

The sum of divisors of a quadratic form

by

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1. Introduction. Let $\tau(n)$ denote the number of divisors of n . Gafurov [3, 4] investigated the sum

$$S_2(X) = \sum_{1 \leq m, n \leq X} \tau(m^2 + n^2),$$

and obtained the asymptotic formula

$$S_2(X) = A_1 X^2 \log X + A_2 X^2 + O(X^{5/3} \log^9 X),$$

where A_1 and A_2 are certain constants. This was improved by Yu [10], who proved that

$$S_2(X) = A_1 X^2 \log X + A_2 X^2 + O_\varepsilon(X^{3/2+\varepsilon})$$

for any fixed positive real number ε .

Calderón and de Velasco [1] studied the divisors of the quadratic form $m_1^2 + m_2^2 + m_3^2$. Let

$$(1.1) \quad S_3(X) = \sum_{1 \leq m_1, m_2, m_3 \leq X} \tau(m_1^2 + m_2^2 + m_3^2).$$

Calderón and de Velasco established the asymptotic formula

$$(1.2) \quad S_3(X) = \frac{8\zeta(3)}{5\zeta(5)} X^3 \log X + O(X^3).$$

Let

$$(1.3) \quad \mathfrak{S}_1 = \sum_{q=1}^{\infty} \frac{1}{q^4} \sum_{\substack{a=1 \\ (a,q)=1}}^q \left(\sum_{x=1}^q e\left(\frac{ax^2}{q}\right) \right)^3,$$

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and

$$(1.4) \quad \mathfrak{S}_2 = \sum_{q=1}^{\infty} \frac{2\gamma - 2 \log q}{q^4} \sum_{\substack{a=1 \\ (a,q)=1}}^q \left(\sum_{x=1}^q e\left(\frac{ax^2}{q}\right) \right)^3.$$

Define

$$\begin{aligned} \mathcal{I}_1 &= \int_{-\infty}^{\infty} \left(\int_0^1 e(x^2\beta) dx \right)^3 \left(\int_0^3 e(-x\beta) dx \right) d\beta, \\ \mathcal{I}_2 &= \int_{-\infty}^{\infty} \left(\int_0^1 e(x^2\beta) dx \right)^3 \left(\int_0^3 (\log x) e(-x\beta) dx \right) d\beta. \end{aligned}$$

Recently, Guo and Zhai [5] improved (1.2) to

$$S_3(X) = \frac{8\zeta(3)}{5\zeta(5)} X^3 \log X + (\mathfrak{S}_1 \mathcal{I}_2 + \mathfrak{S}_2 \mathcal{I}_1) X^3 + O_{\varepsilon}(X^{8/3+\varepsilon}),$$

where ε is an arbitrary positive number.

The purpose of this paper is to prove the following result:

THEOREM 1.1. *Let $S_3(X)$ be defined in (1.1). Then*

$$(1.5) \quad S_3(X) = \frac{8\zeta(3)}{5\zeta(5)} X^3 \log X + (\mathfrak{S}_1 \mathcal{I}_2 + \mathfrak{S}_2 \mathcal{I}_1) X^3 + O(X^2 \log^7 X).$$

It is worth pointing out that the error term $O(X^2 \log^7 X)$ in (1.5) cannot be replaced by $o(X^2 \log X)$. Otherwise, on considering $X = N + 1/3$ and $X = N + 2/3$ with N a positive integer, for $j = 1, 2$ we have

$$S_3(N + j/3) = C_1(N + j/3)^3 \log(N + j/3) + C_2(N + j/3)^3 + o(N^2 \log N).$$

This leads to a contradiction since $S_3(N + 1/3) = S_3(N + 2/3)$. Therefore, the room for further improvement is no more than $O(\log^6 X)$.

2. Preliminaries. Throughout this paper, we assume that X is a sufficiently large real number. Let $P = \lfloor 5X \rfloor$, where $\lfloor x \rfloor$ denotes the greatest integer not exceeding the real number x .

Our initial approach is to install a smooth weight in our problem. Let $w_0(x)$ be a function on $[0, \infty)$ satisfying $w_0(x) = 1$ for $x = [0, 3]$, $w_0(x) = 0$ for $x \geq 4$ and such that $w_0''(x)$ exists for $x > 0$. Let $w_1(x)$ be a function supported on $[1/2, 3]$ for which $w_1(x) = 1$ for $1 < x < 2$ and $w_1''(x)$ exists for $0 < x < 2$. Now we introduce the function

$$w(x) = \begin{cases} w_1(x) & \text{if } x < 1, \\ 1 & \text{if } 1 \leq x \leq 3X^2, \\ w_0(x/X^2) & \text{if } x > 3X^2. \end{cases}$$

Note that $w(x)$ is a smooth function supported on $[1/2, 4X^2]$. By (1.1),

$$S_3(X) = \sum_{1 \leq m_1, m_2, m_3 \leq X} \tau(m_1^2 + m_2^2 + m_3^2) = \sum_{\substack{1 \leq m_1, m_2, m_3 \leq X \\ 1 \leq n \leq 3X^2 \\ m_1^2 + m_2^2 + m_3^2 = n}} \tau(n).$$

Since $m_1, m_2, m_3 \leq X$ and $m_1^2 + m_2^2 + m_3^2 = n$ together imply $n \leq 3X^2$, we conclude that

$$S_3(X) = \sum_{\substack{1 \leq m_1, m_2, m_3 \leq X \\ n \\ m_1^2 + m_2^2 + m_3^2 = n}} \tau(n)w(n).$$

In order to apply the circle method, we introduce the exponential sums

$$f(\alpha) = \sum_{1 \leq m \leq X} e(m^2\alpha), \quad h(\alpha) = \sum_n w(n)\tau(n)e(n\alpha),$$

where we write $e(z)$ for $e^{2\pi iz}$. By orthogonality,

$$(2.1) \quad S_3(X) = \int_0^1 f(\alpha)^3 h(-\alpha) d\alpha.$$

Let g be a smooth, compactly supported function on $(0, \infty)$. The Voronoi-type summation formula asserts that

$$(2.2) \quad \sum_{n=1}^{\infty} \tau(n)e\left(\frac{an}{q}\right)g(n) = q^{-1} \int (\log x + 2\gamma - 2 \log q)g(x) dx + \sum_{n=1}^{\infty} \tau(n)e\left(-\frac{\bar{a}n}{q}\right)\tilde{g}_Y(n) + \sum_{n=1}^{\infty} \tau(n)e\left(\frac{\bar{a}n}{q}\right)\tilde{g}_K(n),$$

where $(a, q) = 1$, \bar{a} denotes the inverse of a modulo q and

$$\tilde{g}_Y(y) = -\frac{2\pi}{q} \int_0^{\infty} g(x)Y_0\left(\frac{4\pi\sqrt{xy}}{q}\right) dx, \quad \tilde{g}_K(y) = \frac{4}{q} \int_0^{\infty} g(x)K_0\left(\frac{4\pi\sqrt{xy}}{q}\right) dx.$$

Here Y_0 and K_0 denote the standard Bessel functions. For the proof of the above well-known formula, one may refer to Iwaniec and Kowalski [8, Section 4.5]. For $0 < x \ll 1$, one has

$$(2.3) \quad Y_0(x) = \frac{2}{\pi} \log \frac{x}{2} + O(1), \quad K_0(x) = \log \frac{2}{x} + O(1)$$

and

$$(2.4) \quad Y_\nu(x), K_\nu(x) \ll_\nu x^{-\nu} \quad \text{for real } \nu > 0.$$

If $\nu \geq 0$ and $x > 1 + \nu^2$, then

$$(2.5) \quad \begin{aligned} Y_\nu(x) &= \sqrt{\frac{2}{\pi x}} \left(\sin(x - \nu\pi/2 - \pi/4) + O\left(\frac{1 + \nu^2}{x}\right) \right), \\ K_\nu(x) &= \sqrt{\frac{2}{\pi x}} e^{-x} \left(1 + O\left(\frac{1 + \nu^2}{x}\right) \right). \end{aligned}$$

Their derivatives satisfy recurrence relations: for any $\nu \geq 0$,

$$(2.6) \quad \frac{d}{dx}(x^{\nu+1}B_{\nu+1}(x)) = \epsilon x^{\nu+1}B_\nu(x)$$

where $\epsilon = 1$ for $B = Y$ and $\epsilon = -1$ for $B = K$.

3. The properties of the Gauss sum. We define

$$S(q, a, n) = \sum_{x=1}^q e\left(\frac{ax^2 + nx}{q}\right) \quad \text{and} \quad S(q, a) = S(q, a, 0).$$

LEMMA 3.1. *Suppose that $(q_1, q_2) = 1$. Then*

$$S(q_1q_2, a_1q_2 + a_2q_1, n) = S(q_1, a_1q_2^2, n)S(q_2, a_2q_1^2, n).$$

LEMMA 3.2. *Suppose that $(2a, q) = 1$. Then*

$$S(q, a, n) = e\left(-\frac{4an^2}{q}\right) \left(\frac{a}{q}\right) S(q, 1).$$

Moreover, $|S(q, 1)| = q^{1/2}$.

LEMMA 3.3. *Suppose that $(2, a) = 1$. Then*

$$|S(2^r, a, n)| \leq 2^{1+r/2}.$$

LEMMA 3.4. *One has*

$$\left| \sum_{\substack{a=1 \\ (a,p)=1}}^p \left(\frac{a}{p}\right) e\left(\frac{a}{p}\right) \right| = p^{1/2}.$$

The proof of Lemma 3.1 is elementary, and those of Lemmas 3.2–3.4 can be found in [7, Chapter 7]. We define

$$T(q; n_1, n_2, n_3, m) = \sum_{\substack{a=1 \\ (a,q)=1}}^q S(q, a, n_1)S(q, a, n_2)S(q, a, n_3)e\left(\frac{\bar{a}m}{q}\right),$$

and write $T(q) = T(q; 0, 0, 0, 0)$.

LEMMA 3.5. *Suppose that $(q_1, q_2) = 1$. Then*

$$T(q_1q_2; n_1, n_2, n_3, m) = T(q_1; n_1, n_2, n_3, m)T(q_2; n_1, n_2, n_3, m).$$

Proof. The proof is standard. We have

$$\begin{aligned} T(q_1q_2; n_1, n_2, n_3, m) &= \sum_{\substack{a=1 \\ (a, q_1q_2)=1}}^{q_1q_2} S(q_1q_2, a, n_1)S(q_1q_2, a, n_2)S(q_1q_2, a, n_3)e\left(\frac{\overline{am}}{q_1q_2}\right) \\ &= \sum_{\substack{a=1 \\ (a_1, q_1)=1}}^{q_1} \sum_{\substack{a_2=1 \\ (a_2, q_2)=1}}^{q_2} \prod_{j=1}^3 S(q_1q_2, a_1q_2 + a_2q_1, n_j)e\left(\frac{\overline{a_1q_2 + a_2q_1m}}{q_1q_2}\right). \end{aligned}$$

By Lemma 3.1, we obtain

$$\begin{aligned} T(q_1q_2; n_1, n_2, n_3, m) &= \sum_{\substack{a=1 \\ (a_1, q_1)=1}}^{q_1} \sum_{\substack{a_2=1 \\ (a_2, q_2)=1}}^{q_2} \prod_{j=1}^3 (S(q_1, a_1q_2^2, n_j)S(q_2, a_2q_1^2, n_j))e\left(\frac{\overline{a_1q_2^2m}}{q_1}\right)e\left(\frac{\overline{a_2q_1^2m}}{q_2}\right) \\ &= \sum_{\substack{a=1 \\ (a_1, q_1)=1}}^{q_1} \sum_{\substack{a_2=1 \\ (a_2, q_2)=1}}^{q_2} \prod_{j=1}^3 (S(q_1, a_1, n_j)S(q_2, a_2, n_j))e\left(\frac{\overline{a_1m}}{q_1}\right)e\left(\frac{\overline{a_2m}}{q_2}\right) \\ &= T(q_1; n_1, n_2, n_3, m)T(q_2; n_1, n_2, n_3, m). \blacksquare \end{aligned}$$

In view of Lemma 3.5, to obtain an upper bound for $T(q; n_1, n_2, n_3, m)$, it suffices to consider $T(p^r; n_1, n_2, n_3, m)$.

LEMMA 3.6. *One has*

$$|T(2^r; n_1, n_2, n_3, m)| \leq 2^{2+5r/2}.$$

Proof. This follows from Lemma 3.3. \blacksquare

LEMMA 3.7. *Suppose that $p > 2$. Then*

$$(3.1) \quad |T(p^r; n_1, n_2, n_3, m)| \leq p^{5r/2},$$

$$(3.2) \quad |T(p; n_1, n_2, n_3, m)| \leq p^2.$$

Proof. By Lemma 3.2, for $p > 2$ one has

$$S(p^r, a, n) = e\left(-\frac{\overline{4an^2}}{p^r}\right)\left(\frac{a}{p^r}\right)S(p^r, 1).$$

Therefore,

$$|T(p^r; n_1, n_2, n_3, m)| \leq \sum_{\substack{a=1 \\ (a, p^r)=1}}^{p^r} p^{3r/2} \leq p^{5r/2}.$$

This confirms (3.1). To prove (3.2), we deduce that

$$\begin{aligned}
 T(p; n_1, n_2, n_3, m) &= S(p, 1)^3 \sum_{\substack{a=1 \\ (a,p)=1}}^p e\left(\frac{-4\bar{a}(n_1^2 + n_2^2 + n_3^2 - 4m)}{p}\right) \left(\frac{a}{p}\right) \\
 &= S(p, 1)^3 \left(\frac{-1}{p}\right) \left(\frac{n_1^2 + n_2^2 + n_3^2 - 4m}{p}\right) \sum_{\substack{b=1 \\ (b,p)=1}}^p e\left(\frac{b}{p}\right) \left(\frac{b}{p}\right).
 \end{aligned}$$

Then (3.2) follows from Lemmas 3.2 and 3.4. ■

LEMMA 3.8. *Suppose that $q = q_1 q_2$ with $(q_1, q_2) = 1$, q_1 square-free and q_2 square-full. Then*

$$|T(q; n_1, n_2, n_3, m)| \leq 4q_1^2 q_2^{5/2}.$$

Proof. This follows from Lemmas 3.5–3.7 immediately. ■

From now on, we assume that q has the decomposition $q = q_1 q_2$ with $(q_1, q_2) = 1$, where q_1 is square-free and q_2 is square-full.

4. Approximations for $f(\alpha)$ and $h(\alpha)$

LEMMA 4.1. *Suppose that $(a, q) = 1$, $q \leq P$ and $|\beta| \leq 1/(qP)$. Then*

$$(4.1) \quad f\left(\frac{a}{q} + \beta\right) = \frac{S(q, a)}{q} \int_0^X e(x^2\beta) dx + \sum_{-3q/2 < b \leq 3q/2} S(q, a, b) E(b, q, \beta),$$

where $E(b, q, \beta)$ satisfies

$$(4.2) \quad \sum_{-3q/2 < b \leq 3q/2} |E(b, q, \beta)| \ll \log(q + 2).$$

Proof. The result is implied by the proof of Theorem 4.1 of Vaughan [9]. One has

$$f\left(\frac{a}{q} + \beta\right) = \sum_{x \leq X} e(x^2\beta) e(ax^2/q) = \sum_{x \leq X} e(x^2\beta) \sum_{\substack{m=1 \\ m \equiv x \pmod{q}}}^q e(am^2/q).$$

Then we can obtain

$$(4.3) \quad f\left(\frac{a}{q} + \beta\right) = \frac{1}{q} \sum_{-q/2 < b \leq q/2} S(q, a, b) F(b),$$

where

$$F(b) = \sum_{x \leq X} e(x^2\beta - bx/q).$$

Note that $|2x\beta - b/q| < 1$ for $x \leq X$. By Lemma 4.2 in [9],

$$(4.4) \quad F(b) = \sum_{-1 \leq h \leq 1} I(b + hq) + E_1(b, q, \beta),$$

where

$$I(b) = \int_0^X e(x^2\beta - bx/q) dx$$

and $E_1(b, q, \beta)$ satisfies

$$(4.5) \quad E_1(b, q, \beta) \ll 1.$$

When $b \neq 0$ and $0 \leq x \leq X$, we have $|2x\beta - b/q| \geq \frac{1}{2}|b/q|$. Then by integration by parts,

$$(4.6) \quad I(b) \ll |q/b| \quad \text{for } b \neq 0.$$

Set $B = 3q/2$. On inserting (4.4) into (4.3), we obtain

$$\begin{aligned} f\left(\frac{a}{q} + \beta\right) &= \frac{S(q, a)}{q} \int_0^X e(x^2\beta) dx \\ &= \frac{1}{q} \sum_{\substack{-B < b \leq B \\ b \neq 0}} S(q, a, b) I(b) + \frac{1}{q} \sum_{-q/2 < b \leq q/2} S(q, a, b) E_1(b, q, \beta). \end{aligned}$$

Define

$$E(b, q, \beta) = \begin{cases} q^{-1}(I(b) + E_1(b, q, \beta)) & \text{if } -q/2 < b \leq q/2 \text{ and } b \neq 0, \\ q^{-1}E_1(b, q, \beta) & \text{if } b = 0, \\ q^{-1}I(b) & \text{otherwise.} \end{cases}$$

We can see that

$$f\left(\frac{a}{q} + \beta\right) = \frac{S(q, a)}{q} \int_0^X e(x^2\beta) dx + \sum_{-B < b \leq B} S(q, a, b) E(b, q, \beta).$$

It follows from (4.5) and (4.6) that

$$\sum_{-B < b \leq B} |E(b, q, \beta)| \ll \log(q + 2). \blacksquare$$

LEMMA 4.2. *Suppose that $(a, q) = 1$, $q \leq P$ and $|\beta| \leq 1/(qP)$. Then*

$$(4.7) \quad \begin{aligned} h\left(\frac{a}{q} + \beta\right) &= q^{-1} \int (\log x + 2\gamma - 2 \log q) e(x\beta) w(x) dx \\ &\quad + \sum_{|n| \neq 0} e(-\bar{a}n/q) \Delta(n, q, \beta), \end{aligned}$$

where $\Delta(n, q, \beta)$ satisfies

$$(4.8) \quad \sum_{|n| \neq 0} |\Delta(n, q, \beta)| \ll q \log^2(q + 2) + |\beta|^2 q^{3/2} X^{7/2}.$$

Proof. On applying (2.2) with $g(x) = e(x\beta)w(x)$, we obtain

$$\begin{aligned} h\left(\frac{a}{q} + \beta\right) &= q^{-1} \int (\log x + 2\gamma - 2 \log q) e(x\beta)w(x) dx \\ &\quad + \sum_{|n| \neq 0} e(-\bar{a}n/q) \Delta(n, q, \beta), \end{aligned}$$

where

$$\Delta(n, q, \beta) = \begin{cases} -\frac{2\pi\tau(n)}{q} \int_0^\infty e(x\beta)w(x) Y_0(4\pi\sqrt{x|n|}/q) dx & \text{if } n \geq 1, \\ \frac{4\tau(|n|)}{q} \int_0^\infty e(x\beta)w(x) K_0(4\pi\sqrt{x|n|}/q) dx & \text{if } n \leq -1. \end{cases}$$

We use B to denote the Bessel function Y or K . In view of (2.6), we deduce from integration by parts that

$$\begin{aligned} \int_0^\infty e(x\beta)w(x) B_0(4\pi\sqrt{x|n|}/q) dx &= \frac{q}{\epsilon 2\pi\sqrt{|n|}} \int_0^\infty (e(x\beta)w(x))' (x^{1/2} B_1(4\pi\sqrt{x|n|}/q)) dx \\ &= \frac{q^2}{(2\pi)^2 |n|} \int_0^\infty (e(x\beta)w(x))'' (x B_2(4\pi\sqrt{x|n|}/q)) dx. \end{aligned}$$

By the definition of $w(x)$, we have $w''(x) \ll |x^{-2}\chi_S(x)|$ and $w'(x) \ll |x^{-1}\chi_S(x)|$, where χ_S denotes the characteristic function of the set $S = [1/2, 1] \cup [3X^2, 4X^2]$. Thus,

$$\begin{aligned} (e(x\beta)w(x))'' &\ll |w''(x)| + |w'(x)\beta| + w(x)|\beta|^2 \\ &\ll x^{-2}\chi_S(x) + x^{-1}\chi_S(x)|\beta| + |\beta|^2\chi_{[1/2, 4X^2]}(x) \\ &\ll x^{-2}\chi_S(x) + |\beta|^2\chi_{[1/2, 4X^2]}(x). \end{aligned}$$

From the above, we deduce that

$$\begin{aligned} \int_0^\infty e(x\beta)w(x) B_0(4\pi\sqrt{x|n|}/q) dx &\ll \frac{q^2}{|n|} \int_{1/2}^{4X^2} (x^{-1}\chi_S(x) + x|\beta|^2) |B_2(4\pi\sqrt{x|n|}/q)| dx. \end{aligned}$$

By (2.4) and (2.5),

$$(4.9) \quad B_2(4\pi\sqrt{x|n|}/q) = \begin{cases} O(q^{1/2}x^{-1/4}|n|^{-1/4}) & \text{if } x \gg q^2/|n|, \\ O(q^2x^{-1}|n|^{-1}) & \text{if } x \ll q^2/|n|. \end{cases}$$

When $|n| > q^2$, we have

$$\begin{aligned} \int_0^\infty e(x\beta)w(x)B_0(4\pi\sqrt{x|n|}/q) dx \\ \ll \frac{q^2}{|n|} \int_{1/2}^{4X^2} (x^{-1}\chi_S(x) + x|\beta|^2)q^{1/2}x^{-1/4}|n|^{-1/4} dx \\ \ll q^{5/2}|n|^{-5/4} + |\beta|^2q^{5/2}X^{7/2}|n|^{-5/4}. \end{aligned}$$

Then we obtain

$$(4.10) \quad \sum_{|n|>q^2} |\Delta(n, q, \beta)| \ll q \log^2(q+2) + |\beta|^2X^{7/2}q \log(q+2).$$

We also have

$$\begin{aligned} \int_0^\infty e(x\beta)w(x)B_0(4\pi\sqrt{x|n|}/q) dx \\ = \frac{q}{\epsilon 2\pi\sqrt{|n|}} \int_0^\infty (2\pi i\beta e(x\beta)w(x) + e(x\beta)w'(x))(x^{1/2}B_1(4\pi\sqrt{x|n|}/q)) dx \\ = \Delta_1 + \Delta_2 + \Delta_3, \end{aligned}$$

where

$$\begin{aligned} \Delta_1 &= \frac{q}{\epsilon 2\pi\sqrt{|n|}} \int_0^\infty (2\pi i\beta e(x\beta)w(x))(x^{1/2}B_1(4\pi\sqrt{x|n|}/q)) dx, \\ \Delta_2 &= \frac{q}{\epsilon 2\pi\sqrt{|n|}} \int_{3X^2}^{4X^2} (e(x\beta)w'(x))(x^{1/2}B_1(4\pi\sqrt{x|n|}/q)) dx, \\ \Delta_3 &= \frac{q}{\epsilon 2\pi\sqrt{|n|}} \int_{1/2}^1 (e(x\beta)w'(x))(x^{1/2}B_1(4\pi\sqrt{x|n|}/q)) dx. \end{aligned}$$

We first handle Δ_1 . By (2.6),

$$\Delta_1 = \frac{q^2}{4\pi^2|n|} \int_0^\infty (2\pi i\beta e(x\beta)w(x))'(x)B_2(4\pi\sqrt{x|n|}/q) dx.$$

Then from (4.9), we get

$$\begin{aligned} \Delta_1 &\ll \frac{q^2}{|n|} \int_{q^2/|n|}^{4X^2} (|\beta| + x|\beta|^2)q^{1/2}x^{-1/4}|n|^{-1/4} dx \\ &\quad + \frac{q^2}{|n|} \int_{1/2}^{q^2/|n|} (|\beta| + x|\beta|^2)q^2x^{-1}|n|^{-1} dx \\ &\ll |\beta|q^{5/2}X^{3/2}|n|^{-5/4} + |\beta|^2q^{5/2}X^{7/2}|n|^{-5/4}. \end{aligned}$$

We apply (2.6) again to deduce that

$$\begin{aligned} \Delta_2 &= \frac{q^2}{4\pi^2|n|} \int_{3X^2}^{4X^2} (e(x\beta)w'(x))'(xB_2(4\pi\sqrt{x|n|}/q)) dx \\ &\ll \frac{q^2}{4\pi^2|n|} \int_{3X^2}^{4X^2} (|\beta| + x^{-1})q^{1/2}x^{-1/4}|n|^{-1/4} dx \\ &\ll |\beta|q^{5/2}X^{3/2}|n|^{-5/4} + q^{5/2}X^{-1/2}|n|^{-5/4}. \end{aligned}$$

When $|n| \leq q^2$, by (2.4) with $\nu = 1$, we have

$$\Delta_3 \ll \frac{q}{\sqrt{|n|}} \int_{1/2}^1 x^{-1/2}qx^{-1/2}|n|^{-1/2} dx \ll \frac{q^2}{|n|}.$$

It follows from the above that, if $|n| \leq q^2$, then

$$\int_0^\infty e(x\beta)w(x)B_0(4\pi\sqrt{x|n|}/q) dx \ll \frac{q^2}{|n|} + |\beta|^2q^{5/2}X^{7/2}|n|^{-5/4}.$$

Therefore,

$$(4.11) \quad \sum_{|n| \leq q^2} |\Delta(n, q, \beta)| \ll q \log^2(q + 2) + |\beta|^2q^{3/2}X^{7/2}.$$

Now (4.8) follows from (4.10) and (4.11). ■

It follows from integration by parts together with trivial bounds that

$$(4.12) \quad \int_0^X e(x^2\beta) dx \ll \frac{X}{\sqrt{1 + X^2|\beta|}}$$

and

$$(4.13) \quad \int (\log x - 2\gamma - 2 \log q)e(x\beta)w(x) dx \ll \frac{X^2(\log q + \log X)}{1 + X^2|\beta|}.$$

LEMMA 4.3. Suppose that $q \leq P$ and $|\beta| \leq 1/(qP)$. Then, for any $v \in \mathbb{Z}$,

$$\begin{aligned} \sum_{\substack{a=1 \\ (a,q)=1}}^q f\left(\frac{a}{q} + \beta\right)^3 h\left(-\frac{a}{q} - \beta\right) e(-\bar{a}v/q) \\ = \frac{1}{q^4} \mathcal{I}(\beta, q) \sum_{\substack{a=1 \\ (a,q)=1}}^q S(q, a)^3 e(-\bar{a}v/q) + D(q, \beta), \end{aligned}$$

where

$$\mathcal{I}(\beta, q) = \left(\int_0^X e(x^2\beta) dx \right)^3 \int (\log x + 2\gamma - 2 \log q) e(-x\beta) w(x) dx$$

and $D(q, \beta)$ satisfies

$$(4.14) \quad \sum_{q \leq P} \int_{|\beta| \leq 1/(qP)} |D(q, \beta)| d\beta \ll X^2 \log^6 X.$$

Proof. Let

$$v(\beta) = \int_0^X e(x^2\beta) dx, \quad \vartheta(\beta, q) = \int (\log x + 2\gamma - 2 \log q) e(x\beta) w(x) dx.$$

Set $B = 3q/2$. Then we introduce

$$\begin{aligned} D_1 &= \sum_{-B < b_1, b_2, b_3 \leq B} \sum_{|n| \neq 0} \left(\prod_{j=1}^3 E(b_j, q, \beta) \right) \Delta(n, q, -\beta) T(q; b_1, b_2, b_3, n-v), \\ D_2 &= \frac{v(\beta)}{q} \sum_{-B < b_1, b_2 \leq B} \sum_{|n| \neq 0} E(b_1, q, \beta) E(b_2, q, \beta) \Delta(n, q, -\beta) \\ &\quad \times T(q; b_1, b_2, 0, n-v), \\ D_3 &= \frac{v^2(\beta)}{q^2} \sum_{-B < b_1 \leq B} \sum_{|n| \neq 0} E(b_1, q, \beta) \Delta(n, q, -\beta) T(q; b_1, 0, 0, n-v), \\ D_4 &= \frac{v^3(\beta)}{q^3} \sum_{|n| \neq 0} \Delta(n, q, -\beta) T(q; 0, 0, 0, n-v), \\ D_5 &= \frac{\vartheta(-\beta, q)}{q} \sum_{-B < b_1, b_2, b_3 \leq B} \left(\prod_{j=1}^3 E(b_j, q, \beta) \right) T(q; b_1, b_2, b_3, -v), \\ D_6 &= \frac{v(\beta)\vartheta(-\beta, q)}{q^2} \sum_{-B < b_1, b_2 \leq B} E(b_1, q, \beta) E(b_2, q, \beta) T(q; b_1, b_2, 0, -v), \\ D_7 &= \frac{v^2(\beta)\vartheta(-\beta, q)}{q^3} \sum_{-B < b_1 \leq B} E(b_1, q, \beta) T(q; b_1, 0, 0, -v). \end{aligned}$$

By Lemmas 4.1–4.2, we get

$$\begin{aligned} \sum_{\substack{a=1 \\ (a,q)=1}}^q f\left(\frac{a}{q} + \beta\right)^3 h\left(-\frac{a}{q} - \beta\right) e(-\bar{a}v/q) \\ = \frac{1}{q^4} \mathcal{I}(\beta, q) \sum_{\substack{a=1 \\ (a,q)=1}}^q S(q, a)^3 e(-\bar{a}v/q) \\ + D_1 + 3D_2 + 3D_3 + D_4 + D_5 + 3D_6 + 3D_7. \end{aligned}$$

We only handle D_1, D_4, D_5 and D_7 . The estimates for D_2, D_3 and D_6 can be handled similarly.

By Lemma 3.8, $T(q; b_1, b_2, b_3, n - v) \ll q_1^2 q_2^{5/2}$. Then by (4.2) and (4.8),

$$D_1 \ll q_1^2 q_2^{5/2} (\log^3 X) (q_1 q_2 \log^2 X + |\beta|^2 q_1^{3/2} q_2^{3/2} X^{7/2}).$$

Therefore,

$$\sum_{q \leq P} \int_{|\beta| \leq 1/(qP)} |D_1| d\beta \ll X^2 \log^6 X.$$

For D_4 , one finds by (4.2), (4.8) and (4.12) that

$$D_4 \ll q_1^2 q_2^{5/2} \frac{X^3}{q_1^3 q_2^3 (1 + X^2 |\beta|)^{3/2}} (q_1 q_2 \log^2 X + |\beta|^2 q_1^{3/2} q_2^{3/2} X^{7/2}).$$

Then one has

$$\sum_{q \leq P} \int_{|\beta| \leq 1/(qP)} |D_4| d\beta \ll X^2 \log^3 X.$$

By (4.2) and (4.13),

$$D_5 \ll q_1^2 q_2^{5/2} (\log^4 X) \frac{X^2}{q_1 q_2 (1 + X^2 |\beta|)}.$$

It follows that

$$\sum_{q \leq P} \int_{|\beta| \leq 1/(qP)} |D_5| d\beta \ll X^2 \log^5 X.$$

By (4.2), (4.12) and (4.13),

$$D_7 \ll q_1^2 q_2^{5/2} (\log^2 X) \frac{X^4}{q_1^3 q_2^3 (1 + X^2 |\beta|)^2}.$$

Thus we have

$$\sum_{q \leq P} \int_{|\beta| \leq 1/(qP)} |D_7| d\beta \ll X^2 \log^4 X.$$

Similarly, we can prove

$$\sum_{q \leq P} \int_{|\beta| \leq 1/(qP)} (|D_2| + |D_3| + |D_6|) d\beta \ll X^2 \log^5 X.$$

Therefore, (4.14) is established. ■

5. The proof of Theorem 1.1. We apply the Hardy–Littlewood–Kloosterman circle method to decompose the integral (2.1), getting

$$S_3(X) = \sum_{q \leq P} \sum_{\substack{a=1 \\ (a,q)=1}}^q \int_{\mathcal{B}(q,a)} f\left(\frac{a}{q} + \beta\right)^3 h\left(-\frac{a}{q} - \beta\right) d\beta,$$

where

$$\mathcal{B}(q, a) = \left[-\frac{1}{q(q+q')}, \frac{1}{q(q+q'')} \right]$$

with q' and q'' satisfying

$$P < q + q', q + q'' \leq q + P, \quad aq' \equiv 1 \pmod{q}, \quad aq'' \equiv -1 \pmod{q}.$$

Note that

$$\left[-\frac{1}{2qP}, \frac{1}{2qP} \right] \subseteq \mathcal{B}(q, a) \subseteq \left[-\frac{1}{qP}, \frac{1}{qP} \right].$$

We exchange the summation over a and the integration over β by the standard technique. One may refer to the proof of Lemma 13 of Estermann [2] for this technique (see also Section 3 of Heath-Brown [6]). We have

$$\begin{aligned} (5.1) \quad & \sum_{\substack{a=1 \\ (a,q)=1}}^q \int_{\beta \in \mathcal{B}(q,a)} (\dots) d\beta \\ &= \int_{|\beta| \leq 1/(qP)} \sum_{|v| \leq P} \sigma(v; \beta, q) \sum_{\substack{a=1 \\ (a,q)=1}}^q (\dots) e(-\bar{a}v/q) d\beta \end{aligned}$$

for some function σ satisfying

$$(5.2) \quad \sigma(v; \beta, q) \ll \frac{1}{1 + |v|}.$$

Therefore,

$$\begin{aligned} S_3(X) &= \sum_{q \leq P} \int_{|\beta| \leq 1/(qP)} \sum_{|v| \leq P} \sigma(v; \beta, q) \\ &\quad \times \sum_{\substack{a=1 \\ (a,q)=1}}^q f\left(\frac{a}{q} + \beta\right)^3 h\left(-\frac{a}{q} - \beta\right) e(-\bar{a}v/q) d\beta. \end{aligned}$$

In light of Lemma 4.3,

$$S_3(X) = \sum_{q \leq P} \int \sum_{|\beta| \leq 1/(qP)} \sum_{|v| \leq P} \sigma(v; \beta, q) \sum_{\substack{a=1 \\ (a,q)=1}}^q \frac{S(q, a)^3}{q^4} \mathcal{I}(\beta, q) e(-\bar{a}v/q) d\beta \\ + \sum_{q \leq P} \int \sum_{|\beta| \leq 1/(qP)} \sum_{|v| \leq P} \sigma(v; \beta, q) D(q, \beta) d\beta,$$

where $\mathcal{I}(\beta, q)$ and $D(q, \beta)$ are as in the lemma. We conclude by (5.2) that

$$(5.3) \quad S_3(X) = \sum_{q \leq P} \sum_{\substack{a=1 \\ (a,q)=1}}^q \int_{\beta \in \mathcal{B}(q,a)} \frac{S(q, a)^3}{q^4} \mathcal{I}(\beta, q) d\beta + O(X^2 \log^7 X).$$

By (5.1), we have

$$\sum_{q \leq P} \sum_{\substack{a=1 \\ (a,q)=1}}^q \int_{\beta \in \mathcal{B}(q,a) \setminus [-1/(2qP), 1/(2qP)]} \frac{S(q, a)^3}{q^4} \mathcal{I}(\beta, q) d\beta \\ = \sum_{q \leq P} \int_{1/(2qP) \leq |\beta| \leq 1/(qP)} \sum_{|v| \leq P} \sigma(v; \beta, q) \sum_{\substack{a=1 \\ (a,q)=1}}^q \frac{S(q, a)^3}{q^4} \mathcal{I}(\beta, q) e(-\bar{a}v/q) d\beta \\ = \sum_{q \leq P} \int_{1/(2qP) \leq |\beta| \leq 1/(qP)} \sum_{|v| \leq P} \sigma(v; \beta, q) \frac{T(q; 0, 0, 0, -v)}{q^4} \mathcal{I}(\beta, q) d\beta.$$

Then we deduce from Lemma 3.8 and (5.2) that

$$\sum_{q \leq P} \sum_{\substack{a=1 \\ (a,q)=1}}^q \int_{\beta \in \mathcal{B}(q,a) \setminus [-1/(2qP), 1/(2qP)]} \frac{S(q, a)^3}{q^4} \mathcal{I}(\beta, q) d\beta \\ \ll \sum_{|v| \leq P} \frac{1}{1 + |v|} \sum_{q \leq P} \int_{1/(2qP) \leq |\beta| \leq 1/(qP)} \frac{q_1^2 q_2^{5/2}}{q^4} |\mathcal{I}(\beta, q)| d\beta.$$

On applying $\mathcal{I}(\beta, q) \ll X^5 (\log X) (1 + X^2 |\beta|)^{-5/2}$, we obtain

$$(5.4) \quad \sum_{q \leq P} \sum_{\substack{a=1 \\ (a,q)=1}}^q \int_{\beta \in \mathcal{B}(q,a) \setminus [-1/(2qP), 1/(2qP)]} \frac{S(q, a)^3}{q^4} \mathcal{I}(\beta, q) d\beta \ll X^2 \log^3 X.$$

Moreover, we have

$$(5.5) \quad \sum_{q \leq P} \frac{T(q)}{q^4} \int_{|\beta| > 1/(2qP)} \mathcal{I}(\beta, q) d\beta \ll \sum_{q \leq P} \frac{|T(q)|}{q^4} X^{3/2} (\log X) q^{3/2} \\ \ll X^2 \log^2 X.$$

From (5.3)–(5.5), we get

$$\begin{aligned}
 (5.6) \quad S_3(X) &= \sum_{q \leq P} \sum_{\substack{a=1 \\ (a,q)=1}}^q \int_{|\beta| \leq 1/(2qP)} \frac{S(q,a)^3}{q^4} \mathcal{I}(\beta, q) d\beta + O(X^2 \log^7 X) \\
 &= \sum_{q \leq P} \frac{T(q)}{q^4} \int_{|\beta| \leq 1/(2qP)} \mathcal{I}(\beta, q) d\beta + O(X^2 \log^7 X) \\
 &= \sum_{q \leq P} \frac{T(q)}{q^4} \int_{-\infty}^{\infty} \mathcal{I}(\beta, q) d\beta + O(X^2 \log^7 X).
 \end{aligned}$$

Let

$$\mathcal{I}_0(\beta, q) = \left(\int_0^X e(x^2\beta) dx \right)^3 \int (\log x + 2\gamma - 2 \log q) e(-x\beta) w_0(x/X^2) dx.$$

By changing variables, we conclude that

$$\begin{aligned}
 (5.7) \quad \mathcal{I}_0(\beta, q) &= X^5 (\log X) \mathcal{J}_1(X^2\beta) \\
 &\quad + X^5 (\mathcal{J}_2(X^2\beta) + (2\gamma - 2 \log q) \mathcal{J}_1(X^2\beta)),
 \end{aligned}$$

where

$$\begin{aligned}
 \mathcal{J}_1(\beta) &= \left(\int_0^1 e(x^2\beta) dx \right)^3 \int e(-x\beta) w_0(x) dx, \\
 \mathcal{J}_2(\beta) &= \left(\int_0^1 e(x^2\beta) dx \right)^3 \int (\log x) e(-x\beta) w_0(x) dx.
 \end{aligned}$$

Note that

$$|\mathcal{I}(\beta, q) - \mathcal{I}_0(\beta, q)| \ll X^3 (\log X) (1 + X^2 |\beta|)^{-3/2}.$$

We deduce that

$$\begin{aligned}
 \sum_{q \leq P} \frac{T(q)}{q^4} \int_{-\infty}^{\infty} (\mathcal{I}(\beta, q) - \mathcal{I}_0(\beta, q)) d\beta \\
 \ll \sum_{q \leq P} \frac{|T(q)|}{q^4} \int_{-\infty}^{\infty} \frac{X^3 \log X}{(1 + X^2 |\beta|)^{3/2}} d\beta.
 \end{aligned}$$

Therefore, by Lemma 3.8,

$$(5.8) \quad \sum_{q \leq P} \frac{T(q)}{q^4} \int_{-\infty}^{\infty} (\mathcal{I}(\beta, q) - \mathcal{I}_0(\beta, q)) d\beta \ll X \log X.$$

By (5.6)–(5.8),

$$\begin{aligned} S_3(X) &= \sum_{q \leq P} \frac{T(q)}{q^4} \int_{-\infty}^{\infty} \mathcal{I}_0(\beta, q) d\beta + O(X^2 \log^7 X) \\ &= \sum_{q \leq P} \frac{T(q)}{q^4} X^3 (\log X) \mathfrak{J}_1 + \sum_{q \leq P} \frac{T(q)}{q^4} X^3 \mathfrak{J}_2 \\ &\quad + \sum_{q \leq P} \frac{T(q)(2\gamma - 2 \log q)}{q^4} X^3 \mathfrak{J}_1 + O(X^2 \log^7 X), \end{aligned}$$

where

$$\mathfrak{J}_1 = \int_{-\infty}^{\infty} \mathcal{J}_1(\beta) d\beta, \quad \mathfrak{J}_2 = \int_{-\infty}^{\infty} \mathcal{J}_2(\beta) d\beta.$$

It is easy to verify

$$\begin{aligned} \sum_{q \leq P} \frac{T(q)}{q^4} &= \mathfrak{S}_1 + O(X^{-1} \log X), \\ \sum_{q \leq P} \frac{T(q)(2\gamma - 2 \log q)}{q^4} &= \mathfrak{S}_2 + O(X^{-1} \log^2 X), \end{aligned}$$

where \mathfrak{S}_1 and \mathfrak{S}_2 are given by (1.3) and (1.4) respectively. Then we finally obtain

$$S_3(X) = \mathfrak{S}_1 \mathfrak{J}_1 X^3 \log X + (\mathfrak{S}_1 \mathfrak{J}_2 + \mathfrak{S}_2 \mathfrak{J}_1) X^3 + O(X^2 \log^7 X).$$

Note that $\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{J}_1, \mathfrak{J}_2$ are constants independent of X . The proof of Theorem 1.1 is complete. ■

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