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The asymptotic behaviour of the counting functions of Ω-sets in arithmetical semigroups

by

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1. Introduction. We study oscillatory properties of the error terms of the counting functions of some subsets of arithmetical semigroups. Quantitative factorization theory, i.e. the study of quantitative properties of such subsets defined by factorization-related conditions, was initiated by Fogels [5] and further developed by Narkiewicz (cf., e.g., [24–26]). Of course, the question of what an arithmetical semigroup exactly is can hardly ever be settled, as the interests of number theorists expand over time, but it seems clear that at the minimum it should include the semigroup of positive integers, the semigroups of non-zero integers and non-zero principal ideals in algebraic number fields and, more generally, the generalized Hilbert semigroups modulo an integral ideal (cf. Subsection 2.1). It is well known that in these cases the arithmetical properties of the semigroup are closely related to the analytical properties of the L-functions associated to the class group, and that these L-functions either belong to or are closely related to elements of the Selberg class.

Therefore we consider (Section 2) a class of semigroups, called L-semigroups, with precisely this property. In particular we define *simple L-semigroups*, which include all the examples given above. The counting function of a subset A of an L-semigroup S is defined, as usual, as

$$A(x) = \sideset{}{'}\sum_{\substack{a\in A\\ N(a)\le x}} 1,$$

where the sum is over non-associated elements of A, and N denotes the norm defined in S. When the zeta function associated to A (cf. Subsection 4.2) is sufficiently regular, the main term and the error term of $A(x)$ may be

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defined analytically (cf. Section 3 for the definitions of the error term and oscillations).

The set $\boldsymbol{G}_k$ of elements of S with at most k distinct lengths of factorizations into irreducibles (or a related set $\overline{\boldsymbol{G}_k} = \boldsymbol{G}_k \setminus \boldsymbol{G}_{k-1}$) was considered by Narkiewicz [24] and Śliwa [38] who studied the main term in the case of the semigroup of non-zero algebraic integers. Kaczorowski [10] gave more precise asymptotics for $\boldsymbol{G}_k(x)$. Geroldinger and Halter-Koch [7, Chapter 9] studied the main term in an abstract setting. The author [29, 31] and Schmid and the author [35] showed, in the case of the semigroup of non-zero algebraic integers, totally positive integers, and some related semigroups, that the error term of $\boldsymbol{G}_k(x)$ has oscillations of size $x^{1/2-\varepsilon}$ for $k \geq 2$ and, conditionally, in the case $k = 1$ (provided some conjectures hold, either combinatorial or analytical, related to multiplicities of zeros of L functions).

In the present paper we show the existence of oscillations of the error term of $\boldsymbol{G}_k(x)$ unconditionally, for all $k \geq 1$. We also obtain slightly larger oscillations and treat more general semigroups, including all the generalized Hilbert semigroups.

THEOREM 1.1. *If S is a simple L-semigroup with class number $h \geq 3$, and k is a positive integer, then the error term of the counting function $\boldsymbol{G}_k(x)$ has oscillations of logarithmic frequency and size $\sqrt{x}(\log x)^{-M}$ for some $M > 0$.*

We also consider the set $B = B(a, q)$ of positive integers without non-trivial divisors in a given arithmetic progression $a \bmod q$. Here "non-trivial divisors" means that we allow the divisor $\gcd(a, q)$, as otherwise the defined set, in the case $\gcd(a, q) \equiv a \pmod{q}$, would become almost trivial. The distribution of numbers with this property was first considered by Banks, Friedlander and Luca [1] who defined a very similar set, say $C = C(a, q)$, of positive integers without divisors $\neq 1$ and congruent to a modulo q. They obtained the asymptotics for the main term of $C(x) = B(x)$ in the case of prime q. Narkiewicz and the author [28] studied the main term of $C(x)$ for a general rational integer modulus. Of course $B(a, q) = C(a, q)$ if q is a prime (and $a \nmid q$), and for general moduli differences occur only in the case $\gcd(a, q) \equiv a \pmod{q}$, $\gcd(a, q) \neq 1$. If $q/\gcd(a, q) \leq 2$, then the set $B(a, q)$ has a very simple, regular structure. In all the remaining cases we show the existence of oscillations of the error term of $B(x)$.

THEOREM 1.2. *Let a, q be positive integers with $q/\gcd(a, q) \geq 3$. Let B denote the set of positive integers that have no divisor other than $\gcd(a, q)$ congruent to a modulo q. Then the error term of the counting function of B has oscillations of logarithmic frequency and size $\sqrt{x}(\log x)^{-M}$ for some $M > 0$.*

We also consider, so to say, "arithmetic progressions of ideals" in an algebraic number field and obtain an analogous, more general, though also more technical result (Theorem 5.4) for the set of ideals without non-trivial divisors in such a progression. It is partly derived from a general property (Corollary 5.3) of sets defined by "forbidden" divisors. We use two analytical theorems recently obtained by the author, quoted in Section 3, and some combinatorial results (Subsection 4.1) related to a family of semigroup subsets called Ω-sets, in particular concerning so-called (r,d)-singular Ω-sets. In Subsection 4.3 we show a general oscillation theorem (Theorem 4.7) for the counting functions of (r,d)-singular Ω-sets.

We write, as usual, $s=\sigma+it$, and $\overline{F}(s)=\overline{F(\overline{s})}$ for a complex function $F(s)$. If G is a function meromorphic in a neighbourhood of ρ, we let $m(\rho,G)$ denote the order of a zero of G at ρ, with $m(\rho,G)=-m$ in the case of a pole of order m, and $m(\rho,G)=0$ if G is regular and non-zero at ρ. We denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the sets of positive integers, integers, and rational, real and complex numbers, respectively, and put $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. In expressions of the form $\log(s-\rho)$ or $(s-\rho)^w$ (where $\rho,w\in\mathbb{C}$), and $\log G(s)$, where G is a function from the Selberg class, we generally mean the principal branches. In Section 2 we define some further notation pertaining to the semigroup S and use it throughout the paper.

2. L-semigroups. We say that a complex function $F(s)$ *has an Euler product* (cf. [3]) if it satisfies

$$F(s)=\exp\Big(\sum_{n=1}^{\infty}b(n)n^{-s}\Big),\qquad \sigma>3/2,\tag{2.1}$$

for some $b(n)\ll n^{\theta}$ with $\theta<1/2$ and $b(n)=0$ unless n is a prime power greater than 1. We recall that the *Selberg class* $\mathcal{S}$ of L-functions (cf. [11, 14–23, 36]) consists of Dirichlet series $F(s)=\sum_{n=1}^{\infty}a(n)n^{-s}$ with $a(n)\ll n^{\varepsilon}$ such that, for some integer $m\geq 0$, the function $(s-1)^mF(s)$ can be analytically continued to an entire function of finite order, $F(s)$ has an Euler product and satisfies the functional equation of the form $\Phi(s)=\omega\overline{\Phi}(1-s)$, where $|\omega|=1$, and

$$\Phi(s)=F(s)Q^s\prod_{j=1}^{r}\Gamma(\lambda_js+\mu_j)$$

for some $Q>0$, $r\geq 0$, $\lambda_j>0$ and $\Re\mu_j\geq 0$, $j=1,\dots,r$. The number $d_F=2\sum_{j=1}^{r}\lambda_j$ is called the *degree* of F and it is well known [3] that it depends on F alone. The zeros of $F(s)$, or their multiplicities, are divided into trivial and non-trivial. The non-trivial ones are simply the zeros of $\Phi(s)$, or equivalently the zeros of $F(s)$ satisfying $\sigma\geq 1/2$ and their images in the reflection across the line $\sigma=1/2$ (cf. also [3]). The following fact is generally known. The second part of it was stated in [16] and the first one follows in the same way as [9, (5.28)].

LEMMA 2.1. *Let $F \in \mathcal{S}$. Then*

$$\log F(s) = \sum_{|\gamma - t| \le 1} \log(s-\rho) + O_F(\log(|t|+2))$$

for all $s = \sigma + it$ such that $-1 \le \sigma \le 2$ and $F(s) \neq 0$, where the sum is over all non-trivial zeros $\rho = \beta + i\gamma$ of F, counted with multiplicity. Moreover,

$$N_F(T) = \frac{d_F}{2\pi} T \log T + c_F T + O_F(\log T), \qquad T \to \infty,$$

where $N_F(T)$ is the number of non-trivial zeros ρ, counted with multiplicity, satisfying $0 \le \gamma \le T$, d_F is the degree of F, and c_F is another constant, depending only on F.

We say that the Euler product (2.1) is *finite* if $b(p^k) = 0$ for all $k \in \mathbb{N}$ and all but finitely many primes p.

LEMMA 2.2. *If $F(s)$ has a finite Euler product, then, for some $\theta < 1/2$, it does not vanish in the half-plane $\sigma > \theta$ and it has an absolutely convergent Dirichlet series expansion there.*

Proof. We have

$$F(s) = \exp\Big(\sum_{j=1}^{m} \sum_{k=1}^{\infty} b(p_j^k) p_j^{-ks}\Big), \qquad \sigma > 3/2,$$

for some primes $p_1 < \cdots < p_m$ and $b(n) \ll n^\theta$, $\theta < 1/2$, so clearly $F(s)$ has a non-vanishing extension in $\Re s > \theta$. For $p = p_1, \ldots, p_m$ the function

$$f_p(z) = \exp\Big(\sum_{k=1}^{\infty} b(p^k) z^k\Big)$$

is regular in $|z| < p^{-\theta}$, so $f_p(z) = 1 + \sum_{k=1}^{\infty} a_p(p^k) z^k$ for some $a_p(p^k)$ that satisfies $a_p(p^k) \ll p^{k^{\theta+\varepsilon}}$ for every $\varepsilon > 0$. Hence $F(s) = \prod_{j=1}^{k} f_{p_j}(p_j^{-s})$ is a product of Dirichlet series absolutely convergent for $\sigma > \theta$. ∎

Let $\mathcal{F}(\mathcal{P})$ denote the free multiplicative monoid generated by a non-empty set $\mathcal{P}$. Let S be a *semigroup with divisor theory* (cf. [6, 8]), i.e. a commutative, cancellative semigroup with a unit and a monoid homomorphism $\varphi : S \to \mathcal{F}(\mathcal{P})$ such that for all $a, b \in S$ the divisibility $\varphi(a) \mid \varphi(b)$ in $\mathcal{F}(\mathcal{P})$ implies $a \mid b$ in S, and every $p \in \mathcal{P}$ equals $\gcd(\varphi(a_1), \ldots, \varphi(a_n))$ for some $a_1, \ldots, a_n \in S$. We assume that the class group $\mathrm{Cl}(S)$ of S is finite and put $h = |\mathrm{Cl}(S)|$. We write S and $\mathrm{Cl}(S)$ mutiplicatively. We assume that the divisor semigroup $\mathcal{F}(\mathcal{P})$ of S is equipped with a multiplicative norm $N : \mathcal{F}(\mathcal{P}) \to \mathbb{N}$, i.e. $N(a) = 1$ implies $a = 1$ for $a \in \mathcal{F}(\mathcal{P})$, and $N(ab) = N(a)N(b)$ for every $a, b \in \mathcal{F}(\mathcal{P})$.

We say that S is an *L-semigroup* if

(i) we have
$$\#\{a \in \mathcal{F}(\mathcal{P}) : N(a) \le x\} \ll_\varepsilon x^{1+\varepsilon}, \qquad x \to \infty,$$
for every $\varepsilon > 0$, and

(ii) for every $\chi \in \widehat{\mathrm{Cl}(S)}$ the Dirichlet series
$$L(s,\chi) = \sum_{a \in \mathcal{F}(\mathcal{P})} \chi(a) N(a)^{-s}$$
satisfies $f_\chi(s)^{-1} L(s,\chi) \in \mathcal{S}$ for some function f_χ with a finite Euler product.

We say that the L-semigroup is *simple* if:

(iii) $L(s,\chi_0)$ has a simple pole at $s = 1$, and

(iv) for all $\chi \in \widehat{\mathrm{Cl}(S)} \setminus \{\chi_0\}$ the function $L(s,\chi)$ is regular and non-vanishing at $s = 1$.

While the notion of an L-semigroup may be viewed simply as a tool to conveniently encapsulate key properties of many semigroups that frequently arise in number theory (though definitely not all, a notable exception being the Selberg class itself), it also raises a number of questions that seem interesting in their own right. For example: are the degrees (in the Selberg class) of all the functions $f_\chi(s)^{-1} L(s,\chi)$, $\chi \in \widehat{\mathrm{Cl}(S)}$, necessarily the same? It is so in all examples of L-semigroups known to the author. In that case we can call this common degree the degree of S, denoted $\deg S$. Then, if S_1, S_2 are L-semigroups with degrees, we have
$$\deg(S_1 \times S_2) = \deg(S_1) + \deg(S_2),$$
where the required structure on the product semigroup $S_1 \times S_2$ is defined in the obvious way. Can we say something similar about other invariants? What can we say about the semigroup-like structure defined by this direct product (say, on isomorphism classes of L-semigroups)? Is f_χ always a Dirichlet polynomial? Are the non-trivial local factors of f_χ^{-1} always identical to the corresponding factors of $f_\chi(s)^{-1} L(s,\chi)$? Is S necessarily simple if it is primitive (not isomorphic to a non-trivial direct product)? The author hopes to return to some of these questions in a future paper.

Kaczorowski and Perelli [21, Non-Vanishing Conjecture] conjectured that $F(1+it) \ne 0$ for all $F \in \mathcal{S}$, $t \in \mathbb{R}$. They also showed that this conjecture is equivalent to a general prime number theorem and that it follows from other well-known conjectures regarding the Selberg class. We say that S *satisfies* (NVC) on the line $\sigma = 1$ if $L(1+it,\chi) \ne 0$ for all $\chi \in \widehat{\mathrm{Cl}(S)}$, $t \in \mathbb{R}$. In (iii)–(iv) above we assumed this for $t = 0$ for simple L-semigroups.

These conditions (for a particular semigroup) are of course much weaker than (NVC) for the entire Selberg class. In particular we have $F(1+it) \neq 0$ for all known examples of L-functions [13], so we are able to verify (NVC) on the line $\Re s = 1$ for all known L-semigroups.

We denote the principal class of $\mathrm{Cl}(S)$ by E, the set of classes that contain at least one prime divisor by G_0, the number of prime divisors of a in the class X (counted according to their multiplicities) by $\Omega_X(a)$, and the class of $d \in \mathcal{F}(\mathcal{P})$ in $\mathrm{Cl}(S)$ by $[d]$. For $a \in \mathcal{F}(\mathcal{P})$ let $\mathrm{Supp}(a) = \{X \in G_0 : \Omega_X(a) > 0\}$.

2.1. Generalized Hilbert semigroups. Let K be an algebraic number field, $\mathcal{O}_K$ its ring of integers, $\mathfrak{f}$ a non-zero ideal, $H^*_{\mathfrak{f}}(K)$ the class group of K modulo $\mathfrak{f}$ in the narrow sense, and H' a subgroup of $H^*_{\mathfrak{f}}(K)$. The semigroup S of non-zero ideals of $\mathcal{O}_K$ whose classes belong to H' is called the *generalized Hilbert semigroup*. It was introduced by Halter-Koch [8, Beispiel 4]. If H' is the trivial subgroup of $H^*_{\mathfrak{f}}(K)$, we call S the *generalized Hilbert semigroup modulo* $\mathfrak{f}$. The divisor theory of S is the embedding of S in $\mathcal{F}(\mathcal{P})$, where $\mathcal{P}$ is the set of all prime ideals not dividing $\mathfrak{f}$, and $\mathrm{Cl}(S)$ may be identified with $H^*_{\mathfrak{f}}(K)/H'$. Characters of $\mathrm{Cl}(S)$ may be identified with characters χ of $H^*_{\mathfrak{f}}(K)$ such that $H' \subseteq \ker\chi$. Let χ be such a character, induced by a primitive character χ^* modulo $\mathfrak{q}$. Then

$$L(s,\chi^*) = \prod_{\substack{\mathfrak{p}\mid\mathfrak{f}\\ \mathfrak{p}\nmid\mathfrak{q}}} (1-\chi^*(\mathfrak{p})N(\mathfrak{p})^{-s})^{-1} L(s,\chi)$$

belongs to the Selberg class (cf. [14, 16]). We have $L(1+it,\chi) \neq 0$, $t \in \mathbb{R}$, $\chi \in \widehat{\mathrm{Cl}(S)}$. The function $L(s,\chi_0)$ has a simple pole at $s = 1$, and, for $\chi \neq \chi_0$, the function $L(s,\chi)$ is regular and non-vanishing at $s = 1$ (cf. [27]). Therefore S is a simple L-semigroup, of degree equal to the degree of the field, satisfying (NVC). Such semigroups are the only examples of simple L-semigroups known to the author. Of course they include, as special cases, the classical Hilbert semigroups (modulo a positive integer f), the reduced multiplicative semigroup of $\mathcal{O}_K$ and the semigroup $\mathcal{I}(\mathcal{O}_K)$ of non-zero ideals.

3. Singularities and oscillations of arithmetical functions. Next we recall some results of [32, 33]. Let $\mathcal{D} \subseteq \mathbb{C}$ be a region and let $C \subseteq \mathcal{D}$ be a discrete closed set contained in some half-plane $\Re s \leq \sigma_1$. We say that a complex function F is *defined in* $\mathcal{D}$ *with possible singularities in* C if

(i) the function F is defined and regular on $\mathcal{D}$ apart from horizontal cuts inside $\mathcal{D}$, extending from the points $\rho \in C$ to the left, up to the boundary of $\mathcal{D}$ or to $\rho - \infty$, and

(ii) F has an extension (possibly branched) onto $\mathcal{D} \setminus C$ covering all points inside the cuts at least twice (from "above" and from "below").

We call F *maximal* if, in addition, the set C above is minimal, i.e. no extension of F exists that satisfies the above with a proper subset of C in place of C. In that case we call C the *set of singularities* of F. Let $\mathrm{Br}(\mathcal{D})$ denote the set of all maximal functions defined in $\mathcal{D}$ with possible singularities. As shown in [32] every function F with possible singularities has a unique maximal extension (that we can identify with F) and the set $\mathrm{Br}(\mathcal{D})$ is an integral domain. By a branch of $G(s)^w$, where $w \in \mathbb{C}$ and G is a function regular in $\mathcal{D}$ with all zeros contained in some half-plane $\Re s \le \sigma_1$, we mean any function of the form $e^{wh(s)}$, where $h(s)$ is a function defined in $\mathcal{D}$ with possible singularities such that $e^{h(s)} = G(s)$ identically. We denote by $\mathrm{Hol}(\mathcal{D})$ the ring of functions holomorphic in $\mathcal{D}$ and by $\mathrm{Hol}^{\mathbb{C}}(\mathcal{D})$ the subring of $\mathrm{Br}(\mathcal{D})$ generated by $\mathrm{Hol}(\mathcal{D})$ and by (all) branches of functions of the form $G(s)^w$.

THEOREM 3.1 ([32]). *Let $T > 0$ and let $\mathcal{D}$ be a region containing the set*

$$\{s \in \mathbb{C} : \Re s \ge 1/2,\ |\Im s| \ge T\} \cup \{s \in \mathbb{C} : \Re s > 1,\ |\Im s| \le T\}.$$

Let $F_1, \dots, F_n \in \mathcal{S}$ with $\log F_1, \dots, \log F_n$ linearly independent over $\mathbb{Q}$ and let $P \in \mathrm{Hol}^{\mathbb{C}}(\mathcal{D})[X_1, \dots, X_n]$ with $\deg P > 0$. Then the function $f(s) = P(\log F_1(s), \dots, \log F_n(s), s)$ has infinitely many singularities in $\mathcal{D}$ with $\Re s \ge 1/2$.

Let $f : (0, \infty) \to \mathbb{R}$ be a function with locally bounded variation satisfying $f(x) = (f(x-0) + f(x+0))/2$ for all $x \in (0, \infty)$. We recall that the *Mellin transform* [40] of f, defined as

$$F(s) = \int_0^\infty f(x) x^{-s-1}\, dx,$$

is absolutely convergent in the strip $\{s \in \mathbb{C} : \sigma_1 < \Re s < \sigma_2\}$, where $\sigma_1 = \inf\{\sigma : f(x) = O(x^\sigma),\ x \to \infty\}$ and $\sigma_2 = \sup\{\sigma : f(x) = O(x^\sigma),\ x \to 0^+\}$. In [33] the author has defined F to be *weakly bounded* if $-\infty < \sigma_1 < \sigma_2 \le \infty$ and it satisfies conditions equivalent to:

(i) F has an extension to some region containing the strip

$$\{s \in \mathbb{C} : \sigma_0 \le \Re s \le \sigma_1\} \tag{3.1}$$

for some $\sigma_0 < \sigma_1$, with possible singularities of the form

$$F(s) = \sum_{j=1}^{m} (s-\rho)^{-b_j} (\log(s-\rho))^{c_j} h_j(s),$$

where b_js are complex numbers, c_js are non-negative integers and each h_j is regular in a neighbourhood of ρ,

(ii) the set of singularities C is contained in (3.1),

(iii) for some integer-valued function $m(\rho)$ supported on C, with $m(\rho) > 0$ for all $\rho \in C \setminus \mathbb{R}$, and for $N(T) = \sum_{0 \le \Im\rho \le T} m(\rho)$, we have $N(T+1) - N(T) = O(\log T)$ for all $T \ge 2$, and

(iv) there exist numbers $a, M > 0$ such that for every $s = \sigma + it \in \mathcal{D}$ with $\sigma > \sigma_0$ and $t \geq 2$ there is at least one branch of F satisfying

$$F(s) \ll t^M \exp\Big(a \sum_{|\gamma - t| \leq 1} m(\rho) \big|\log|s-\rho|\big|\Big),$$

where the sum is taken over all $\rho = \beta + i\gamma \in C$ satisfying $|\gamma - t| \leq 1$.

Let $\mathcal{C}_0$ be a contour of integration starting at a point $\theta > \sigma_0$, going just below the real line, around any singularities of F on $[\sigma_0, \sigma_1]$, crossing the real line to the right of σ_1, and then going just above the real line, back to the beginning. Then

$$\mathcal{M}(x) = \frac{1}{2\pi i} \int_{\mathcal{C}_0} x^s F(s)\, ds$$

is called the *main term* of $f(x)$ and $E(x) = f(x) - \mathcal{M}(x)$ the *error term* [10, 14, 33]. We say that $E(x)$ *has oscillations of logarithmic frequency and size* $g(x)$ if there exists a sequence $x_n \nearrow \infty$ such that

$$(-1)^n E(x_n) \gg g(x_n), \qquad \log x_n \ll n \quad \text{and} \quad \log x_{n+1} \sim \log x_n.$$

The main result in [33] implies the following.

Theorem 3.2 ([33, Corollary to Theorem 1.1]). *Let $f(x)$ be a real function with weakly bounded Mellin transform $F(s)$, and let $\mathcal{D}$ and θ be as above. If the function $F(s)$ has a singularity at $\rho = \beta + i\gamma \in \mathcal{D}$, $\gamma \neq 0$, $\theta < \beta < \sigma_2$, then the error term of $f(x)$ has oscillations of logarithmic frequency and size $x^\beta (\log x)^{-M}$ for some $M \in \mathbb{R}$.*

Finally the following result is useful when showing the existence of a singularity at a particular point.

Lemma 3.3 ([30, Lemma 5]). *Let F be a function of the form*

$$F(s) = \sum_{i=1}^{N} (s-\rho)^{w_i} P_i(\log(s-\rho))$$

where $N \geq 0$, $w_i \in \mathbb{C}$, $\rho \in \mathbb{C}$, and P_i are polynomials with coefficients regular in the neighbourhood of ρ. Then F can be uniquely represented in the form

$$F(s) = \sum_{j=1}^{N'} (s-\rho)^{w'_j} Q_j(\log(s-\rho))$$

with N', w'_j, and Q_j being as m, w_j and P_j above, but w'_j pairwise non-congruent modulo $\mathbb{Z}$ and the coefficients of Q_j not all 0 at ρ. Each w'_j is congruent modulo $\mathbb{Z}$ to one of the w_is.

4. Ω-sets in L-semigroups

4.1. Ω-sets in semigroups with divisor theory. We use some combinatorial tools from [34], but we phrase them in another language to make their application straightforward. For $Y \in \mathrm{Cl}(S)$, $U \subseteq G_0$, and $\alpha : G_0 \to \mathbb{N}_0$ such that $\alpha(X) = 0$ for all $X \in U$, let

$$\boldsymbol{\Omega}(U, \alpha) = \{a \in \mathcal{F}(\mathcal{P}) : \Omega_X(a) = \alpha(X) \text{ for all } X \in G_0 \setminus U\}$$

and

$$\boldsymbol{\Omega}_Y(U, \alpha) = Y \cap \boldsymbol{\Omega}(U, \alpha).$$

A subset $A \subseteq \mathcal{F}(\mathcal{P})$ is called an *Ω-set* if the value of the characteristic function of A on a depends only on the values of $\Omega_X(a)$, $X \in G_0$. The *rank* of A, denoted $\mathrm{rk}\, A$, is the smallest number r such that A is contained in a finite union of sets of the form $\boldsymbol{\Omega}_Y(U, \alpha)$ with $|U| \le r$. In addition $\mathrm{rk}\, \emptyset = -\infty$. The *degree* of A, denoted $\deg A$, is the supremum of all values of $\sum_X \alpha(X)$ over Y, U, α such that

$$\mathrm{rk}(\boldsymbol{\Omega}_Y(U, \alpha) \cap A) = \mathrm{rk}\, A.$$

The *number of layers* of A is the maximum length $l = l(A)$ of an interleaved chain of divisors $a_1 \,|\, b_1 \,|\, a_2 \,|\, b_2 \,|\, \cdots \,|\, b_{l-1} \,|\, a_l$ in $\mathcal{F}(\mathcal{P})$, all having the same class in $\mathrm{Cl}(S)$ and such that $a_1, \ldots, a_l \in A$ and $b_1, \ldots, b_{l-1} \notin A$. Finally, the *in-class divisor closure* of A is

$$\mathrm{Div}(A) = \{d \in \mathcal{F}(\mathcal{P}) : d \,|\, a \text{ for some } a \in A,\ [d] = [a]\}.$$

In case $A \subseteq \varphi(S)$ the set $A_1 = \varphi^{-1}(A) \subseteq S$ is also called an Ω-set. The rank, degree and other properties of A_1 are defined to be those of A.

Proposition 4.1 ([34]). *Let A be a non-empty Ω-set. We have $l(A) < \infty$ if and only if the characteristic function of A can be represented in the form*

$$A = \sum_{j=1}^{n} \gamma_j \cdot \boldsymbol{\Omega}_{Y_j}(U_j, \alpha_j), \tag{4.1}$$

where sets are tacitly identified with their characteristic functions, the sets $\boldsymbol{\Omega}_{Y_j}(U_j, \alpha_j)$ are non-empty and pairwise distinct, $Y_j \in \mathrm{Cl}(S)$, and $\gamma_j \in \mathbb{Z} \setminus \{0\}$ $(j = 1, \ldots, n)$. The representation (4.1) *is then unique up to order and we have*

$$\mathrm{rk}\, A = \max_j |U_j|, \qquad \deg A = \max_{j:\, |U_j| = \mathrm{rk}\, A} \sum_{X \in G_0} \alpha_j(X),$$

and $\gamma_{j_0} = 1$ whenever $\boldsymbol{\Omega}_{Y_{j_0}}(U_{j_0}, \alpha_{j_0})$ is inclusion-maximal among all the $\boldsymbol{\Omega}_{Y_j}(U_j, \alpha_j)$.

If A is an Ω-set with $l(A) < \infty$, $r \in \mathbb{N}_0$ and $d \in \mathbb{N}$, then we say that A is *(r, d)-singular* if the unique representation (4.1) satisfies the following conditions:

(i) There is at least one j such that

$$|U_j| = r \quad \text{and} \quad \sum_X \alpha_j(X) = d. \tag{4.2}$$

(ii) The sign of γ_j is the same for every j satisfying (4.2).
(iii) For each j not satisfying (4.2) we have $|U_j| < r$ or $\sum_X \alpha_j(X) < d$.

The *absolute degree* of A, for $l(A) < \infty$, is defined based on (4.1) as

$$\deg_0(A) = \begin{cases} \max_j \sum_{X \in G_0} \alpha_j(X), & A \neq \emptyset, \\ 0 & A = \emptyset. \end{cases}$$

THEOREM 4.2 ([34]). *Let A be an Ω-set such that* $\mathrm{Div}(A) = A$. *Then A is (r,d)-singular for some $r \geq 0$ and $d > 0$ if and only if* $\deg_0(A) > 0$.

4.2. The counting function of an Ω-set. We let χ and ψ denote characters of $\mathrm{Cl}(S)$ and number these characters as $\chi_0, \ldots, \chi_{h-1}$. Let $F_\chi(s) = f_\chi(s)^{-1} L(s,\chi)$, $\chi \in \mathrm{Cl}(S)$, and $F_i(s) = F_{\chi_i}(s)$, $i = 0, \ldots, h-1$. We assume, as we may, that the numbering of characters is such that χ_0 is the principal character and $\log F_0(s), \ldots, \log F_u(s)$, for some $u \leq h-1$, is a maximal linearly independent (over $\mathbb{Q}$) subset of $\log F_0(s), \ldots, \log F_{h-1}(s)$. We also use shorthand notation

$$\begin{aligned} l_0(s) &= (\log F_0(s), \ldots, \log F_u(s)), \\ l_1(s) &= (\log F_1(s), \ldots, \log F_u(s)), \\ l_2(s) &= (\log F_0(s), \ldots, \log F_u(s), \log F_0(2s), \ldots, \log F_u(2s)). \end{aligned}$$

Suppose $\theta \in (2/5, 1/2)$ is sufficiently close to $1/2$ so that all the $f_\chi(s)$ are absolutely convergent and non-zero in $\sigma \geq \theta$. We denote by $\mathcal{A}$ the ring of Dirichlet series absolutely convergent in $\sigma > \theta$. We write, e.g., $P(l_2(s), s)$ for the value at s of a polynomial $P \in \mathcal{A}[x_0, \ldots, x_{2u+1}]$ applied to these logarithms. Let $\langle \chi \,|\, U \rangle = \frac{1}{h} \sum_{X \in U} \chi(X)$. For any set $A \subseteq \mathcal{F}(\mathcal{P})$ we can define its zeta functions as

$$\begin{aligned} Z(s,A) &= \sum_{a \in A} N(a)^{-s}, & \sigma > 1, \\ Z(s,\chi,A) &= \sum_{a \in A} \chi(a) N(a)^{-s}, & \chi \in \mathrm{Cl}(S),\ \sigma > 1. \end{aligned}$$

We extend the notation $Z(s,A)$ also to subsets of other semigroups with a norm.

LEMMA 4.3. *Suppose S is simple.*

(i) *Each class $X \in \mathrm{Cl}(S)$ contains infinitely many prime divisors.*
(ii) *For $Y \in \mathrm{Cl}(S)$, $U \subseteq \mathrm{Cl}(S)$, $\alpha : \mathrm{Cl}(S) \to \mathbb{N}_0$ with $\alpha|_U = 0$, and $X_0 = \prod_{X \notin U} X^{\alpha(X)}$ the set $\mathbf{\Omega}_Y(U,\alpha)$ is non-empty if and only if $X_0 Y^{-1} \in \langle U \rangle$.*

(iii) *Under the notation of* (ii) *we have*

$$Z(s,\mathbf{\Omega}_Y(U,\alpha))=\Big(\prod_{X\notin U}P_{X,\alpha(X)}(l_2(s),s)\Big)$$
$$\cdot\frac{1}{h}\sum_{\chi}\chi(X_0Y^{-1})\Big(\prod_{X\in U}H_{X,\chi(X)}(s)\Big)G_{U,\chi}(s),\qquad\sigma>1,$$

where

$$G_{U,\chi}(s)=\prod_{\psi}L(s,\psi)^{\langle\chi\psi^{-1}|U\rangle}L(2s,\psi)^{\frac{1}{2}\langle\chi^2\psi^{-1}|U\rangle}L(2s,\psi^2)^{-\frac{1}{2}\langle\chi\psi^{-2}|U\rangle},$$
$$\chi\in\widehat{\mathrm{Cl}(S)},$$

the function $H_{X,z}(s)$ *(for* $z\in\mathbb{C}$, $|z|\le 1$*) is such that* $\log H_{X,z}(s)$ *is in* $\mathcal{A}$, *and* $P_{X,m}\in\mathcal{A}[x_0,\dots,x_{2u+1}]$ *(for* $m\in\mathbb{N}_0$*) is a polynomial of degree* m, *with the coefficient of* x_0^m *equal to* $\frac{1}{h^m m!}$. *We have*

$$\lim_{\sigma\to1^+}\Big(\prod_{X\in U}H_{X,1}(\sigma)\Big)G_{U,\chi_0}(\sigma)(\sigma-1)^{|U|/h}\in(0,\infty).\tag{4.3}$$

(iv) *Under the notation of* (iii), *for every* $U\subseteq\mathrm{Cl}(S)$ *we have*

$$Z(s,\mathbf{\Omega}(U,0))=\Big(\prod_{X\in U}H_{X,1}(s)\Big)G_{U,\chi_0}(s),\qquad\sigma>1.$$

Proof. We have $Z(s,\mathbf{\Omega}_Y(U,\alpha))=\frac{1}{h}\sum_\chi\overline{\chi(Y)}\,Z(s,\chi,\mathbf{\Omega}(U,\alpha))$ and, for every $\chi\in\widehat{\mathrm{Cl}(S)}$,

$$Z(s,\chi,\mathbf{\Omega}(U,\alpha))=\Big(\prod_{X\in U}Z_X(s,\chi(X))\Big)\Big(\prod_{X\notin U}\chi(X)^{\alpha(X)}Z_{X,\alpha(X)}(s)\Big),\tag{4.4}$$

where

$$Z_X(s,z)=\sum_{\substack{a\in\mathcal{F}(\mathcal{P})\\ p|a\Rightarrow p\in X}}z^{\Omega(a)}N(a)^{-s},\qquad\sigma>1,$$

and

$$Z_{X,m}(s)=\sum_{\substack{a\in\mathcal{F}(\mathcal{P})\\ p|a\Rightarrow p\in X\\ \Omega(a)=m}}N(a)^{-s},\qquad\sigma>1.$$

Let $P_X(s)=\sum_{p\in\mathcal{P}\cap X}N(p)^{-s}$, $\sigma>1$, $X\in\mathrm{Cl}(S)$. We have (cf. [12]),

$$
\begin{aligned}
(4.5)\qquad P_X(s)&=\frac{1}{h}\sum_{\chi}\overline{\chi(X)}\sum_{k=1}^{\infty}\frac{\mu(k)}{k}\log L(ks,\chi^k)\\
&=\frac{1}{h}\sum_{\chi}\overline{\chi(X)}\bigl(\log L(s,\chi)-\tfrac12\log L(2s,\chi^2)\bigr)+R_{1,X}(s)\\
&=\frac{1}{h}\sum_{\chi}\overline{\chi(X)}\bigl(\log F_\chi(s)-\tfrac12\log F_{\chi^2}(2s)\bigr)+R_{2,X}(s),
\end{aligned}
$$

for some $R_{1,X}(s),R_{2,X}(s)\in\mathcal{A}$. This implies (i) and (ii). We have

$$Z_X(s,z)=\prod_{p\in\mathcal{P}\cap X}\frac{1}{1-zp^{-s}},\qquad \sigma>1,\ |z|\le1,$$

hence

$$\log Z_X(s,z)=\sum_{m=1}^{\infty}\frac{z^m}{m}P_X(ms),\qquad \sigma>1,\ |z|\le 1$$

and

$$
\begin{aligned}
&Z_X(s,\chi(X))\\
&=\Bigl(\prod_{\psi}L(s,\psi)^{\frac1h\chi\psi^{-1}(X)}L(2s,\psi)^{\frac{1}{2h}\chi^2\psi^{-1}(X)}L(2s,\psi^2)^{-\frac{1}{2h}\chi\psi^{-2}(X)}\Bigr)F_{X,\chi(X)}(s),\\
&\hfill \sigma>1,
\end{aligned}
$$

where

$$
\begin{aligned}
\log H_{X,z}(s)=zR_{1,X}(s)&+\frac{z^2}{2}\Bigl(P_X(2s)-\frac1h\sum_\chi\overline{\chi(X)}\log L(2s,\chi)\Bigr)\\
&+\sum_{m=3}^{\infty}\frac{z^m}{m}P_X(ms),\qquad \sigma>1,\ |z|\le1,
\end{aligned}
$$

hence $\log H_{X,z}(s)\in\mathcal{A}$. We also obtain

$$
\begin{aligned}
(4.6)\qquad &Z_{X,m}(s)\\
&=\sum_{k\ge0}\frac{1}{k!}\sum_{\substack{m_1=1\\ m_1+\cdots+m_k=m}}^{\infty}\cdots\sum_{m_k=1}^{\infty}\frac{1}{m_1\ldots m_k}P_X(m_1s)\ldots P_X(m_ks),\qquad \sigma>1.
\end{aligned}
$$

The semigroup S is simple, so each $\log L(s,\chi_j)$ for $j=u+1,\ldots,h-1$ is a linear combination of $\log L(s,\chi_1),\ldots,\log L(s,\chi_u)$ (i.e. without $\log L(s,\chi_0)$) because of the behaviour at 1. Hence, substituting $P_X(m_is)$ in (4.6) using (4.5) we get

$$Z_{X,m}(s)=P_{X,m}(l_2(s),s),$$

where $P_{X,m}$ is a polynomial with the properties stated in (iii). The limit (4.3) is non-zero and finite, hence it is positive, because

$$\Big(\prod_{X\in U} H_{X,1}(\sigma)\Big)G_{U,\chi_0}(\sigma) = \zeta(\sigma, \mathbf{\Omega}(U,0)) > 0, \qquad \sigma > 1.$$

We have shown (iii) and, substituting $Z(s, \mathbf{\Omega}(U,\alpha)) = Z(s, \chi_0, \mathbf{\Omega}(U,\alpha))$ in (4.4), also (iv). ■

Lemma 4.3 and Proposition 4.1, by the classical Tauberian theorem of Delange and Ikehara (cf. also the formulation by Geroldinger and Halter-Koch [7, Theorem 8.2.5]), imply the following.

Corollary 4.4. *If S is a simple L-semigroup and $A \subseteq \mathcal{F}(\mathcal{P})$ is a non-empty Ω-set with $l(A) < \infty$, then the counting function of A satisfies*

$$A(x) \asymp cx(\log x)^{\operatorname{rk}(A)/h-1}(\log\log x)^d,$$

where

$$d = \begin{cases} \deg A, & \operatorname{rk} A > 0,\\ \deg A - 1, & \operatorname{rk} A = 0,\end{cases}$$

and $c > 0$ depends only on A. If, in addition, S satisfies (NVC) *on the line $\Re s = 1$, then $\asymp$ may be replaced with $\sim$.*

4.3. Oscillations of the counting function

Lemma 4.5. *Suppose S is a simple L-semigroup and $A \subseteq \mathcal{F}(\mathcal{P})$ is a non-empty Ω-set with $l(A) < \infty$. The zeta function $Z(s,A)$ has an analytic continuation with possible singularities to an open half-plane containing the half-plane $\Re s \ge 1/2$. If A is (r,d)-singular for some $r \ge 0$ and $d > 0$, then $Z(s,A)$ has a singularity outside the real line in the half-plane $\Re s \ge 1/2$.*

Proof. The first part follows from Lemma 4.3 and Proposition 4.1. Suppose A is (r,d)-singular. Using the unique representation (4.1) and Lemma 4.3 we have

$$Z(s,A) = \sum_{j=1}^{n} \gamma_j \cdot \Big(\prod_{X\notin U_j} P_{X,\alpha_j(X)}(l_2(s), s)\Big) \cdot \frac{1}{h}\sum_{\chi} \chi(X_j)\Big(\prod_{X\in U_j} F_{X,\chi(X)}(s)\Big)G_{U_j,\chi}(s),$$

where $X_j = \prod_{X\notin U} X^{\alpha_j(X)}Y_j^{-1} \in \langle U\rangle$, $j = 1,\dots,n$. Clearly

$$Z(s,A) = P(l_0(s), s)$$

for some $P \in \mathrm{Hol}^{\mathbb{C}}(\mathcal{D}')[x_0,\dots,x_u]$ where $\mathcal{D}' = \mathcal{D} \setminus [2/5, 1/2]$. Let $P_{X,m,1} \in \mathrm{Hol}(\mathcal{D}')[x_0,\dots,x_u]$ denote the polynomial obtained from $P_{X,m}$ by substituting the functions $\log F_0(2s),\dots,\log F_u(2s)$ for the variables $x_{u+1},\dots,x_{2u+1}$. Let $P^{(d)} \in \mathrm{Hol}^{\mathbb{C}}(\mathcal{D}')[x_1,\dots,x_u]$ and $P_j^{(d)} \in \mathrm{Hol}(\mathcal{D}')[x_1,\dots,x_u]$ denote the coefficients of x_0^d in the polynomials P and $\prod_{X\notin U_j} P_{X,\alpha_j(X),1}$, respectively. We have

$$P^{(d)}(l_1(s),s)=\sum_{j=1}^{n}\frac{\gamma_j}{h}P_j^{(d)}(l_1(s),s)\sum_{\chi}\chi(X_j)\Big(\prod_{X\in U_j}H_{X,\chi(X)}(s)\Big)G_{U_j,\chi}(s).$$

Since A is (r,d)-singular, we can write $\{1,\ldots,n\}=I_1\cup I_2\cup I_3$, where I_1 is the set of all j such that $\sum_X\alpha_j(X)<d$, I_2 is the set of all $j\in\{1,\ldots,n\}$ such that $|U_j|<r$, and I_3 is the set of all j such that $|U_j|=r$ and $\sum_X\alpha_j(X)=d$. Let $\omega\in\{1,-1\}$ match the sign of γ_j for $j\in I_3$ (all these signs are equal by the definition of (r,d)-singular sets). We have $P_j^{(d)}=0$ for $j\in I_1$. For every j, X and χ the function $P_j^{(d)}(l_1(s),s)$ is bounded close to $s=1$ and we have

$$H_{X,\chi(X)}(s)\sim 1,\quad s\to 1,$$
$$G_{U_j,\chi}(s)\sim(s-1)^{-\langle\chi|U_j\rangle},\quad s\to 1.$$

For $j\in I_2$ and arbitrary χ, as well as for $j\in I_3$ and $U\not\subseteq\ker\chi$, we have $\langle\chi\,|\,U_j\rangle<r/h$, so

$$\lim_{\sigma\to1^+}P_j^{(d)}(l_1(\sigma),\sigma)\Big(\prod_{X\in U_j}H_{X,\chi(X)}(\sigma)\Big)G_{U_j,\chi}(s)(\sigma-1)^{r/h}=0.$$

Finally, for $j\in I_3$ and $U_j\subseteq\ker\chi$ we have $\chi(X_j)=1$, $\omega\gamma_j>0$, $P_j^{(d)}=\prod_{X\notin U_j}(h^{\alpha_j(X)}\alpha_j(X)!)^{-1}$ (a constant), and $G_{U_j,\chi}=G_{U_j,\chi_0}$. The number of such χ for a given U_j is $h/|\langle U_j\rangle|$. We obtain

$$\lim_{\sigma\to1^+}\omega P^{(d)}(l_1(\sigma),\sigma)(\sigma-1)^{r/h}$$
$$=\sum_{j\in I_3}\frac{\omega\gamma_jP_j^{(d)}}{|\langle U_j\rangle|}\lim_{\sigma\to1^+}\Big(\prod_{X\in U_j}H_{X,1}(\sigma)\Big)G_{U_j,\chi_0}(\sigma)(\sigma-1)^{r/h}>0,$$

so $P^{(d)}$ cannot vanish identically, and thus $\deg P\ge d>0$. By Theorem 3.1 the function $Z(s,A)$ has the required singularity. ■

LEMMA 4.6. *Let $S\subseteq\mathcal{F}(\mathcal{P})$ be a simple L-semigroup and $A\subseteq\mathcal{F}(\mathcal{P})$ an Ω-set with $l(A)<\infty$. Then the Mellin transform $s^{-1}Z(s,A)$ of the function $\frac12(A(x-0)+A(x+0))$ is weakly bounded.*

Proof. Let (4.1) be the unique representation of A. Let $\sigma_0\in(2/5,1/2)$ be sufficiently close to $1/2$ so that all the $f_\chi(s)$ are absolutely convergent and non-zero in $\sigma\ge\sigma_0$ and $L(s,\chi)L(2s,\chi)$ has no zeros on the segment $[\sigma_0,1/2)$, $\chi\in\widehat{\mathrm{Cl}(S)}$. Let C denote the set of all zeros and poles of $L(ns,\chi)$ for $\sigma_0\le\sigma\le1$, $\chi\in\widehat{\mathrm{Cl}(S)}$, $n=1,2$. Let $\mathcal{D}\subseteq\mathbb{C}$ be a region containing the strip $\sigma_0\le\sigma\le1$, contained in the strip $2/5\le\sigma\le1$, and not containing any zeros of $L(ns,\chi)$ for $2/5 \le \sigma < \sigma_0$, $\chi \in \widehat{\mathrm{Cl}(S)}$, $n = 1, 2$. The assertion follows from Proposition 4.1 and Lemmas 2.1, 2.2 and 4.3. ∎

THEOREM 4.7. *Let $S \subseteq \mathcal{F}(\mathcal{P})$ be a simple L-semigroup and $A \subseteq \mathcal{F}(\mathcal{P})$ an Ω-set with $l(A) < \infty$. If A is (r,d)-singular for some $r \ge 0$, $d > 0$, then the error term of the counting function of A has oscillations of logarithmic frequency and size $\sqrt{x}(\log x)^{-M}$ for some $M > 0$.*

Proof. By Theorem 3.2 and Lemma 4.6, it suffices to show that $s^{-1}Z(s,A)$ has a singularity in the half-plane $\sigma \ge 1/2$, not on the real line. This, in turn, follows from Lemma 4.5. ∎

5. Applications

5.1. Proof of Theorem 1.1. We recall that $G_0 \subseteq \mathrm{Cl}(S)$ denotes the set of classes that contain prime divisors. The sets $\boldsymbol{G}_k$ are closely related to so-called half-factorial subsets of G_0. One of the equivalent definitions of a half-factorial set $U \subseteq G_0$ is that we should have $\boldsymbol{\Omega}_E(U,0) \subseteq \boldsymbol{G}_1$ (cf., e.g., Skula [37], Śliwa [38] and, for the modern terminology, Geroldinger and Halter-Koch [7]). We have the following.

THEOREM 5.1 ([34]). *Let k be a positive integer. The set $\boldsymbol{G}_k$ is (r,d)-singular for some $r \in \mathbb{N}_0$ and $d \in \mathbb{N}$ if and only if the set G_0 is not half-factorial.*

Since S is a simple L-semigroup, we have $G_0 = \mathrm{Cl}(S)$ by Lemma 4.3. As shown by Carlitz [2] and, in greater generality, by Skula [37, Proposition 3.2], in this case G_0 is half-factorial if and only if $h \le 2$. Hence Theorem 5.1 implies the assertion.

5.2. Elements without divisors in a given Ω-set. Let $F \subseteq \mathcal{F}(\mathcal{P})$ denote a non-empty Ω-set. Then the set

$$A = \mathcal{F}(\mathcal{P}) \setminus F\mathcal{F}(\mathcal{P}), \tag{5.1}$$

i.e. the set of elements of $\mathcal{F}(\mathcal{P})$ without divisors in F, is also an Ω-set. Let $F_{\min}$ denote the set of minimal elements (with respect to division) of F.

THEOREM 5.2 ([34]). *If $F \subseteq \mathcal{F}(\mathcal{P})$ is an Ω-set, then the set* (5.1) *is (r,d)-singular for some $r \in \mathbb{N}_0$ and $d \in \mathbb{N}$ if and only if*

$$\max_{b \in F_{\min}} \max_{X} \Omega_X(b) > 1. \tag{5.2}$$

COROLLARY 5.3. *If S is a simple L-semigroup and $F \subseteq \mathcal{F}(\mathcal{P})$ is an Ω-set such that* (5.2) *holds, then the error term of the counting function of* (5.1) *has oscillations of logarithmic frequency and size $\sqrt{x}(\log x)^{-M}$ for some $M > 0$.*

5.3. Ideals without divisors in a given arithmetic progression. Let K be an algebraic number field, $\mathcal{O}_K$ its ring of integers, and $\zeta_K(s)$ the Dedekind zeta function. Let $\mathfrak{a},\mathfrak{q}$ be non-zero integral ideals of $\mathcal{O}_K$. We write $\mathfrak{a}'\equiv\mathfrak{a} \pmod{\mathfrak{q}}$ if, for some totally positive $a,a'\in\mathcal{O}_K$ with $a\equiv a'\equiv 1 \pmod{\mathfrak{q}}$ we have $a\mathfrak{a}=a'\mathfrak{a}'$. In particular, when $\mathfrak{a}$ is relatively prime to $\mathfrak{q}$, this congruence is equivalent to $\mathfrak{a}'$ being in the class $[\mathfrak{a}]$ of $H^*_{\mathfrak{q}}(K)$. We say that an ideal $\mathfrak{b}\in\mathcal{I}(\mathcal{O}_K)$ *has no non-trivial divisors congruent to* $\mathfrak{a}$ *modulo* $\mathfrak{q}$ if for every $\mathfrak{a}'\equiv\mathfrak{a} \pmod{\mathfrak{q}}$ with $\mathfrak{a}'\neq\gcd(\mathfrak{a},\mathfrak{q})$, we have $\mathfrak{a}'\nmid\mathfrak{b}$. Let $B(\mathfrak{a},\mathfrak{q})$ denote the set of all ideals $\mathfrak{b}\in\mathcal{I}(\mathcal{O}_K)$ that have no non-trivial divisors congruent to $\mathfrak{a}$ modulo $\mathfrak{q}$.

THEOREM 5.4. *Let $\mathfrak{a},\mathfrak{q}\in\mathcal{I}(\mathcal{O}_K)$, $\mathfrak{d}=\gcd(\mathfrak{a},\mathfrak{q})$, $\mathfrak{a}_1=\mathfrak{d}^{-1}\mathfrak{a}$, and $\mathfrak{f}=\mathfrak{d}^{-1}\mathfrak{q}$. Let S be the generalized Hilbert semigroup modulo $\mathfrak{f}$. If*

(i) *$[\mathfrak{a}_1]=E$ and $|\mathrm{Cl}(S)|\geq 2$, or*
(ii) *$[\mathfrak{a}_1]\neq E$ and $\mathrm{Cl}(S)\not\cong C(2)^m$, $m\in\mathbb{N}$, or*
(iii) *$[\mathfrak{a}_1]\neq E$ and there exist some $\chi\in\widehat{\mathrm{Cl}(S)}$ and a simple zero ρ of $L(s,\chi)$ such that $\rho\in\mathbb{C}\setminus\mathbb{R}$, $\Re\rho\geq 1/2$, $\chi([\mathfrak{a}_1])\neq 0$, and*

$$L(\rho,\psi)\neq 0,\qquad \psi\in\widehat{\mathrm{Cl}(S)}\setminus\{\chi\},$$

then the error term of the counting function of the set $B=B(\mathfrak{a},\mathfrak{q})$ has oscillations of logarithmic frequency and size $\sqrt{x}(\log x)^{-M}$ for some $M>0$.

Proof. We have $B=\mathcal{I}(\mathcal{O}_K)\setminus\mathfrak{d}F\mathcal{I}(\mathcal{O}_K)$, where $F=[\mathfrak{a}_1]\setminus\{\mathcal{O}_K\}$ if $[\mathfrak{a}_1]=E$, and $F=[\mathfrak{a}_1]$ otherwise. Hence

$$Z(s,B)=(1-N(\mathfrak{d})^{-s})\zeta_K(s)+\prod_{\mathfrak{p}\mid\mathfrak{f}}(1-N(\mathfrak{p})^{-s})^{-1}Z(s,A), \tag{5.3}$$

where $A=\mathcal{F}(\mathcal{P})\setminus F\mathcal{F}(\mathcal{P})$. It follows from (5.3) and Lemma 4.6 that the counting function of B has a weakly bounded Mellin transform, so by Theorem 3.2 it suffices to show that the function $Z(s,B)$ has a singularity outside the real line in the half-plane $\Re s\geq 1/2$. If (5.2) is satisfied, then the set A is (r,d)-singular for some $r\geq 0$ and $d>0$ by Theorem 5.2, so, by (5.3) and Lemma 4.5, an appropriate singularity exists and the assertion follows. We check the condition (5.2) in several cases.

CASE 1: $[\mathfrak{a}_1]=E$, $h\geq 2$. Then $F=E\setminus\{\mathcal{O}_K\}$, so $F_{\min}$ is equal to the set $\mathcal{A}(S)$ of irreducibles of S. If $\mathfrak{p}\in\mathcal{P}$ and $X=[\mathfrak{p}]$ is of order $m>1$, then $\mathfrak{p}^m\in\mathcal{A}(S)$ and $\Omega_X(\mathfrak{p}^m)=m$, so (5.2) holds.

CASE 2: $[\mathfrak{a}_1]\neq E$ and $\mathrm{Cl}(S)\cong C(2)^m$ for some $m\in\mathbb{N}$. Then $F=[\mathfrak{a}_1]$. Suppose $\mathfrak{b}\in F$ and $\mathfrak{p},\mathfrak{q}\in\mathcal{P}$ are such that $[\mathfrak{p}]=[\mathfrak{q}]$ and $\mathfrak{p}\mathfrak{q}\mid\mathfrak{b}$. Then $[(\mathfrak{p}\mathfrak{q})^{-1}\mathfrak{b}]=[\mathfrak{b}]$, so $(\mathfrak{p}\mathfrak{q})^{-1}\mathfrak{b}\in F$. Hence we have $\Omega_X(\mathfrak{b})\leq 1$ for all $\mathfrak{b}\in F_{\min}$ and $X\in\mathrm{Cl}(S)$, i.e. (5.2) fails.

CASE 3: $[\mathfrak{a}_1] \neq E$ and $\mathrm{Cl}(S) \ncong C(2)^m$, $m \in \mathbb{N}$. Then $F = [\mathfrak{a}_1]$ again. If $[\mathfrak{a}_1]$ is a square in $\mathrm{Cl}(S)$, then we can find some $\mathfrak{r} \in \mathcal{P}$ such that $[\mathfrak{r}]^2 = [\mathfrak{a}_1]$, so $\mathfrak{r}^2 \in F_{\min}$ and (5.2) holds. Otherwise let $X \in \mathrm{Cl}(S)$, $Y = X^{-2}[\mathfrak{a}_1]$, and let $\mathfrak{p} \in \mathcal{P} \cap X$ and $\mathfrak{q} \in \mathcal{P} \cap Y$. We have $\mathfrak{p}^2\mathfrak{q} \in F_{\min}$ (implying (5.2)) if and only if $X, Y, X^2, XY \neq E$, i.e. $\{X, X^2\} \cap \{E, [\mathfrak{a}_1]\} = \emptyset$, for which a sufficient condition is

$$X^2 \notin \{E, [\mathfrak{a}_1]^2\}, \tag{5.4}$$

since $[\mathfrak{a}_1]$ is not a square. Such an X may always be found when $\mathrm{Cl}(S)$ has an element of order ≥ 5, so suppose

$$\mathrm{Cl}(S) \cong C(2)^a \oplus C(4)^b \oplus C(3)^c, \qquad b + c > 0.$$

The number of squares in this group is $2^b 3^c$, so if $2^b 3^c > 2$, then again we can find X satisfying (5.4). Otherwise $b = 1$ and $c = 0$. If $\operatorname{ord}[\mathfrak{a}_1] = 4$, then $\mathfrak{r}^3 \in F_{\min}$ for $\mathfrak{r} \in \mathcal{P} \cap [\mathfrak{a}_1]^{-1}$, so (5.2) holds again. Otherwise $\operatorname{ord}[\mathfrak{a}_1] = 2$, so (5.4) holds for any X of order 4.

It remains to consider Case 2 when the assumption (iii) holds for some χ and ρ. Similarly to [1] and [28, proof of Lemma 7] we have

$$A = \bigcup_{i=1}^{m} \mathbf{\Omega}(H_i, \alpha_i), \tag{5.5}$$

where, for every $i = 1, \ldots, m$, the set H_i is a subgroup of $\mathrm{Cl}(S)$ not containing $[\mathfrak{a}_1]$, $\alpha_i : \mathrm{Cl}(S) \to \mathbb{N}_0$, $\alpha_i|_{H_i} = 0$, and the set $\mathbf{\Omega}(H_i, \alpha_i)$ is non-empty and inclusion-maximal in (5.5). By the inclusion-exclusion principle we obtain

$$A = \sum_{j=1}^{n} \gamma_j \cdot \mathbf{\Omega}(U_j, \beta_j)$$

for some integer γ_j, $\mathbf{\Omega}(U_j, \beta_j) \neq \emptyset$, $\beta_j : \mathrm{Cl}(S) \to \mathbb{N}_0$, all U_j pairwise distinct, each U_j equal to an intersection of one or more of the H_i, and, whenever $U_j = H_i$ for some i, j, we have $\gamma_j = 1$ and $\beta_j = \alpha_i$. As each set $\mathbf{\Omega}(U_j, \beta_j)$ is a union of disjoint sets of the form $\mathbf{\Omega}_X(U_j, \beta_j)$, it follows that

$$\deg_0(A) \geq \max_{1 \leq i \leq m} \sum_X \alpha_i(X).$$

Since (5.2) is false and $\mathrm{Div}(A) = A$, Theorems 4.2 and 5.2 imply

$$\deg_0(A) = 0,$$

so that all the α_is and β_js must be zero. If $H \subseteq \mathrm{Cl}(S)$ is a subgroup with $[\mathfrak{a}_1] \notin H$, then $\mathbf{\Omega}(H, 0) \subseteq A$. Consequently the H_is are all the inclusion-maximal subgroups of $\mathrm{Cl}(S)$ not containing $[\mathfrak{a}_1]$. We have $|\ker \chi| = 2^{m-1}$, so

$\ker\chi$ is one of the H_i, say $\ker\chi = H_1$. By the final assertion of Lemma 4.3 we have

$$Z(s,A) = \sum_{j=1}^{n} \gamma_j \Big(\prod_{X\in U_j} H_{X,1}(s)\Big) G_{U_j,\chi_0}(s)$$

and $G_{U_j,\chi_0}(s) = (s-\rho)^{\langle \chi^{-1} \mid U_j\rangle} G_j(s)$ for some $G_j(s)$ non-zero and regular in a neighbourhood of ρ, $j = 1,\dots,n$. We have

$$\langle \chi^{-1} \mid U_j\rangle = \begin{cases} |U_j|/2^m, & U_j \subseteq \ker\chi,\\ 0 & \text{otherwise,}\end{cases}$$

hence $\langle \chi^{-1} \mid U_1\rangle = 1/2$ and $\langle \chi^{-1} \mid U_j\rangle \in [0,1/4]$ for $j = 2,\dots,m$. It follows from Lemma 3.3 that $Z(s,A)$ has the required singularity, and so does $Z(s,B)$. ∎

5.4. Proof of Theorem 1.2. Let $f = q/\gcd(a,q)$. We apply Theorem 5.4 with $K = \mathbb{Q}$, $\mathfrak{a} = a\mathbb{Z}$ and $\mathfrak{q} = q\mathbb{Z}$, so S is the Hilbert semigroup modulo f. The assertion follows directly except for the case $\mathrm{Cl}(S) \cong \Phi(f) \cong C(2)^m$, $m \in \mathbb{N}$, where $\Phi(f)$ denotes the group of units of the ring $\mathbb{Z}/f\mathbb{Z}$. In that case we have $f \in \{3,4,6,8,12,24\}$. Let χ_1 denote the primitive Dirichlet character modulo 3, χ_2 the primitive character modulo 4, and χ_3 the primitive odd character modulo 8. With the help of the PARI/GP [39] system, the ComputeL package by Dokchitser [4], and some scripts of the present author [29], we can find ρ_1, ρ_2, ρ_3 such that for each $i = 1,2,3$ the number ρ_i is a simple zero of $L(s,\chi_i)$, $\Im\rho_i \neq 0$, $\Re\rho_i \geq 1/2$, and $L(\rho_i,\psi^*) \neq 0$ for all characters ψ modulo 24, with $\psi^* \neq \chi_i$, for example:

$$\rho_1 \approx 1/2 + 8.03973715568146i,$$
$$\rho_2 \approx 1/2 + 6.02094890469760i,$$
$$\rho_3 \approx 1/2 + 3.57615483678759i.$$

We have $\ker\chi_1 = \{1\}$ in $\Phi(3)$ and in $\Phi(6)$, and $\ker\chi_2 = \{1\}$ in $\Phi(4)$, hence also $\ker\chi_1 \cap \ker\chi_2 = \{1\}$ in $\Phi(12)$. Similarly $\ker\chi_2 \cap \ker\chi_3 = \{1\}$ in $\Phi(8)$ and $\ker\chi_1 \cap \ker\chi_2 \cap \ker\chi_3 = \{1\}$ in $\Phi(24)$. Therefore the hypotheses of (iii) in Theorem 5.4 are satisfied for $f \in \{3,4,6,8,12,24\}$ and the assertion follows again.

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