/2} \bigg\}. \end{aligned}$$

Furthermore,

$$\begin{aligned} I_1 &\ll x^{1/2+\varepsilon} + x^{1/2+\varepsilon} \log T \max_{T_1\le T} \bigg\{ \frac{1}{T_1} \Big(\int_{T_1/2}^{T_1} |\zeta(1/2+\varepsilon+it)|^{12}\, dt \Big)^{1/12} \\ &\times \Big(\max_{T_1/2\le t\le T_1} |L(\mathrm{sym}^2 f, 1/2+\varepsilon+it)|^{2/5} \int_{T_1/2}^{T_1} |L(\mathrm{sym}^2 f, 1/2+\varepsilon+it)|^{2}\, dt \Big)^{5/12} \\ &\times \Big(\int_{T_1/2}^{T_1} |L(\mathrm{sym}^4 f, 1/2+\varepsilon+it)|^{2}\, dt \Big)^{1/2} \bigg\}. \end{aligned}$$

After applying Lemmas 2.5–2.7, we have

$$I_1 \ll x^{1/2+\varepsilon} + x^{1/2+\varepsilon} T^{\frac{1}{6}+(\frac{2}{5}\times\frac{11}{16}+\frac{3}{2})\times\frac{5}{12}+\frac{5}{4}-1+\varepsilon} \ll x^{1/2+\varepsilon} T^{37/32+\varepsilon}. \tag{4.2}$$

For the integrals over the horizontal segments, we use Lemmas 2.2 and 2.5 to get

$$\begin{aligned} I_2 + I_3 &\ll \int_{1/2+\varepsilon}^{b} x^{\sigma} |L(\mathrm{sym}^2 f \times \mathrm{sym}^2 f, \sigma + iT)| T^{-1} \, d\sigma \\ &\ll \max_{1/2+\varepsilon \le \sigma \le b} x^{\sigma} T^{(\frac{1}{3}+\frac{11}{8}+\frac{5}{2})(1-\sigma)+\varepsilon} T^{-1} + x^{1+\varepsilon}/T \\ &\ll \max_{1/2+\varepsilon \le \sigma \le b} \left(\frac{x}{T^{101/24}}\right)^{\sigma} T^{101/24-1+\varepsilon} + \frac{x^{1+\varepsilon}}{T} \\ &\ll \left(\frac{x}{T^{101/24}}\right)^{b} T^{101/24-1+\varepsilon} + \left(\frac{x}{T^{101/24}}\right)^{1/2+\varepsilon} T^{101/24-1+\varepsilon} + \frac{x^{1+\varepsilon}}{T} \\ &\ll \frac{x^{1+\varepsilon}}{T} + x^{1/2+\varepsilon} T^{53/48+\varepsilon}. \end{aligned} \tag{4.3}$$

From (4.1)–(4.3), we have

$$\sum_{n \le x} \lambda_f^2(n^2) = c_2 x + O(x^{1+\varepsilon}/T) + O(x^{1/2+\varepsilon} T^{37/32+\varepsilon}). \tag{4.4}$$

On taking $T = x^{16/69}$ in (4.4), we conclude that

$$\sum_{n \le x} \lambda_f^2(n^2) = c_2 x + O(x^{53/69+\varepsilon}).$$

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