ACTA ARITHMETICA
120.4 (2005)

A characterization of some q-multiplicative functions

by

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1. INTRODUCTION

1.1. DEFINITION. Let $\mathbb{N}$ be the set of non-negative integers, and let $q > 1$ be an integer. To every element n of $\mathbb{N}$, one can associate a unique representation

$$n = \sum_{k=0}^{\infty} a_k(n) q^k, \qquad 0 \le a_k(n) \le q-1.$$

Following Gelfond [2], a complex-valued arithmetic function f such that $f(0 \cdot q^k) = 1$ for all $k \ge 0$ and

$$f(n) = \prod_{k \ge 0} f(a_k(n) q^k)$$

is called a *q-multiplicative function.*

1.2. Introductory remarks. Since the first investigations of Delange [1], the study of q-additive functions, and q-multiplicative functions of modulus 1 has been developed by many authors. Apparently, the case of q-multiplicative functions not of modulus 1 does not seem to have been so popular, and concerning this topic, we can cite, as recent references, an article of Spilker [6] and another one of Lee [4], both relating to the almost-periodicity of q-multiplicative functions. In this article, we shall give some results concerning a class of q-multiplicative functions satisfying a growth condition.

2. RESULTS

We shall prove the following results:

THEOREM 1. *Let f be a non-negative q-multiplicative function. Then* (i)&(ii)$\Leftrightarrow$(iii)&(iv), *where*

2000 *Mathematics Subject Classification*: 11A25, 11N64, 11N56.
Key words and phrases: mean-value, q-multiplicative functions.

(i) $0 < \limsup_{x\to\infty} \frac{1}{x} \sum_{n\le x} f(n) < \infty$,

(ii) *if* $I(\cdot)$ *is the characteristic function of a subset of* $\mathbb{N}$ *then*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{0\le n<x} I(n) = 0 \;\Rightarrow\; \lim_{x\to\infty} \frac{1}{x} \sum_{0\le n<x} I(n)f(n) = 0,$$

(iii) $\sum_{r\in\mathbb{N}} \sum_{0\le a\le q-1} (1 - f(aq^r))^2 < \infty$,

(iv) $\limsup_{k\to\infty} \sum_{0\le r\le k} \sum_{0\le a\le q-1} (f(aq^r) - 1) < \infty$.

We also have

Theorem 2. *Let* f *be a non-negative* q*-multiplicative function satisfying conditions* (i) *and* (ii) *of Theorem* 1*. Then, for all* $r \ge 0$, $f(\cdot)^r$ *satisfies the same conditions.*

Now, for y in $\mathbb{N}$, we define a function $F_{y-}(\cdot)$ by

$$F_{y-}(n) = \Big(\prod_{0\le k\le y-1} f(a_k(n)q^k)\Big)\Big(\prod_{0\le j\le y-1} \frac{1}{q} \sum_{0\le a\le q-1} f(aq^j)\Big)^{-1}.$$

We have the following result:

Proposition 3. *Let* f *be a non-negative* q*-multiplicative function satisfying conditions* (i) *and* (ii) *of Theorem* 1*. Then, given any* $\varepsilon > 0$*, there exists a* $Y(\varepsilon)$ *in* $\mathbb{N}$ *such that if* $y \ge Y(\varepsilon)$*, then*

$$\limsup_{x\to\infty} \frac{1}{x} \sum_{0\le n\le x} \left| F_{y-}(n) - \frac{f(n)}{\prod_{0\le r\le \log x/\log q} \frac{1}{q} \sum_{0\le a\le q-1} f(aq^r)} \right| \le \varepsilon,$$

which implies that

$$\lim_{x\to\infty} \Big(\frac{1}{x} \sum_{0\le n\le x-1} f(n)\Big)\Big(\prod_{0\le r\le \log x/\log q} \frac{1}{q} \sum_{0\le a\le q-1} f(aq^r)\Big)^{-1} = 1.$$

Remark 1. Condition (ii) can be replaced, for instance, by: for any $\varepsilon > 0$, there exists $\eta > 0$ such that, if $I(\cdot)$ is the characteristic function of a subset of $\mathbb{N}$ then

$$\limsup_{x\to\infty} \frac{1}{x} \sum_{0\le n<x} I(n) \le \eta \;\Rightarrow \limsup_{x\to\infty} \frac{1}{x} \sum_{0\le n<x} I(n)f(n) \le \varepsilon.$$

The next result completes the first one in the general case.

Theorem 4. *Let f be a complex-valued q-multiplicative function. Define a q-multiplicative function f^* of modulus* 1 *or* 0 *by*

$$f^*(n) = \begin{cases} f(n)|f(n)|^{-1} & \text{if } f(n) \neq 0, \\ 0 & \text{if } f(n) = 0. \end{cases}$$

Suppose that

(i) $0 < \limsup_{x\to\infty} \frac{1}{x} \sum_{n \leq x} |f(n)| < \infty$,

(ii) *if $I(\cdot)$ is the characteristic function of a subset of $\mathbb{N}$ then*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{0 \leq n < x} I(n) = 0 \;\Rightarrow\; \lim_{x\to\infty} \frac{1}{x} \sum_{0\leq n<x} I(n) f(n) = 0.$$

Then

($\mathcal{S}$) *the non-negative q-multiplicative function $|f(\cdot)|$ satisfies* (ii).

Under conditions (i), (ii) *and*

(iii) $0 < \limsup_{x\to\infty} \left| \frac{1}{x} \sum_{q^r \leq n \leq x} f(n) \right| < \infty$ *for some $r \geq 0$,*

we have not only ($\mathcal{S}$) but also

($\mathcal{S}'$) $$\sum_{k\geq 0} \sum_{0\leq a \leq q-1} (1 - \operatorname{Re} f^*(aq^k)) < \infty.$$

Moreover, ($\mathcal{S}$) $\Leftrightarrow$ (i)&(ii), *and* ($\mathcal{S}$)&($\mathcal{S}'$) $\Leftrightarrow$ (i)&(ii)&(iii).

3. PROOFS

3.1. Proof of Theorem 1. The steps of the proof are the following:

1) we remark that there is a natural associated structure of a compact space Z_q equipped with a probability measure μ;
2) we study the structure of the open sets of this space, and prove that they are disjoint unions of “elementary” components;
3) we build a (pre-)measure ν on these open sets;
4) we remark that it defines a Borel measure, still denoted by ν;
5) this Borel measure is absolutely continuous with respect to μ;
6) we give an explicit formula for $d\nu/d\mu$ and get Proposition 3;
7) from classical results of probability theory, we deduce Theorems 1 and 2.

Step 1: *Compact space associated to a q-multiplicative function.* Let $q > 1$ be an integer, and f a q-multiplicative function. We denote by Z_q the compact space $(\mathbb{Z}/q\mathbb{Z})^{\mathbb{N}}$ equipped with the measure $\mu = \bigotimes_N \mu_q$, where μ_q

is the uniform measure on the discrete space $\mathbb{Z}/q\mathbb{Z}$. An element a of Z_q can be written as $a = (a_0, a_1, \ldots)$, $0 \le a_k \le q-1$, $k \ge 0$, and an integer is an element of Z_q which has only a finite number of digits different from zero. For $a = (a_0, a_1, \ldots) \in Z_q$ and $k \ge 0$ we set

$$x_{k-}(a) = \{a_j\}_{0\le j\le k-1}, \qquad x_{k+}(a) = \{a_j\}_{j\ge k}.$$

These are two sequences of random variables on Z_q. We have the identity

$$\prod_{0\le j\le k-1} \frac{1}{q} \sum_{0\le a\le q-1} f(aq^j) = \int_{Z_q} f(x_{k-})\, d\mu.$$

STEP 2: *Open sets in Z_q.* We denote by $(a, k(a))$ the arithmetical progression $\{a + q^{k(a)}n\}_{n\in\mathbb{N}}$, where $a, k(a) \in \mathbb{N}$ satisfy $k(a) \ge \log a/\log q$, and by $I_{a,k(a)}$ its characteristic function. Note that $I_{a,k(a)}$ is the restriction to $\mathbb{N}$ of the characteristic function, still denoted $I_{a,k(a)}$, of the elementary open subset $O_{(a,k(a))}$ of Z_q defined by

$$O_{(a,k(a))} = (x_{k(a)-}(a), x_{k(a)+}(Z_q)),$$

and that this function is continuous, which implies that

$$\lim_{x\to\infty} \frac{1}{x} \sum_{0\le n<x} I_{a,k(a)}(n) = \mu(O_{(a,k(a))}).$$

We have the following lemma:

LEMMA 5. *Let O be an open set in Z_q, and I_O its characteristic function. Then there exists a subset $A(O)$ of $\mathbb{N}$ such that I_O can be written as $I_O = \sum_{a\in A(O)} I_{a,k(a)}$, i.e. O can be written as the disjoint union $\bigcup_{a\in A(O)} O_{(a,k(a))}$.*

Proof. If O is an open set, then for a given a in O, there exists an elementary open set $O_{(x_{k(a)-}(a),k(a))}$ such that $O_{(x_{k(a)-}(a),k(a))} \subseteq O$. So, $O = \bigcup_{a\in O} O_{(x_{k(a)-}(a),k(a))}$. Now, if $O_{(x_{k(a)-}(a),k(a))} \cap O_{(x_{k(b)-}(b),k(b))} \ne \emptyset$, then one of these two sets is contained in the other. As a consequence, O can be written as a disjoint union $\bigcup_{c\in A(O)} O_{(c,k(c))}$, and so $I_O = \sum_{c\in A(O)} I_{c,k(c)}$. ∎

STEP 3: *Definition of a measure ν on the open sets of Z_q.* Given a non-negative q-multiplicative function f such that

$$0 < S = \limsup_{x\to\infty} \frac{1}{x} \sum_{0\le n<x} f(n) < \infty,$$

we can define a measure ν on the open sets of Z_q in the following way.

First, we remark that

$$0 < S' = \limsup_{k\to\infty} \frac{1}{q^k} \sum_{0\le n\le q^k-1} f(n) < \infty. \tag{1}$$

For let x_i be a sequence such that

$$\frac{1}{2}\,S \le \frac{1}{x_i}\sum_{0\le n<x_i} f(n).$$

Then a fortiori,

$$\frac{1}{2}\,S \le \frac{1}{x_i}\sum_{0\le n\le q^{\log_q(x_i)+1}-1} f(n)$$

and so

$$\left(\frac{q^{\log_q(x_i)+1}}{x_i}\right)^{-1}\left(\frac{1}{2}\,S\right) \le \frac{1}{q^{k(x_i)+1}}\sum_{0\le n\le q^{k(x_i)+1}-1} f(n).$$

Since $(q^{\log_q(x_i)+1}/x_i)^{-1} \ge 1/q$, this shows that there is some $S' \ge \frac{1}{2q}S$, hence > 0, such that

$$0 < S' \le \limsup_{k\to\infty}\frac{1}{q^k}\sum_{0\le n\le q^k-1} f(n) < \infty.$$

Now, for a given $I_{a,k(a)}$, if $k \ge k(a)$, we have

$$\begin{aligned}
&\frac{1}{q^k}\sum_{0\le n\le q^k-1} f(n)I_{a,k(a)}(n)\\
&\qquad = \frac{f(a)}{\sum_{0\le n\le q^{k(a)}-1} f(n)}\left(\frac{1}{q^k}\sum_{0\le n\le q^k-1} f(n)\right)\\
&\qquad = f(a)\Big(\prod_{0\le k\le k(a)-1}\sum_{0\le b\le q-1} f(bq^k)\Big)^{-1}\left(\frac{1}{q^k}\sum_{0\le n\le q^k-1} f(n)\right),
\end{aligned}$$

and so we shall define $\nu(I_{a,k(a)})$ by

$$\nu(I_{a,k(a)}) = \frac{1}{S'}\limsup_{k\to\infty}\frac{1}{q^k}\sum_{0\le n\le q^k-1} f(n)I_{a,k(a)}(n),$$

i.e.

$$\nu(I_{a,k(a)}) = \frac{1}{S'}\,f(a)\Big(\prod_{0\le k\le k(a)-1}\sum_{0\le b\le q-1} f(bq^k)\Big)^{-1}\limsup_{k\to\infty}\frac{1}{q^k}\sum_{0\le n\le q^k-1} f(n),$$

which gives

$$\begin{aligned}
\nu(I_{a,k(a)}) &= \frac{1}{S'}\,f(a)\Big(\prod_{0\le k\le k(a)-1}\sum_{0\le b\le q-1} f(bq^k)\Big)^{-1}S'\\
&= f(a)\Big(\prod_{0\le k\le k(a)-1}\sum_{0\le b\le q-1} f(bq^k)\Big)^{-1}.
\end{aligned}$$

Remark 2. ν is well defined due to the very special structure of the open sets of Z_q.

REMARK 3. By (1), there exists a sequence K of positive integers k such that

$$\lim_{\substack{k\in K\\ k\to\infty}} \frac{1}{q^k} \sum_{0\le n\le q^k-1} f(n) = \limsup_{r\to\infty} \frac{1}{q^r} \sum_{0\le n\le q^r-1} f(n).$$

We fix such a sequence. The important point in the choice of K is not the mere existence of the lim sup, but the fact that the sequence of averages $q^{-k}\sum_{0\le n\le q^k-1} f(n)$, $k\in K$, has a limit point not equal to zero. This remark will be useful for the proof of Theorem 4.

STEP 4: *ν is a Borel measure.* We now consider the set $\mathcal{A}$ of complex-valued continuous functions defined on Z_q by

$$\mathcal{A} = \Big\{ h = \sum_{l_a\in L} l_a I_{a,k(a)};\ L \text{ finite}, l_a \in \mathbb{C} \Big\}.$$

This is an algebra of step functions, and we can assume that $I_{a,k(a)} I_{a',k(a')} = 0$ if $(a,k(a)) \ne (a',k(a'))$. By the Stone–Weierstrass theorem ([3, p. 101, note 1.a]), this algebra is dense for the uniform topology in the set of complex-valued continuous functions on Z_q. We define $\nu(h)$ by $\nu(h) = \sum_{l_a\in L} l_a \nu(I_{a,k(a)})$. Note that this definition agrees with the definition of $\nu(I_{a,k(a)})$ given above and does not depend on the way h is written, since

$$\begin{aligned}
&\frac{1}{q^k} \sum_{0\le n\le q^k-1} f(n)h(n) \\
&\quad = \frac{1}{q^k} \sum_{0\le n\le q^k-1} f(n) \sum_{l_a\in L} l_a I_{a,k(a)}(n) \\
&\quad = \Big(\sum_{l_a\in L} l_a f(a) \Big(\prod_{0\le k\le k(a)-1} \sum_{0\le b\le q-1} f(bq^k) \Big)^{-1} \Big) \frac{1}{q^k} \sum_{0\le n\le q^k-1} f(n),
\end{aligned}$$

and so

$$\begin{aligned}
&\frac{1}{S'} \lim_{\substack{k\in K\\ k\to\infty}} \frac{1}{q^k} \sum_{0\le n\le q^k-1} f(n)h(n) \\
&= \frac{1}{S'} \Big(\sum_{l_a\in L} l_a f(a) \Big(\prod_{0\le k\le k(a)-1} \sum_{0\le b\le q-1} f(bq^k) \Big)^{-1} \Big) \lim_{\substack{k\in K\\ k\to\infty}} \frac{1}{q^k} \sum_{0\le n\le q^k-1} f(n) \\
&= \sum_{l_a\in L} l_a f(a) \Big(\prod_{0\le k\le k(a)-1} \sum_{0\le b\le q-1} f(bq^k) \Big)^{-1} \\
&= \nu(h) = \sum_{l_a\in L} l_a \nu(I_{a,k(a)}).
\end{aligned}$$

Observe also that $\nu(1) = 1$. Now, it is immediate that, given $\varepsilon > 0$, if $h, h' \in \mathcal{A}$ satisfy $\sup_{t\in Z_q} |h'(t) - h(t)| \le \varepsilon$, then $|\nu(h'-h)| \le \varepsilon$, since $h'-h$ can be written as $\sum_{l_a\in L} l_a I_{a,k(a)}$ with $I_{a,k(a)} I_{a',k(a')} = 0$ if $(a, k(a)) \ne (a', k(a'))$, and so $|l_a| \le \varepsilon$. Hence we get

$$|h' - h| = \sum_{l_a\in L} |l_a| I_{a,k(a)} \le \sum_{l_a\in L} \varepsilon I_{a,k(a)},$$

which gives

$$\nu(|h' - h|) \le \sum_{l_a\in L} |l_a| \nu(I_{a,k(a)}) \le \varepsilon \sum_{l_a\in L} \nu(I_{a,k(a)}) \le \varepsilon\nu(1) \le \varepsilon \cdot 1 = \varepsilon.$$

As a consequence, ν defines a continuous linear form on the set of complex-valued continuous functions defined on Z_q. By the Riesz representation theorem ([3, p. 129, (11.37)]), this shows that ν is a Borel measure on Z_q.

Step 5: *Absolute continuity of ν with respect to μ.* Let B be a Borel subset of Z_q. Then, given $\varepsilon > 0$, there exists an open set O and a compact set K such that $K \subseteq B \subseteq O$ and $\mu(O - K) \le \varepsilon$. Since $\nu(1) = 1$ and ν is defined on the open sets of Z_q, we know that $\nu(K)$ can be defined by $\nu(K) = 1 - \nu(Z_q - K)$, and to prove that B is ν-measurable, using the Lusin criterion ([5, p. 68, (vii)]), it will be sufficient to show that given a sequence $\{O_j\}_{j\in\mathbb{N}^*}$ of open sets such that $\lim_{j\to\infty} \mu(O_j) = 0$, we have $\lim_{j\to\infty} \nu(O_j) = 0$.

Assume the contrary, i.e. that there exists a sequence $\{O_j\}_{j\in\mathbb{N}^*}$ of open sets such that $\lim_{j\to\infty} \mu(O_j) = 0$ and $\nu(O_j) \ge 2\lambda > 0$ for some $\lambda > 0$. Due to the structure of the open sets of Z_q described above, any O_j can be written as a disjoint union $\bigcup_{a\in A(O_j)} O_{(a,k(a))}$. Since $\nu(O_j) = \sum_{a\in A(O_j)} \nu(O_{(a,k(a))})$ and each term of this sum is non-negative, we can find an α_j such that the open set $O_{j,\alpha_j} = \bigcup_{a\in A(O_j),\, k(a)\le\alpha_j} O_{(a,k(a))}$ satisfies $\nu(O_{j,\alpha_j}) \ge \lambda$. Note that the characteristic function I_j of O_{j,α_j} is periodic with period q^{α_j} and that $\lim_{j\to\infty} \mu(O_{j,\alpha_j}) = 0$ since from $O_{j,\alpha_j} \subseteq O_j$, we have $\mu(O_{j,\alpha_j}) \le \mu(O_j)$, and $\lim_{j\to\infty} \mu(O_j) = 0$.

From now on, to simplify notation, we write O_j for O_{j,α_j}.

Recalling (1), let X_j be a sequence of positive integers such that

$$S' = \lim_{j\to\infty} \frac{1}{q^{X_j}} \sum_{0\le n\le q^{X_j}-1} f(n),$$

and moreover, $X_j - \alpha_j$ and $X_{j+1} - X_j$ tend to infinity as $j \to \infty$. Observe that this implies that q^{α_j} divides q^{X_j}. Then define a subset of $\mathbb{N}$, with characteristic function I, by $I(n) = I_{j-1}(n)$ for $q^{X_j} \le n < q^{X_{j+1}}$.

We will prove that

$$\lim_{x\to\infty}\frac{1}{x}\sum_{0\le n<x} I(n)=0. \tag{2}$$

Indeed, given x, there exists a unique i such that $q^{X_i}\le x<q^{X_{i+1}}$. We have

$$\begin{aligned}\sum_{0\le n<x} I(n) &= \sum_{0\le n<q^{X_{i-1}}} I(n)+\sum_{q^{X_{i-1}}\le n<q^{X_i}} I(n)+\sum_{q^{X_i}\le n<x} I(n)\\ &= \sum_{0\le n<q^{X_{i-1}}} I(n)+\sum_{q^{X_{i-1}}\le n<q^{X_i}} I_{i-1}(n)+\sum_{q^{X_i}\le n<x} I_i(n).\end{aligned}$$

Now,

$$\sum_{0\le n<q^{X_{i-1}}} I(n)\le q^{X_{i-1}},$$

$$\begin{aligned}\sum_{q^{X_{i-1}}\le n<q^{X_i}} I_{i-1}(n) &\le \frac{q^{X_i}-q^{X_{i-1}}}{q^{\alpha_{i-1}}}\sum_{0\le n\le q^{\alpha_{i-1}}-1} I_{i-1}(n)\\ &= (q^{X_i}-q^{X_{i-1}})\mu(O_{i-1}),\end{aligned}$$

since I_{i-1} is a periodic function with period $q^{\alpha_{i-1}}$. Moreover, using the q^{α_i}-periodicity of I_i, we have

$$\sum_{q^{X_i}\le n<x} I_i(n)\le \sum_{q^{X_i}\le n<q^{\alpha_i}([x/q^{\alpha_i}]+1)} I_i(n)=\left(\left[\frac{x}{q^{\alpha_i}}\right]+1-q^{X_i-\alpha_i}\right)\sum_{0\le n<q^{\alpha_i}} I_i(n).$$

Hence

$$\sum_{q^{X_i}\le n<x} I_i(n)\le\left(\left[\frac{x}{q^{\alpha_i}}\right]+1-\frac{q^{X_i}}{q^{\alpha_i}}\right)(q^{\alpha_i}\mu(O_i))$$

and therefore

$$\sum_{q^{X_i}\le n<x} I_i(n)\le (x+q^{\alpha_i}-q^{X_i})\mu(O_i)\le x\mu(O_i).$$

So, for x such that $q^{X_i}\le x<q^{X_{i+1}}$, we have

$$\sum_{0\le n<x} I(n)\le q^{X_{i-1}}+(q^{X_i}-q^{X_{i-1}})\mu(O_{i-1})+x\mu(O_i),$$

which gives

$$\begin{aligned}\frac{1}{x}\sum_{0\le n<x} I(n) &\le \frac{q^{X_{i-1}}}{x}+\frac{q^{X_i}-q^{X_{i-1}}}{x}\mu(O_{i-1})+\mu(O_i)\\ &\le \frac{q^{X_{i-1}}}{q^{X_i}}+\mu(O_{i-1})+\mu(O_i)\end{aligned}$$

since $q^{X_i}\le x$. But $X_i-X_{i-1}\to\infty$ as $i\to\infty$, and $\mu(O_j)=o(1)$ as $j\to\infty$. As a consequence, we get (2).

We shall now prove that

$$\limsup_{x\to\infty}\frac{1}{x}\sum_{0\le n<x} f(n)I(n)\ge \lambda S'>0. \tag{3}$$

We have

$$\begin{aligned}\sum_{0\le n<q^{X_{j+1}}} f(n)I(n) &= \sum_{0\le n<q^{X_j}} f(n)I(n)+\sum_{q^{X_j}\le n<q^{X_{j+1}}} f(n)I(n)\\ &= \sum_{0\le n<q^{X_j}} f(n)(I(n)-I_j(n))+\sum_{0\le n<q^{X_{j+1}}} f(n)I_j(n)\\ &\ge \sum_{0\le n<q^{X_{j+1}}} f(n)I_j(n)-\sum_{0\le n<q^{X_j}} f(n).\end{aligned}$$

Now, by condition (i) of Theorem 1, we have $\sum_{0\le n<q^{X_j}} f(n)=O(q^{X_j})$. Moreover,

$$\begin{aligned}\sum_{0\le n<q^{X_{j+1}}} f(n)I_j(n) &= \Big(\sum_{0\le n<q^{\alpha_j}} f(n)I_j(n)\Big)\sum_{0\le n<q^{X_{j+1}-\alpha_j}} f(q^{\alpha_j}n)\\ &= \Big\{\Big(\sum_{0\le n<q^{\alpha_j}} f(n)I_j(n)\Big)\Big(\sum_{0\le n<q^{\alpha_j}} f(n)\Big)^{-1}\Big\}\\ &\quad\times\Big\{\Big(\sum_{0\le n<q^{\alpha_j}} f(n)\Big)\Big(\sum_{0\le n<q^{X_{j+1}-\alpha_j}} f(q^{\alpha_j}n)\Big)\Big\}\\ &= \nu(O_j)\sum_{0\le n<q^{X_{j+1}}} f(n).\end{aligned}$$

By choice of the X_j,

$$\limsup_{x\to\infty}\frac{1}{x}\sum_{0\le n<x} f(n)I(n)\ge \liminf \nu(O_j)\,\frac{1}{q^{X_{j+1}}}\sum_{0\le n<q^{X_{j+1}}} f(n),$$

and since $\nu(O_j)\ge\lambda$, we get (3). This contradicts hypothesis (ii) of Theorem 1, and so ν is absolutely continuous with respect to μ.

STEP 6: *Explicit derivative of the measure ν.* Since ν is a probability measure absolutely continuous with respect to μ, the Radon–Nikodym theorem ([3, p. 144, (12.17)]) shows that there exists a non-negative integrable function, say h, such that if B is a Borel subset of Z_q, then $\nu(B)=\int_B h\,d\mu$. We have defined on Z_q the two sequences of random variables $x_{k-}(a)=\{a_j\}_{0\le j\le k-1}$ and $x_{k+}(a)=\{a_j\}_{j\ge k}$ for $a=(a_0,a_1,\ldots)\in Z_q$. Now, given some a in Z_q, we consider the sequence of open subsets O_k of Z_q defined by $O_k=(x_{k-}(a),x_{k+}(Z_q))$. Each characteristic function I_{O_k} is continuous and since $\mu(O_k)=1/q^k$, we have

$$(4)\qquad \frac{\nu(O_k)}{\mu(O_k)} = f(x_{k-}(a))\Big(\prod_{0\le r\le k-1}\frac{1}{q}\sum_{0\le b\le q-1} f(bq^r)\Big)^{-1}$$
$$= \frac{1}{\mu(O_k)}\int_{O_k} h(t)\,d\mu(t) = \frac{1}{\mu(O_k)}\int_{Z_q} h(t)I_{O_k}(t)\,d\mu(t)$$
$$= \int_{x_{k+}(Z_q)} h(x_{k-}(a), x_{k+}(t))\,d\mu(x_{k+}(t)).$$

By a direct application of a classical result of Jessen ([7, p. 108]), we find that the quotient (4) converges in $\mathcal{L}^1(Z_q,\mu)$ and μ-almost surely to h.

Remark 4. As a consequence, we obtain Proposition 3, since by the Cauchy criterion, given any $\varepsilon > 0$, there exists a $Y(\varepsilon)$ such that if $z \ge y \ge Y(\varepsilon)$, then

$$\int_{Z_q}\left|\frac{f(x_{y-}(t))}{\prod_{0\le r\le y-1} q^{-1}\sum_{0\le b\le q-1} f(bq^r)} - \frac{f(x_{z-}(t))}{\prod_{0\le r\le z-1} q^{-1}\sum_{0\le b\le q-1} f(bq^r)}\right| d\mu(t) \le \varepsilon,$$

which can be written as

$$\frac{1}{q^z}\sum_{0\le n\le q^z-1}\left|\frac{f(x_{y-}(n))}{\prod_{0\le r\le y-1} q^{-1}\sum_{0\le b\le q-1} f(bq^r)} - \frac{f(n)}{\prod_{0\le r\le z-1} q^{-1}\sum_{0\le b\le q-1} f(bq^r)}\right| \le \varepsilon,$$

which implies immediately that

$$\lim_{x\to\infty}\Big(\frac{1}{x}\sum_{0\le n\le x-1} f(n)\Big)\Big(\prod_{0\le r\le \log_q x-1}\frac{1}{q}\sum_{0\le b\le q-1} f(bq^r)\Big)^{-1} = 1.$$

Step 7 (*The end*!)

Step 7.1: *Consequence of the continuity of* ν

Lemma 6. *If ν is continuous, then $1/2 \le f(aq^k) \le 3/2$ except for a finite set of aq^k, and*

$$\limsup_{k\to\infty}\sum_{r=0}^{k}\sum_{0\le a\le q-1}(1-f(aq^r))^2 < \infty.$$

Proof. First of all, we remark that since f satisfies condition (ii) of Theorem 1, and by (1), we have

$$\operatorname{card}\{(a,k);\ 0\le a\le q-1,\ k\ge 0,\ f(aq^k)=0\} < \infty.$$

For we have

$$\begin{aligned}\frac{1}{q^k}\,&\mathrm{card}\{n;\, 0 \le n \le q^k-1,\, f(n) \ne 0\}\\ &= \prod_{0\le r\le k-1} \frac{1}{q}\,\mathrm{card}\{(a,r);\; f(aq^r) \ne 0,\, 0 \le a \le q-1\}\\ &= \prod_{0\le r\le k-1} \Bigl(1-\frac{1}{q}\,\mathrm{card}\{(a,r);\; f(aq^r) = 0,\, 0 \le a \le q-1\}\Bigr),\end{aligned}$$

and this is $o(1)$ if

$$\limsup_{k\to\infty} \sum_{0\le r\le k-1} \frac{1}{q}\,\mathrm{card}\{(a,r);\; f(aq^r) = 0,\, 0 \le a \le q-1\} = \infty,$$

which implies that

$$\limsup_{k\to\infty} \frac{1}{q^k} \sum_{0\le n\le q^k-1} f(n) = 0,$$

a contradiction with (1).

As a consequence, there exists some k such that the restriction of f to $q^k\mathbb{N}$ is never zero. To simplify notation, we shall assume that $f(aq^k)$ is never zero *ab initio*.

Now, since the limit of the sequence

$$f(x_{k-}(a))\Bigl(\prod_{0\le r\le k-1} \frac{1}{q} \sum_{0\le b\le q-1} f(bq^r)\Bigr)^{-1}$$

(see (4)) exists μ-almost surely, applying the three series theorem ([7, p. 88, Corollaire 1]) to the logarithm of this sequence, we deduce that for any $c > 0$,

$$\sum_{\{(a,k);\, |\log(f(aq^k)/q^{-1}\sum_{0\le b\le q-1} f(bq^k)|>c\}} q^{-1} < \infty,$$

and since $f(0\cdot q^r) = 1$ for all r, this shows that

$$\Bigl|\log\Bigl(\frac{1}{q}\sum_{0\le b\le q-1} f(bq^k)\Bigr)\Bigr| \le c$$

except for a finite number of k, and similarly, from

$$\sum_{\{(a,k);\, |\log(f(aq^k)/q^{-1}\sum_{0\le b\le q-1} f(bq^k)|>c\}} q^{-1} < \infty,$$

we conclude that $|\log f(aq^k)| \le 2c$ except for a finite number of a and k. Since c can be chosen as small as we want, there exists some κ such that for

$k \geq \kappa$, we have

$$(5) \qquad \frac{1}{2} \leq \frac{1}{q} \sum_{0 \leq b \leq q-1} f(bq^k) \leq \frac{3}{2} \quad \text{and} \quad \frac{1}{2} \leq f(aq^k) \leq \frac{3}{2}$$

for all a. As above, to simplify notation, we shall assume that this holds *ab initio*.

Now, it is a famous result of Kakutani ([7, p. 109]) that ν is absolutely continuous if and only if the product

$$(6) \qquad \prod_{0 \leq k \leq y} \frac{(q^{-1} \sum_{0 \leq b \leq q-1} \sqrt{f(bq^k)})^2}{q^{-1} \sum_{0 \leq b \leq q-1} f(bq^k)}$$

tends to a positive limit as $y \to \infty$. Since it is a product of positive numbers less than or equal to 1, this is equivalent to

$$\sum_{k \geq 0} \frac{1}{q^{-1} \sum_{0 \leq b \leq q-1} f(bq^k)} \left(\frac{1}{q} \sum_{0 \leq b \leq q-1} f(bq^k) - \left(\frac{1}{q} \sum_{0 \leq b \leq q-1} \sqrt{f(bq^k)} \right)^2 \right) < \infty,$$

and by (5) it means that

$$\sum_{k \geq 0} \left(\frac{1}{q} \sum_{0 \leq b \leq q-1} f(bq^k) - \left(\frac{1}{q} \sum_{0 \leq b \leq q-1} \sqrt{f(bq^k)} \right)^2 \right) < \infty.$$

By a classical formula of Lagrange, this is exactly

$$(7) \qquad \frac{1}{2q^2} \sum_{k=0}^{\infty} \sum_{0 \leq a, b \leq q-1} (\sqrt{f(aq^k)} - \sqrt{f(bq^k)})^2 < \infty.$$

Now, since $f(0 \cdot q^k) = 1$ for all k, this is equivalent to

$$\sum_{k=0}^{\infty} \sum_{0 \leq a \leq q-1} (1 - \sqrt{f(aq^k)})^2 < \infty,$$

and by (5), this can be written as

$$\sum_{k=0}^{\infty} \sum_{0 \leq a \leq q-1} (1 - f(aq^k))^2 < \infty. \quad \blacksquare$$

Step 7.2: *Proof of Theorem 2.* We remark that the statement is evident for $r = 0$. Now, if $0 < r \leq 1$, it will be sufficient to prove it for $r = 1/2$. For if

$$(8) \qquad 0 < \limsup_{x \to \infty} \frac{1}{x} \sum_{0 \leq n < x} f(n)^{1/2} < \infty,$$

then using the Hölder inequality, for $1/2 < r < 1$ we get

$$0 < \limsup_{x \to \infty} \frac{1}{x} \sum_{0 \leq n < x} f(n)^r < \infty,$$

and also if I is the characteristic function of a subset of $\mathbb{N}$ then

$$\lim_{x\to\infty}\frac{1}{x}\sum_{0\le n<x} I(n)=0 \;\Rightarrow\; \lim_{x\to\infty}\frac{1}{x}\sum_{0\le n<x} I(n)f(n)^r=0.$$

So, the conclusion will be satisfied in the range $]1/2,1[\,\cup\,\{1\}$, and by iteration, in $]1/2^2,1/2[\,\cup\,\{1/2\}\cup\,]1/2,1]$. The case $r=1/2$ will be solved shortly using the Hölder inequality, and so, the conclusion will be satisfied in $\bigcup_{k>0}]1/2^k,1]$, i.e. in $]0,1]$.

Now, (8) is an immediate consequence of the absolute continuity of ν with respect to μ, for the product (6) converges to a positive number, say $\mathcal{L}$, as $y\to\infty$, and so, for y large enough,

$$2\mathcal{L}^{-1/2}\prod_{0\le k\le y}\left(\frac{1}{q}\sum_{0\le b\le q-1} f(bq^k)\right)^{1/2} \ge \prod_{0\le k\le y}\frac{1}{q}\sum_{0\le b\le q-1}\sqrt{f(bq^k)}$$
$$\ge \frac{1}{2}\mathcal{L}^{-1/2}\prod_{0\le k\le y}\left(\frac{1}{q}\sum_{0\le b\le q-1} f(bq^k)\right)^{1/2},$$

which yields

$$0<\limsup_{k\to\infty}\frac{1}{q^k}\sum_{0\le n\le q^k-1} f(n)^{1/2}<\infty.$$

To obtain the result for $r>1$, it will be sufficient to prove it for the exponent 2. For if it holds for 2, it will hold for all positive powers of 2, and hence for all $r\ge 1$ by the Hölder inequality. Now, by (5) and (7), we have

$$\sum_{k=0}^{\infty}\sum_{0\le a,b\le q-1}(f(aq^k)-f(bq^k))^2$$
$$\le \sum_{k=0}^{\infty}\sum_{0\le a,b\le q-1}(\sqrt{f(aq^k)}-\sqrt{f(bq^k)})^2(\sqrt{f(aq^k)}+\sqrt{f(bq^k)})^2$$
$$\le \left(2\cdot\frac{3}{2}\right)^2\sum_{k=0}^{\infty}\sum_{0\le a,b\le q-1}(\sqrt{f(aq^k)}-\sqrt{f(bq^k)})^2<\infty.$$

Since, by the Lagrange formula,

$$\frac{1}{2q^2}\sum_{k=0}^{\infty}\sum_{0\le a,b\le q-1}(f(aq^k)-f(bq^k))^2$$
$$=\sum_{k\ge 0}\left(\left(\frac{1}{q}\sum_{0\le b\le q-1} f(bq^k)^2\right)-\left(\frac{1}{q}\sum_{0\le b\le q-1} f(bq^k)\right)^2\right)$$

and since $1/2\le f(bq^k)\le 3/2$, this gives

$$\sum_{k\ge 0}\frac{1}{q^{-1}\sum_{0\le b\le q-1} f(bq^k)^2}\cdot\frac{1}{q}\sum_{0\le b\le q-1} f(bq^k)^2-\frac{1}{q}\Big(\sum_{0\le b\le q-1} f(bq^k)\Big)^2<\infty,$$

and so the product (6) converges to a positive limit, say $\mathcal{L}'$, as $y\to\infty$. We can now conclude in the same way as above in the case $r=1/2$.

STEP 7.3: *End of proof of Theorem 1.* First, we remark that

$$\limsup_{k\to\infty}\prod_{0\le r\le k}\frac{1}{q}\sum_{0\le a\le q-1} f(aq^r)=\prod_{0\le r\le k}\bigg(1-\frac{1}{q}\sum_{0\le a\le q-1}(1-f(aq^r))\bigg).$$

Now, since

$$0<S'=\limsup_{k\to\infty}\prod_{0\le r\le k}\frac{1}{q}\sum_{0\le a\le q-1} f(aq^r)<\infty$$

and logarithm is a continuous increasing function on $]0,\infty[$, we get

$$\begin{aligned}\log\limsup_{k\to\infty}&\prod_{0\le r\le k}\bigg(1-\frac{1}{q}\sum_{0\le a\le q-1}(1-f(aq^r))\bigg)\\ &=\limsup_{k\to\infty}\log\prod_{0\le r\le k}\bigg(1-\frac{1}{q}\sum_{0\le a\le q-1}(1-f(aq^r))\bigg)=\log S',\end{aligned}$$

and since $-1/2\le 1-f(aq^r)\le 1/2$, we obtain

$$\limsup_{k\to\infty}\sum_{0\le r\le k}\frac{-1}{q}\sum_{0\le a\le q-1}(1-f(aq^r))+O\bigg(\frac{1}{q}\Big(\sum_{0\le a\le q-1}(1-f(aq^r))\Big)^2\bigg)=\log S'.$$

Now, we remark that

$$\frac{1}{q}\Big(\sum_{0\le a\le q-1}(1-f(aq^r))\Big)^2\le\frac{1}{q}\sum_{0\le a\le q-1}(1-f(aq^r))^2,$$

and since

$$\sum_{r\in\mathbb{N}}\sum_{0\le a\le q-1}(1-f(aq^r))^2<\infty,$$

we conclude that

$$\limsup_{k\to\infty}\sum_{0\le r\le k}\frac{-1}{q}\sum_{0\le a\le q-1}(1-f(aq^r))<\infty,$$

i.e.

$$\limsup_{k\to\infty}\sum_{0\le r\le k}\sum_{0\le a\le q-1}(f(aq^r)-1)<\infty.$$

Hence we have shown that conditions (iii) and (iv) of Theorem 1 hold.

Conversely, assuming that (iii) and (iv) hold, we deduce immediately that $-1/2\le 1-f(aq^r)\le 1/2$ if r is large enough. It is harmless to assume that it is so for all r. Now, we reverse the argument:

$$\limsup_{k\to\infty} \sum_{0\le r\le k} \sum (f(aq^r)-1) < \infty$$

implies that

$$\limsup_{k\to\infty} \sum_{0\le r\le k} \frac{-1}{q} \sum_{0\le a\le q-1} (1-f(aq^r)) < \infty,$$

and since $\sum_{r\in\mathbb{N}} \sum_{0\le a\le q-1}(1-f(aq^r))^2 < \infty$, we find that

$$\limsup_{k\to\infty} \log \prod_{0\le r\le k} \left(1-\frac{1}{q}\sum_{0\le a\le q-1}(1-f(aq^r))\right) < \infty.$$

Now, since logarithm is a continuous increasing function on $]0,\infty[$, we have

$$\limsup_{k\to\infty} \log \prod_{0\le r\le k} \left(1-\frac{1}{q}\sum_{0\le a\le q-1}(1-f(aq^r))\right) = \log \limsup_{k\to\infty} \prod_{0\le r\le k} \left(1-\frac{1}{q}\sum_{0\le a\le q-1}(1-f(aq^r))\right)$$

and so

$$0 < \limsup_{k\to\infty} \prod_{0\le r\le k} \frac{1}{q} \sum_{0\le a\le q-1} f(aq^r) < \infty.$$

The same computation as above shows that the product (6) tends to a positive limit as $y\to\infty$, and so, by the Kakutani Theorem, the sequence of functions

$$f(x_{k-}(\cdot))\left(\prod_{0\le r\le k-1} \frac{1}{q}\sum_{0\le b\le q-1} f(bq^r)\right)^{-1} \tag{9}$$

converges in $\mathcal{L}^1(Z_q,\mu)$. As a consequence, by the Cauchy criterion, given any $\varepsilon>0$, there exists a $Y(\varepsilon)$ such that if $z\ge y\ge Y(\varepsilon)$, we have

$$\int_{Z_q} \left| \frac{f(x_{y-}(t))}{\prod\limits_{0\le r\le y-1} q^{-1} \sum\limits_{0\le b\le q-1} f(bq^r)} - \frac{f(x_{z-}(t))}{\prod\limits_{0\le r\le z-1} q^{-1} \sum\limits_{0\le b\le q-1} f(bq^r)} \right| d\mu(t) \le \varepsilon q^{-1},$$

which can be written as

$$\frac{1}{q^z} \sum_{0\le n\le q^z-1} \left| \frac{f(x_{y-}(n))}{\prod\limits_{0\le r\le y-1} q^{-1} \sum\limits_{0\le b\le q-1} f(bq^r)} - \frac{f(n)}{\prod\limits_{0\le r\le z-1} q^{-1} \sum\limits_{0\le b\le q-1} f(bq^r)} \right| \le \varepsilon q^{-1}.$$

Denoting by z the expression $[\log x/\log q]+1$, if $I(\cdot)$ is the characteristic function of a subset of $\mathbb{N}$ and $\lim_{x\to\infty} x^{-1}\sum_{0\le n<x} I(n) = 0$, we have

$$\begin{aligned}
&\frac{1}{x}\sum_{0\le n\le x}\left|\frac{f(x_{y-}(n))}{\prod\limits_{0\le r\le y-1} q^{-1}\sum\limits_{0\le b\le q-1} f(bq^r)} - \frac{f(n)}{\prod\limits_{0\le r\le z} q^{-1}\sum\limits_{0\le b\le q-1} f(bq^r)}\right| I(n)\\
&\le \frac{q^z}{x}\cdot\frac{1}{q^z}\sum_{0\le n\le q^z-1}\left|\frac{f(x_{y-}(n))}{\prod\limits_{0\le r\le y-1} q^{-1}\sum\limits_{0\le b\le q-1} f(bq^r)} - \frac{f(n)}{\prod\limits_{0\le r\le z} q^{-1}\sum\limits_{0\le b\le q-1} f(bq^r)}\right| I(n)\\
&\le q\cdot\frac{1}{q^z}\sum_{0\le n\le q^z-1}\left|\frac{f(x_{y-}(n))}{\prod\limits_{0\le r\le y} q^{-1}\sum\limits_{0\le b\le q-1} f(bq^r)} - \frac{f(n)}{\prod\limits_{0\le r\le z} q^{-1}\sum\limits_{0\le b\le q-1} f(bq^r)}\right|\\
&\le q\cdot q^{-1}\varepsilon \le \varepsilon.
\end{aligned}$$

Now, we remark that

$$\frac{f(x_{y-}(n))}{\prod\limits_{0\le r\le y} q^{-1}\sum\limits_{0\le b\le q-1} f(bq^r)} \le C(y) < \infty,$$

and so

$$\begin{aligned}
&\left|\frac{1}{x}\sum_{0\le n\le x}\frac{f(x_{y-}(n))}{\prod\limits_{0\le r\le y} q^{-1}\sum\limits_{0\le b\le q-1} f(bq^r)} I(n)\right.\\
&\qquad\qquad\left. - \frac{1}{x}\sum_{0\le n\le x}\frac{f(n)}{\prod\limits_{0\le r\le z} q^{-1}\sum\limits_{0\le b\le q-1} f(bq^r)} I(n)\right|\\
&\quad = \left|\frac{1}{\prod\limits_{0\le r\le z} q^{-1}\sum\limits_{0\le b\le q-1} f(bq^r)}\frac{1}{x}\sum_{0\le n\le x} f(n)I(n) + o(1)\right| \quad \text{as } x\to\infty\\
&\quad \le \varepsilon,
\end{aligned}$$

since $x^{-1}C(y)\sum_{0\le n\le x} I(n) = o(1)$ as $x\to\infty$. Hence

$$\limsup_{x\to\infty}\frac{1}{\prod\limits_{0\le r\le z} q^{-1}\sum\limits_{0\le b\le q-1} f(bq^r)}\sum_{0\le n\le x} f(n)I(n) \le \varepsilon,$$

which gives

$$\limsup_{x\to\infty}\sum_{0\le n\le x} f(n)I(n) \le \varepsilon \limsup_{x\to\infty}\prod_{0\le r\le z}\frac{1}{q}\sum_{0\le b\le q-1} f(bq^r) \le \varepsilon\Lambda.$$

Hence

$$\limsup_{x\to\infty}\sum_{0\le n\le x} f(n)I(n) = 0. \blacksquare$$

3.2. Proof of Theorem 4. Most of the arguments given above which rely on classical probability theory apply in this general case of complex-valued q-multiplicative functions, and so the details will be given only when necessary.

STEP 1: $(\mathcal{S})$ *holds.* This is a consequence of the following result:

PROPOSITION 7. *Let f be an arithmetical function satisfying the condition*

$$0 < S = \limsup_{x\to\infty} \frac{1}{x} \sum_{0\le n<x} |f(n)| < \infty.$$

Assume that for any sequence $I(n)$ with values 0 or 1 we have

$$\lim_{x\to\infty} \frac{1}{x} \sum_{0\le n<x} I(n) = 0 \;\Rightarrow\; \lim_{x\to\infty} \left|\frac{1}{x} \sum_{0\le n<x} I(n)f(n)\right| = 0.$$

Then also

$$\lim_{x\to\infty} \frac{1}{x} \sum_{0\le n<x} I(n) = 0 \;\Rightarrow\; \lim_{x\to\infty} \frac{1}{x} \sum_{0\le n<x} I(n)|f(n)| = 0.$$

Proof. Let M be a positive integer. We can assume that $I(n)$ takes the value 0 when $f(n) = 0$. If $f(n) \not\equiv 0$, we denote by f^* the arithmetical function $f \cdot |f|^{-1}$. Now, when f^* is of modulus 1, for integers k in $[0, M-1]$, we define a sequence $I_{k,M}(n)$ with values 0 or 1 by $I_{k,M}(n) = 1$ if $\arg f^*(n) \in [2\pi k/M, 2\pi(k+1)/M[$, and 0 elsewhere. It is clear that $I(n) = \sum_{0\le k\le M-1} I_{k,M}(n)$. Now, we remark that

$$\begin{aligned}\frac{1}{x} \sum_{0\le n<x} I(n)|f(n)| &= \frac{1}{x} \sum_{0\le n<x} \Big(\sum_{0\le k\le M-1} I_{k,M}(n)\Big)|f(n)| \\ &= \frac{1}{x} \sum_{0\le k\le M-1} \Big(\sum_{0\le n<x} I_{k,M}(n)|f(n)|\Big) \\ &= \frac{1}{x} \sum_{0\le k\le M-1} \left|e^{2i\pi k/M} \sum_{0\le n<x} I_{k,M}(n)|f(n)|\right|.\end{aligned}$$

Observe that

$$\begin{aligned} e^{2i\pi k/M} \sum_{0\le n<x} I_{k,M}(n)|f(n)| &= \sum_{0\le n<x} I_{k,M}(n)|f(n)|e^{2i\pi k/M} \\ &= \sum_{0\le n<x} I_{k,M}(n)|f(n)|(e^{2i\pi k/M} - f^*(n)) + \sum_{0\le n<x} I_{k,M}(n)|f(n)|f^*(n) \\ &= \sum_{0\le n<x} I_{k,M}(n)|f(n)|(e^{2i\pi k/M} - f^*(n)) + \sum_{0\le n<x} I_{k,M}(n)f(n).\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{1}{x}\sum_{0\le n<x} I(n)|f(n)| \\
&\quad = \frac{1}{x}\Big|\sum_{0\le k\le M-1}\Big(\sum_{0\le n<x} I_{k,M}(n)|f(n)|(e^{2i\pi k/M}-f^*(n)) \\
&\qquad\qquad\qquad\qquad + \frac{1}{x}\sum_{0\le n<x} I_{k,M}(n)f(n)\Big)\Big| \\
&\quad \le \frac{1}{x}\sum\sum_{0\le k\le M-1}\Big(\sum_{0\le n<x} I_{k,M}(n)|f(n)|\,|e^{2i\pi k/M}-f^*(n)|\Big) \\
&\qquad\qquad\qquad\qquad + \frac{1}{x}\Big|\sum_{0\le n<x} I(n)f(n)\Big|,
\end{aligned}$$

and this can be written as

$$\begin{aligned}
&\frac{1}{x}\sum_{0\le n<x} I(n)|f(n)| \\
&\quad \le \frac{1}{x}\sum_{0\le k\le M-1}\Big(\sum_{0\le n<x} I_{k,M}(n)|f(n)|\,|e^{2i\pi k/M}-f^*(n)|\Big) + o(1), \qquad x\to\infty.
\end{aligned}$$

Now, we remark that

$$I_{k,M}(n)|f(n)|\,|e^{2i\pi k/M}-f^*(n)| = I_{k,M}(n)|f(n)|O(1/M)$$

with the O uniform in M, since $\arg f^*(n)\in[2\pi k/M, 2\pi(k+1)/M[$. This gives

$$\begin{aligned}
&\frac{1}{x}\sum_{0\le k\le M-1}\Big(\sum_{0\le n<x} I_{k,M}(n)|f(n)|\,|e^{2i\pi k/M}-f^*(n)|\Big) \\
&\qquad = O\Big(\frac{1}{M}\Big)\cdot\frac{1}{x}\sum_{0\le k\le M-1}\Big(\sum_{0\le n<x} I_{k,M}(n)|f(n)|\Big) \\
&\qquad = O\Big(\frac{1}{M}\Big)\cdot\frac{1}{x}\sum_{0\le n<x}\Big(\sum_{0\le k\le M-1} I_{k,M}(n)\Big)|f(n)| \\
&\qquad = O\Big(\frac{1}{M}\Big)\cdot\frac{1}{x}\sum_{0\le n<x} I(n)|f(n)| \\
&\qquad \le O\Big(\frac{1}{M}\Big)\cdot\frac{1}{x}\sum_{0\le n<x} |f(n)| = O\Big(\frac{1}{M}\Big)\cdot O(1) = O\Big(\frac{1}{M}\Big),
\end{aligned}$$

since by hypothesis, $x^{-1}\sum_{0\le n<x}|f(n)| = O(1)$. Hence

$$\frac{1}{x}\sum_{0\le n<x} I(n)|f(n)| = O\Big(\frac{1}{M}\Big) + o(1), \qquad x\to\infty,$$

and since M can be as large as we want, we get

$$\lim_{x\to\infty}\frac{1}{x}\sum_{0\le n<x} I(n)|f(n)| = 0. \ \blacksquare$$

STEP 2. This is only a simple remark:

PROPOSITION 8. *If for some* $r \ge 0$,

$$0 < \limsup_{x\to\infty}\left|\frac{1}{x}\sum_{q^r-1\le n\le x} f(n)\right| < \infty,$$

then

$$0 < \limsup_{k\to\infty}\left|\frac{1}{q^k}\sum_{q^r-1\le n\le q^k-1} f(n)\right| < \infty.$$

Proof. First, we may assume that $r = 0$, since the shifted function $n \mapsto f(q^r n)$ is q-multiplicative. Now, the result is due to the structure of the formula for the summatory function of a q-multiplicative function. For if x is a positive integer, written as $x = \sum_{0\le r\le k} a_r q^r$ with $a_k \ne 0$, we have

$$S_x(f) = \sum_{0\le n\le x} f(n) = \Big(\sum_{0\le a\le a_k-1} f(aq^k)\Big)\Big(\prod_{0\le j\le k-1}\sum_{0\le a\le q-1} f(aq^j)\Big) + f(a_k q^k)\sum_{0\le n\le x-a_k q^k} f(n).$$

This gives

$$|S_x(f)| \le \Big(\sum_{0\le a\le a_k-1} |f(aq^k)|\Big)\Big|\prod_{0\le j\le k-1}\sum_{0\le a\le q-1} f(aq^j)\Big| + |f(a_k q^k)|\Big|\sum_{0\le n\le x-a_k q^k} f(n)\Big|.$$

Since $|f(\cdot)|$ satisfies the hypothesis of Theorem 1, the conclusion of Step 7.1 gives

$$\sum_{k=0}^{\infty}\sum_{0\le a\le q-1}(1-|f(aq^k)|)^2 < \infty,$$

and so

$$|S_x(f)| \le a_k(1+o(1))\Big|\prod_{0\le j\le k-1}\sum_{0\le a\le q-1} f(aq^j)\Big| + (1+o(1))\Big|\sum_{0\le n\le x-a_k q^k} f(n)\Big|.$$

Iterating, we find that if

$$\limsup_{k\to\infty}\left|\frac{1}{q^k}\sum_{0\le n\le q^k-1} f(n)\right| = 0,$$

then

$$\limsup_{x\to\infty}\left|\frac{1}{x}\,S_x(f)\right| = 0,$$

which contradicts the hypothesis. ∎

STEP 3. A simple modification of the argument presented in Step 4 of the proof of Theorem 1 leads to the fact that if as above, we define on Z_q a sequence of random variables $x_{k-}(a) = (a_j q^j)_{0\le j\le k}$ for $a = (a_0, a_1, \ldots) \in Z_q$, then the sequence of functions (9) converges in $\mathcal{L}^1(Z_q, \mu)$ and μ-almost surely to some limit g.

STEP 4: ($\mathcal{S}'$) *holds.* First, we recall that in Step 7.1 above, we have proved that it is harmless to assume that $f(aq^k)$ is never zero. A consequence is that the limit of the sequence of functions (9), which converges in $\mathcal{L}^1(Z_q, \mu)$ and μ-a.s., is positive μ-a.s. For if we denote this limit by $\Phi(\cdot)$, we have μ-a.s.,

$$\Phi(t) = \prod_{r\ge 0} |f(a_k(t))|\biggl(\frac{1}{q}\sum_{0\le b\le q-1} |f(bq^r)|\biggr)^{-1},$$

and so, μ-a.s.,

$$\int \Phi(t)\,d\mu(x_{k-}(t)) = \prod_{k\le r} |f(a_k(t))|\biggl(\frac{1}{q}\sum_{0\le b\le q-1} |f(bq^r)|\biggr)^{-1}.$$

A classical result of Jessen ([7, p. 108]) shows that $\int \Phi(t)\,d\mu(x_{k-}(t))$ converges in $\mathcal{L}^1(Z_q, \mu)$ and μ-a.s. to $\int \Phi(t)\,d\mu(t)$, i.e. to 1. Hence we see that $\prod_{k\le r} |f(a_k(t))|(q^{-1}\sum_{0\le b\le q-1} |f(bq^r)|)^{-1}$ tends to 1 μ-a.s. as $k\to\infty$, which implies immediately that $\Phi(t)$ is positive μ-a.s.

Now, since the sequence of functions (9) converges in $\mathcal{L}^1(Z_q, \mu)$, we infer that

$$\int_{Z_q} \biggl| f(x_{k-}(a))\biggl(\prod_{0\le r\le k-1}\frac{1}{q}\sum_{0\le b\le q-1} f(bq^r)\biggr)^{-1}\biggr|\,d\mu = \biggl(\prod_{0\le r\le k}\frac{1}{q}\sum_{0\le b\le q-1} |f(bq^r)|\biggr)\biggl|\prod_{0\le r\le k}\frac{1}{q}\sum_{0\le b\le q-1} f(bq^r)\biggr|^{-1}$$

has a positive finite limit. This implies that

$$\frac{f(x_{k-}(a))}{\prod\limits_{0\le r\le k-1} q^{-1}\sum\limits_{0\le b\le q-1} f(bq^r)}\cdot\frac{\prod\limits_{0\le r\le k-1} q^{-1}\sum\limits_{0\le b\le q-1} |f(bq^r)|}{|f(x_{k-}(a))|} \times\frac{\bigl|\prod\limits_{0\le r\le k-1} q^{-1}\sum\limits_{0\le b\le q-1} f(bq^r)\bigr|}{\prod\limits_{0\le r\le k-1} q^{-1}\sum\limits_{0\le b\le q-1} |f(bq^r)|}$$

converges μ-a.s., since each of the three factors of this product does. Since $f(x_{k-}(a)) = f^*(x_{k-}(a))|f(x_{k-}(a))|$, this product is equal to $f^*(x_{k-}(a))\varpi_k$, where ϖ_k is defined by

$$\prod_{0\le r\le k-1} \frac{1}{q} \sum_{0\le b\le q-1} f(bq^r) = \overline{\varpi}_k \Big| \prod_{0\le r\le k-1} \frac{1}{q} \sum_{0\le b\le q-1} f(bq^r) \Big|.$$

So, $|\varpi_k| = 1$, and $f^*(x_{k-}(a))\varpi_k$ converges μ-a.s. to limit $F^*(a)$; consequently, the symmetrized sequence $f_k^{*\mathrm{s}}(a,b)$ defined by $f^*(x_{k-}(a))\overline{f^*(x_{k-}(b))}$ converges μ^2-a.s. to $F^*(a)\overline{F^*(b)}$. Since all these functions have modulus 1, there exists an open set O such that $\int_O F^*(a)\overline{F^*(b)}\,d\mu^2(a,b) \neq 0$, and due to the structure of the open sets of Z_q, the same holds for an elementary set $(r,k(r)) \times (s,k(s))$. This implies that

$$\lim_{k\to\infty} \int_{(r,k(r))\times(s,k(s))} f_k^{*\mathrm{s}}\,d\mu^2 \neq 0,$$

and computing the value of this integral shows that there exists some t in $\mathbb{N}$ such that

$$\text{(10)} \qquad \lim_{k\to\infty} \Big| \prod_{t\le r\le k} \frac{1}{q} \sum_{0\le b\le q-1} f^*(bq^r) \Big|^2 \text{ exists and is not zero.}$$

Using the Lagrange identity (for complex numbers), we see immediately that this is equivalent to

$$\lim_{k\to\infty} \sum_{k\ge t} \sum_{0\le a\le q-1} (1 - \operatorname{Re} f^*(aq^k)) < \infty,$$

and as a consequence,

$$\lim_{k\to\infty} \sum_{k\ge 0} \sum_{0\le a\le q-1} (1 - \operatorname{Re} f^*(aq^k)) < \infty.$$

This is assertion $(\mathcal{S}')$. ∎

STEP 5. It remains to prove that

1) $(\mathcal{S}) \Leftrightarrow$ (i)&(ii),
2) $(\mathcal{S})$&$(\mathcal{S}') \Leftrightarrow$ (i)&(ii)&(iii).

The proof of 1) is immediate, since if we have $(\mathcal{S})$, we know, by Theorem 1, that for any r positive,

$$0 < \limsup_{x\to\infty} \frac{1}{x} \sum_{n\le x} |f(n)|^r < \infty,$$

and as a consequence, if $I(\cdot)$ is the characteristic function of a subset of $\mathbb{N}$ and $\lim_{x\to\infty} \frac{1}{x}\sum_{0\le n<x} I(n)=0$, then

$$\limsup_{x\to\infty}\left|\frac{1}{x}\sum_{0\le n<x} I(n)f(n)\right| \le \limsup_{x\to\infty}\frac{1}{x}\sum_{0\le n<x} I(n)|f(n)| = 0$$

by applying the Hölder inequality for some exponent $r>1$.

It remains to prove that if conditions $(\mathcal{S})$ and $(\mathcal{S}')$ are fulfilled, then (iii) holds true.

Since

$$\sum_{k\ge 0}\sum_{0\le a\le q-1}(1-\operatorname{Re} f^*(aq^k)) < \infty,$$

using the Lagrange identity (for complex numbers), we deduce that there exists some t in $\mathbb{N}$ such that (10) holds. This implies that the sequence of functions $F^*_{y-}(x)$ defined on Z_q by

$$F^*_{y-}(x) = \Big(\prod_{t\le k\le y} f^*(a_k(x)q^k)\Big)\Big(\prod_{t\le j\le y}\frac{1}{q}\sum_{0\le a\le q-1} f^*(aq^j)\Big)^{-1}$$

is a bounded martingale convergent in $\mathcal{L}^\infty(Z_q,d\mu)$. Similarly, the sequence of functions $F_{y-}(x)$ defined on Z_q by

$$F_{y-}(x) = \Big(\prod_{t\le k\le y} |f(a_k(x)q^k)|\Big)\Big(\prod_{t\le j\le y}\frac{1}{q}\sum_{0\le a\le q-1} |f(aq^j)|\Big)^{-1}$$

is a martingale convergent in $\mathcal{L}^1(Z_q,d\mu)$.

Hence the sequence $F^*_{y-}(x)F_{y-}(x)$ converges in $\mathcal{L}^1(Z_q,d\mu)$. Now, since

$$\begin{aligned}\lim_{y\to\infty}\int |F^*_{y-}(x)F_{y-}(x)|\,d\mu(x) \\ &= \lim_{y\to\infty}\int F_{y-}(x)\Big(\prod_{t\le j\le y}\Big|\frac{1}{q}\sum_{0\le a\le q-1} f^*(aq^j)\Big|\Big)^{-1} d\mu(x) \\ &= \lim_{y\to\infty}\Big(\prod_{t\le j\le y}\Big|\frac{1}{q}\sum_{0\le a\le q-1} f^*(aq^j)\Big|\Big)^{-1} \ne 0,\end{aligned}$$

there exists an open set O such that

$$\lim_{y\to\infty}\int_O F^*_{y-}(x)F_{y-}(x)\,d\mu(x)\ne 0,$$

and so there exists an elementary set $O_{(a,k(a))}$ such that

$$\lim_{y\to\infty}\int_{O_{(a,k(a))}} F^*_{y-}(x)F_{y-}(x)\,d\mu(x)\ne 0.$$

This implies that the limit of the product

$$\left(\prod_{k(a)\le k\le y}\frac{1}{q}\sum_{0\le a\le q-1} f(a_k(x)q^k)\right)\cdot\left(\prod_{k(a)\le j\le y}\frac{1}{q}\sum_{0\le a\le q-1} f^*(aq^j)\right)^{-1} \times\left(\prod_{k(a)\le j\le y}\frac{1}{q}\sum_{0\le a\le q-1} |f(aq^j)|\right)^{-1}$$

exists and is not zero, and a fortiori, the limit of

$$\left|\prod_{k(a)\le k\le y}\frac{1}{q}\sum_{0\le a\le q-1} f(a_k(x)q^k)\right|\cdot\left|\prod_{k(a)\le j\le y}\frac{1}{q}\sum_{0\le a\le q-1} f^*(aq^j)\right| \times\left(\prod_{k(a)\le j\le y}\frac{1}{q}\sum_{0\le a\le q-1} |f(aq^j)|\right)^{-1}$$

exists and is not zero. Now, since

$$\lim_{y\to\infty}\left|\prod_{k(a)\le j\le y}\frac{1}{q}\sum_{0\le a\le q-1} f^*(aq^j)\right|$$

exists and is not zero, and

$$0<\limsup_{y\to\infty}\prod_{k(a)\le j\le y}\frac{1}{q}\sum_{0\le a\le q-1} |f(aq^j)|<\infty,$$

we get

$$0<\limsup_{y\to\infty}\left|\prod_{k(a)\le k\le y}\frac{1}{q}\sum_{0\le a\le q-1} f(a_k(x)q^k)\right|<\infty,$$

and so there exists some $r\ge 0$ such that

$$0<\limsup_{x\to\infty}\left|\frac{1}{x}\sum_{q^r\le n\le x} f(n)\right|<\infty. \blacksquare$$

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Received on 5.2.2004
and in revised form on 10.2.2005 (4713)