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Factors of a perfect square

by

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1. Introduction and main result. In [ErdRos], Erdős and Rosenfeld considered the differences between the divisors of a positive integer n. They exhibited infinitely many integers with four "small" differences and posed the question whether any positive integer can have at most a bounded number of "small" differences. Specifically, they asked

Question 1.1. *Is there an absolute constant K such that, for every c, the number of divisors of n between $\sqrt{n} - c\sqrt[4]{n}$ and $\sqrt{n} + c\sqrt[4]{n}$ is at most K for $n > n_0(c)$?*

They also mentioned a conjecture of Ruzsa which is a stronger question:

Question 1.2. *Given $\epsilon > 0$, is there a constant K_ϵ such that, for any positive integer n, the number of divisors of n between $n^{1/2} - n^{1/2-\epsilon}$ and $n^{1/2} + n^{1/2-\epsilon}$ is at most K_ϵ?*

In this paper, we consider both questions when n is a perfect square. In particular, we have

Theorem 1.3. *For every $c \geq 3$, any perfect square $n = N^2$ can have at most five divisors between $\sqrt{n} - c\sqrt[4]{n}$ and $\sqrt{n} + c\sqrt[4]{n}$ for $n > e^{Cc^6(\log c)^5}$ where C is some sufficiently large constant independent of c.*

Corollary 1.4. *Any sufficiently large perfect square $n = N^2$ has at most five divisors between $\sqrt{n} - \sqrt[4]{n}\,(\log n)^{1/7}$ and $\sqrt{n} + \sqrt[4]{n}\,(\log n)^{1/7}$.*

Proof. When $n = N^2$ is sufficiently large (in terms of C), we have $n > e^{Cc^6(\log c)^5}$ with $c = (\log n)^{1/7} \geq 3$. Pick this c for Theorem 1.3. Then the theorem implies Corollary 1.4.

Theorem 1.3 answers Question 1.1 for perfect squares with $K = 5$ while Corollary 1.4 shows that we can take the range for the divisors to be slightly longer than $\sqrt[4]{n}$. Based on the proof of Theorem 1.3, every perfect square n

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with five divisors between $\sqrt{n}-c\sqrt[4]{n}$ and $\sqrt{n}+c\sqrt[4]{n}$ comes from solutions to Pell equations. For example, consider the Pell equation $X^2-2Y^2=2$. It has integral solutions (X_k,Y_k) generated by $X_k+\sqrt{2}\,Y_k=(3+2\sqrt{2})^k(2+\sqrt{2})$. We can verify by induction that X_k are even, Y_k are odd and $(X_k-2)(X_k+2)=2(Y_k-1)(Y_k+1)$. Now consider the perfect square $n=(X_k-2)^2(X_k+2)^2=4(Y_k-1)^2(Y_k+1)^2$. It has divisors $2(Y_k-1)^2$, $(X_k-2)^2$, $(X_k-2)(X_k+2)$, $(X_k+2)^2$, $2(Y_k+1)^2$ that are between $\sqrt{n}-5\sqrt[4]{n}$ and $\sqrt{n}+5\sqrt[4]{n}$. This shows that the constant $K=5$ is best possible for Question 1.1 restricted to perfect squares.

2. Proof of Theorem 1.3. Throughout the proof, we assume that $n>e^{Cc^6(\log c)^5}$ for some $C>4$. We will specify C at the end of the proof. Suppose $N^2=(N-d_1)(N+e_1)=(N-d_2)(N+e_2)=\cdots=(N-d_r)(N+e_r)$ where $N, N-d_i, N+e_j$ are all the divisors of N^2 that lie in $[N-cN^{1/2}, N+cN^{1/2}]$, where $1\le d_1<d_2<\cdots<d_r\le cN^{1/2}$ and $1\le e_1<e_2<\cdots<e_r\le cN^{1/2}$ are positive integers. As $N^2=(N-d_i)(N+e_i)$, $0<e_id_i=(e_i-d_i)N$. So we must have $e_i>d_i$, say $l_i:=e_i-d_i$ for some positive integer l_i. From $(d_i+l_i)d_i=l_iN$, we have

$$l_i=\frac{d_i^2}{N-d_i}\le\frac{c^2N}{N-cN^{1/2}}\le 2c^2 \tag{2.1}$$

for $N\ge 4c^2$ (which is true when $n>e^{Cc^6(\log c)^5}$).

As $(N-d_i)(N+e_i)=N^2$ is a perfect square, $N-d_i$ and $N+e_i$ have the same squarefree part. Hence

$$N-d_i=a_ix_i^2,\qquad N+e_i=a_iy_i^2,\qquad N=a_ix_iy_i,$$

where a_i is some squarefree integer. Adding the first two equations and subtracting twice the third one, we get

$$l_i=e_i-d_i=a_i(x_i-y_i)^2, \tag{2.2}$$

which gives $1\le a_i\le l_i\le 2c^2$. Similarly, by adding the first two equations and twice the third one, we get

$$4N+l_i=a_i(x_i+y_i)^2. \tag{2.3}$$

Suppose $n=N^2$ has more than five divisors in the interval $[\sqrt{n}-c\sqrt[4]{n}, \sqrt{n}+c\sqrt[4]{n}]$. Then $r\ge 3$ and (2.3) is true for $i=1,2,3$. Subtracting among these three equations and letting $z_i=x_i+y_i$, we arrive at a pair of Pell equations:

$$\begin{cases} a_1z_1^2-a_2z_2^2=l_1-l_2,\\ a_1z_1^2-a_3z_3^2=l_1-l_3.\end{cases} \tag{2.4}$$

Here we need a result of Turk [Tur] on simultaneous Pell equations:

THEOREM 2.1. *Let a, b, c, d be squarefree positive integers with $a \neq b$ and $c \neq d$ and let e and f be any integers. If $af = ce$ then we also assume that $abcd$ is not a perfect square. Then every positive integer solution of*

$$\begin{cases} ax^2 - by^2 = e, \\ cx^2 - dz^2 = f \end{cases}$$

satisfies

$$\max(x, y, z) < e^{C\alpha^2 (\log \alpha)^3 \gamma \log \gamma}$$

where $\alpha = \max(a, b, c, d)$, $\beta = \max(|e|, |f|, 3)$, $\gamma = \max(\alpha \log \alpha, \log \beta)$ and C is a large absolute constant.

From (2.3) and $a_i \leq 2c^2$, we have $z_i \geq \sqrt{N}/c > 2c^2$ if $N > 4c^6$ (which is true if $n > e^{Cc^6 (\log c)^5}$). If $a_1 = a_2$, then (2.4) gives $a_1 |z_1^2 - z_2^2| = |l_1 - l_2| \leq 2c^2$. Clearly $z_1 \neq z_2$, for otherwise $l_1 = l_2$, which forces $x_1 = x_2$ and $y_1 = y_2$ after substituting $l_1 = l_2$ into (2.2) and (2.3). Hence $z_1, z_2 \leq |z_1 + z_2| \leq a_1 |z_1 - z_2| \, |z_1 + z_2| \leq 2c^2$, which contradicts $z_i > 2c^2$. Therefore this case cannot happen. Similarly, $a_1 = a_3$ cannot happen. Now if $a_1 a_2 a_1 a_3$ is a perfect square, then $a_2 = a_3$ since they are all squarefree numbers. Subtracting the two equations in (2.4), we get $a_2 z_2^2 - a_2 z_3^2 = l_2 - l_3$, which implies $z_2, z_3 \leq 2c^2$ by a similar argument to the above. This contradicts $z_i > 2c^2$.

Therefore we can apply Theorem 2.1 to (2.4) with $\alpha \leq 2c^2$, $\beta \leq 2c^2$, $\gamma \leq 6c^2 \log c$ and get $z_i < e^{C' c^6 (\log c)^5}$ for some sufficiently large absolute constant C', which contradicts $z_i \geq \sqrt{N}/c > e^{C' c^6 (\log c)^5}$ if $n = N^2 > e^{3C' c^6 (\log c)^5}$. This proves Theorem 1.3 with absolute constant $C = \max(3C', 5)$.

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