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Localised Bombieri–Vinogradov theorems in imaginary quadratic fields

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1. Introduction. Let K be an imaginary quadratic field with ring of integers $\mathcal{O}_K$. For $y \in \mathbb{R}$, define $\langle y\rangle$ to be such that $\langle y\rangle \equiv y \bmod 1$ and $-1/2 \le \langle y\rangle < 1/2$. Given $0 < \ell \le 1/2$ and $0 \le \Theta < 1$ define

$$(1.1)\qquad \mathcal{S}(x,\Theta,\ell) = \bigl\{\alpha \in \mathcal{O}_K : \bigl||\alpha|^2 - x\bigr| < x\ell,\ -\ell < \langle(\arg\alpha)/2\pi - \Theta\rangle < \ell\bigr\}.$$

If $2\ell < 1/\omega$, where ω is the number of units in K, this set contains only unassociated integers. The main result of this paper is a Bombieri–Vinogradov type result for the prime integers in $\mathcal{S}$. In previous papers, such as [4], the distribution of prime ideals within variants of (1.1) has been studied. To define such a variant we need first let γ be a Grössencharacter on the ideals of K (see Hecke [9, 10] for an explicit construction in a general number field or Knapowski [19] for within a quadratic field). This character, of infinite order, has the property that $\gamma((\alpha)) = (\alpha/|\alpha|)^\omega$ for all $\alpha \in \mathcal{O}_K$. For ideals $\mathfrak{a}$ define $0 \le \vartheta(\mathfrak{a}) < 1$ by $\gamma(\mathfrak{a}) = e^{2\pi i\vartheta(\mathfrak{a})}$. Thus we can define

$$\mathcal{S}_1(x,\Theta,\ell) = \{\mathfrak{a} : |\mathrm{N}\mathfrak{a} - x| \le \ell x, -\ell \le \langle\vartheta(\mathfrak{a}) - \Theta\rangle \le \ell\}.$$

Note that for $\mathfrak{a} = (\alpha)$, where $\alpha \in \mathcal{O}_K$, we have $\langle\vartheta(\mathfrak{a}) - \Theta\rangle = \langle(\omega\arg\alpha)/2\pi - \Theta\rangle$. If we used $\mathcal{S}_1$ in place of $\mathcal{S}$, the integers found would be localised only up to multiplication by units.

Recall (from Landau [20]) that $\mathfrak{a} \equiv \mathfrak{b} \bmod \mathfrak{q}$ means there exist integers $\alpha, \beta \in \mathcal{O}_K$ with $((\alpha\beta), \mathfrak{q}) = 1$, $(\alpha)\mathfrak{a} = (\beta)\mathfrak{b}$ and $\mathfrak{q} \mid (\alpha - \beta)$. We use $h(\mathfrak{q})$ to denote the number of distinct congruence classes of ideals mod $\mathfrak{q}$, and $\phi(\mathfrak{q})$ to denote the number of congruence classes of integers coprime to $\mathfrak{q}$. Let Λ be von Mangoldt's function on the ideals of K in which case, for integers α, we have

$$\Lambda(\alpha) = \begin{cases} \log \mathrm{N}\pi & \text{if } (\alpha) = (\pi)^k \text{ for some prime } \pi,\\ 0 & \text{otherwise.}\end{cases}$$

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THEOREM 1. *For all $A > 0$ and $y \le 1/2\omega$ there exist positive $c_i(A)$, $1 \le i \le 4$, such that*

$$\sum_{\mathrm{N}\mathfrak{q}\le Q} \max_{(\beta,\mathfrak{q})=1} \max_{\substack{x\le z\\ 0\le\Theta<1}} \max_{\ell\le y} \Bigg| \sum_{\substack{\alpha\equiv\beta \bmod \mathfrak{q}\\ \alpha\in\mathcal{S}(x,\Theta,\ell)}} \Lambda(\alpha) - \frac{1}{\phi(\mathfrak{q})} \sum_{\substack{\alpha\in\mathcal{S}(x,\Theta,\ell)\\ (\alpha,\mathfrak{q})=1}} \Lambda(\alpha)\Bigg| \ll \frac{y^2 z}{\log^A z} \tag{1.2}$$

provided $y^2 z \ge z^{7/12}\log^{c_1(A)} z$ and

$$Q \le \begin{cases} y^2 z^{1/2}\log^{-c_2(A)} z & \text{if } y^2 z \ge z^{3/5}\log^{c_3(A)} z,\\ y^2 z^{9/20}\log^{-c_4(A)} z & \text{otherwise.}\end{cases}$$

The regions $\mathcal{S}(x,\Theta,\ell)$ arise from the methods used but, in $\mathbb{Q}(i)$ for example, they may be replaced by quite general geometric regions. Let $\mathcal{R}$ be a subset of the unit disc in $\mathbb{C}$ whose boundary $\partial\mathcal{R}$ is Lipschitz parametrisable (see p. 128 of Lang [21]). For $v, w \in \mathbb{C}$ define

$$\mathcal{R}_{w,v} = w + v\mathcal{R} = \{u \in \mathbb{C} : \exists s \in \mathcal{R},\ u = w + vs\},$$

often referred to as *homogeneously expanding domains*. Note that there exists $\gamma = \gamma(\mathcal{R})$ such that if $|v| \le \gamma|w|$ then $\mathcal{R}_{w,v}$ contains only unassociated integers.

THEOREM 2. *Let $A > 0$ be given. Then, in $K = \mathbb{Q}(i)$, we have*

$$\sum_{\mathrm{N}\mathfrak{q}\le Q} \max_{(\beta,\mathfrak{q})=1} \max_{1<|w|^2\le X} \max_{|v|\le h|w|} \Bigg| \sum_{\substack{\alpha\in\mathcal{R}_{w,v}\\ \alpha\equiv\beta \bmod \mathfrak{q}}} \Lambda(\alpha) - \frac{1}{\phi(\mathfrak{q})} \sum_{\substack{\alpha\in\mathcal{R}_{w,v}\\ (\alpha,\mathfrak{q})=1}} \Lambda(\alpha)\Bigg| \ll \frac{h^2 X}{\log^A X}, \tag{1.3}$$

provided $h \le \gamma$, $h^2 X \ge X^{7/12}\log^{c_1(A)} X$ and

$$Q \le \begin{cases} h^2 X^{1/2}\log^{-c_2(A)} X & \text{when } h^2 X \ge X^{3/5}\log^{c_3(A)} X,\\ h^2 X^{9/20}\log^{-c_4(A)} X & \text{otherwise.}\end{cases}$$

For the record, by the methods of Section 11 where we deal with the condition $\alpha \in \mathcal{R}_{w,v}$, we can prove a version of the Siegel–Walfisz Theorem for homogeneously expanding domains. Let $|\mathcal{R}|$ denote the area of $\mathcal{R}$. Then for all A and $C > 0$ there exists $c = c(A, C)$ such that for $(\beta, \mathfrak{q}) = 1$ and $\mathrm{N}\mathfrak{q} < \log^C |w|$ we have

$$\sum_{\substack{\alpha\in\mathcal{R}_{w,v}\\ \alpha\equiv\beta \bmod \mathfrak{q}}} \Lambda(\alpha) = \frac{4|v|^2|\mathcal{R}|}{\pi\phi(\mathfrak{q})}\left(1 + O\left(\frac{1}{\log^A |w|}\right)\right) \tag{1.4}$$

as $|w| \to \infty$ for $|w|\gamma > |v| \ge |w|^{7/12}\log^c |w|$.

Amongst many applications these Bombieri–Vinogradov type results can be used to estimate errors arising from the use of sieve methods. We leave such uses to future papers though here we might mention that a direct application of Selberg's sieve in $\mathbb{Q}(i)$ (see Rieger [28], or [5]) along with (1.3) gives upper bounds for $|\{\pi \in \mathcal{R}_{w,v} : \pi + \alpha \text{ prime}\}|$, for all $\alpha \in \mathbb{Z}[i]$. In [6] we were interested in the size of discs around each prime π containing no other prime. To this end we defined $\varrho(\pi) = \min\{|\pi - \pi'| : \pi' \neq \pi\}$ and showed that, for Φ a positive function satisfying $\Phi(x) \to \infty$ as $x \to \infty$, we have $\varrho(\pi) < \Phi(\pi)\log^{1/2}|\pi|$ for almost all π. In fact, this bound holds for almost all π in $\mathcal{R}_{w,v}$, for all w and v. In the opposite direction, summing over α the sieve bounds obtained earlier, the methods of [6] now show that $\varrho(\pi) \gg \log^{1/2}|\pi|$ for a positive proportion of the primes in $\mathcal{R}_{w,v}$, for all $|w|\gamma \geq |v| \geq |w|^{7/12}\log^c|w|$.

Bombieri–Vinogradov type results have been given in number fields by Huxley [14] and Wilson [30] for ideals, Hinz [11] for integers and Johnson [17] for ideal numbers.

Results for rational primes in short intervals have been given by Jutila [18], Huxley & Iwaniec [16], Ricci [27], Perelli, Pintz and Salerno [24, 25], Zhan [32], and Timofeev [29].

Our hybrid result is of the same quality as Timofeev's. His work [29] not only has stronger results and applies to more general functions than earlier papers but it also has aspects that enable the present generalisation. For instance, it makes no use of either the Pólya–Vinogradov Theorem or approximate functional equation for appropriate L-functions, both seen in [24]. The complication of detail of the proof in the present paper over [29] comes from the need (see Section 10) to introduce smooth functions.

Much of this paper forms a large part of the second author's Ph.D. thesis.

2. Weight functions. A result of Bombieri, Friedlander & Iwaniec [1] (Corollary to Lemma 9) can be used to derive the following.

LEMMA 3. *For all* $0 < \ell \leq 1$, $0 < \Delta \leq 1/2$ *and* $x_0 \in \mathbb{R}$, *there exists a continuous function* $u(x) = u(x, x_0, \ell, \Delta) : \mathbb{R} \to [0,1]$, *differentiable to all orders*, *such that*

$$u(x) = \begin{cases} 1, & |x - x_0| \leq \ell(1-\Delta), \\ 0, & |x - x_0| \geq \ell(1+\Delta). \end{cases}$$

Furthermore, $u^{(j)}(x) \ll (cj)^{2j}(\ell\Delta)^{-j}$ *for all* $x \in \mathbb{R}$ *and* $j \geq 1$, *and* $\int_{-\infty}^{\infty} u(x)\,dx = 2\ell$.

Define $g_{\ell,\Delta}(y) = u(y, 1, \ell, \Delta)$ for all $y \in \mathbb{R}$, and $f_{\ell,\Delta}(y) = u(y, 0, \ell, \Delta)$ for $-1/2 \leq y \leq 1/2$ and extended to all $\mathbb{R}$ by periodicity. For $z \in \mathbb{C}$ define

$$v_{x,\Theta,\ell,\Delta}(z) = f_{\ell,\Delta}((\arg z)/2\pi - \Theta)g_{\ell,\Delta}(|z|^2/x).$$

Note that, for $z = \alpha$, where $\alpha \in \mathcal{O}_K$, this is *not* a function of the ideal (α). For any function F on the integers of $\mathcal{O}_K$ and $(\beta, \mathfrak{q}) = 1$ define, on the integers α coprime to $\mathfrak{q}$,

$$F_{\beta,\mathfrak{q}}(\alpha) = \begin{cases} \left(1 - \dfrac{1}{\phi(\mathfrak{q})}\right) F(\alpha) & \text{if } \alpha \equiv \beta \bmod \mathfrak{q}, \\ -\dfrac{1}{\phi(\mathfrak{q})} F(\alpha) & \text{otherwise,} \end{cases}$$

along with $F_{\beta,\mathfrak{q}}(\alpha) = 0$ when $(\alpha, \mathfrak{q}) \neq 1$. For our main result, Theorem 4, we introduce an additional averaging, useful in applications within Sieve Theory. So, if h is a function of ideals, then define a truncated convolution by

$$H_L(\alpha) = \sum_{\substack{\mathrm{N}\mathfrak{a} \leq L \\ \mathfrak{a}\mathfrak{b} = (\alpha)}} h(\mathfrak{a}) \Lambda(\mathfrak{b}) = (h_L * \Lambda)(\alpha),$$

say, on the integers of $\mathcal{O}_K$. We combine these definitions to give $H_{L,\beta,\mathfrak{q}}$, seen in

Theorem 4. *Let h be a function on the ideals of K that satisfies*

$$\sum_{\mathrm{N}\mathfrak{a} \leq x} \frac{h(\mathfrak{a})^2}{\mathrm{N}\mathfrak{a}} \ll \log^\kappa x, \tag{2.1}$$

for some $\kappa \geq 0$. For all $A, B > 0$ and $y \leq 1/2\omega$ we have

$$\sum_{\mathrm{N}\mathfrak{q} \leq Q} \max_{(\beta,\mathfrak{q})=1} \max_{\substack{z\mathcal{L}^{-B} \leq x \leq z \\ 0 \leq \Theta < 1}} \max_{y\mathcal{L}^{-B} < \ell \leq y} \Big| \sum_{\alpha} v_{x,\Theta,\ell,\Delta}(\alpha) H_{L,\beta,\mathfrak{q}}(\alpha) \Big| \ll \frac{y^2 z}{\mathcal{L}^A}, \tag{2.2}$$

provided Δy is sufficiently small and we have one of

(a) $Q \leq y^2 z^{1/2} \Delta^2 \mathcal{L}^{c_1}$, $L \leq z(y\Delta)^{12/5} \mathcal{L}^{-c_2}$ *and* $y^2 z \geq z^{9/14} \Delta^{-2} \mathcal{L}^{c_3}$,
(b) $Q \leq y^2 z^{1/2} \Delta^2 \mathcal{L}^{c_4}$, $L \leq z(y\Delta)^{18/5} \mathcal{L}^{-c_5}$ *and* $z^{9/14} \Delta^{-2} \mathcal{L}^{c_3} \geq y^2 z \geq z^{3/5} \Delta^{-2} \mathcal{L}^{c_6}$,
(c) $Q \leq y^2 z^{9/20} \Delta^2 \mathcal{L}^{c_7}$, $L \leq z(y\Delta)^{108/25} \mathcal{L}^{-c_8}$ *and* $z^{3/5} \Delta^{-2} \mathcal{L}^{c_6} \geq y^2 z \geq z^{7/12+\varepsilon} \Delta^{-2}$,
(d) $Q \leq y^2 z^{9/20} \Delta^2 \mathcal{L}^{c_7}$, $L \leq \mathcal{L}^C$ *and* $y^2 z \geq z^{7/12} \mathcal{L}^{c(C)} \Delta^{-2}$ *for any* $C > 0$.

In particular, *from case* (c), *when* $y^2 z = z^{7/12+\varepsilon} \Delta^{-2}$ *we have* $Q \leq z^{1/30-\varepsilon}$ *and* $L \leq z^{1/10-\varepsilon}$.

Here, as throughout this paper, $\mathcal{L} = \log z$.

In the rational case Timofeev has given a Bombieri–Vinogradov type theorem with a very general convolution, though not one with as long a summation of an unknown function such as h. Theorem 4 both generalises and extends a result due to Wu [31], though we cannot base our arguments, as he does, on [24].

From case (d) we will derive

THEOREM 5. *For all $A > 0$ and $y \le 1/2\omega$ we have*

$$\sum_{\mathrm{N}\mathfrak{q}\le Q} \max_{(\beta,\mathfrak{q})=1} \max_{\substack{x\le z\\ 0\le\Theta<1}} \max_{0<\ell\le y} \Big|\sum_{\alpha} v_{x,\Theta,\ell,\Delta}(\alpha)\Lambda_{\beta,\mathfrak{q}}(\alpha)\Big| \ll \frac{y^2 z}{\mathcal{L}^A}, \tag{2.3}$$

provided that $y^2 z \ge z^{7/12}\Delta^{-2}\mathcal{L}^{c_1(A)}$, Δ is sufficiently small, and

$$Q \le \begin{cases} y^2 z^{1/2}\Delta^2\mathcal{L}^{-c_2(A)} & \text{if } y^2 z \ge z^{3/5}\Delta^{-2}\mathcal{L}^{c_3(A)},\\ y^2 z^{9/20}\Delta^2\mathcal{L}^{-c_4(A)} & \text{otherwise.}\end{cases} \tag{2.4}$$

Though we do, in Section 11, give a method for stripping the weights and deriving Theorem 2, in most applications we engineer the problem to include weights before applying Theorem 5.

3. Reduction to character sums. The condition $\alpha \equiv \beta$ within $H_{L,\beta,\mathfrak{q}}$ is dealt with using characters $\widehat{\chi}$ mod $\mathfrak{q}$. We use the same notation for the primitive character, mod $\mathfrak{n}$ where $\mathfrak{n} \mid \mathfrak{q}$, inducing $\widehat{\chi}$. Dropping subscripts on v we see, by orthogonality of characters, that for $(\beta,\mathfrak{q}) = 1$,

$$\sum_{\alpha} v(\alpha)H_{L,\beta,\mathfrak{q}}(\alpha) = \frac{1}{\phi(\mathfrak{q})} \sum_{\substack{\widehat{\chi} \bmod \mathfrak{n}\\ \text{primitive}\\ \mathfrak{n}\mid\mathfrak{q},\ \mathfrak{n}\ne(1)}} \overline{\widehat{\chi}}(\beta) \sum_{(\alpha,\mathfrak{q})=1} v(\alpha)H_L(\alpha)\widehat{\chi}(\alpha). \tag{3.1}$$

From the definition of f as $f(y) = u(y,0,\ell,\Delta)$ for $-1/2 \le y \le 1/2$, it is differentiable to all orders, periodic with period 1, and integrable over a unit interval. Thus it has a Fourier series

$$f(\theta) = \sum_{m=-\infty}^{\infty} a_m e(m\theta) \quad \text{with} \quad a_m = \int_{-1/2}^{1/2} f(\theta)e(-m\theta)\,d\theta \tag{3.2}$$

for all $m \in \mathbb{Z}$, where $e(\alpha) = e^{2\pi i\alpha}$ for all $\alpha \in \mathbb{R}$. The coefficients satisfy $a_0 = 2\ell$, $a_m \ll \ell$ for all $m \in \mathbb{Z}$ and

$$a_m \ll |m|^{-100} \quad \text{for } |m| \ge \frac{c_1}{\Delta\ell}\log^3\left(\frac{1}{\Delta\ell}\right) \tag{3.3}$$

and $\Delta\ell$ sufficiently small. The latter result follows from integration by parts. This final bound on a_m shows that the series in (3.2) can be truncated with an arbitrarily small error.

Let $B_L(Q,z,y)$ denote the left hand side of (2.2). Because we have $\ell \ge y\mathcal{L}^{-B}$ in $B_L(Q,z,y)$ we need never take the point of truncation larger than

$$W = \frac{c_1\mathcal{L}^{3+B}}{\Delta y}, \tag{3.4}$$

provided $\log(1/\Delta\ell) \ll \mathcal{L}$. In what follows, the errors of truncation are disregarded and we simply say, for example, that the inner sum in (3.1) can be replaced by

$$\sum_{|m|\le W} a_m e(-m\Theta) \sum_{(\alpha,\mathfrak{q})=1} g(\mathrm{N}\alpha/x) H_L(\alpha)\lambda^m(\alpha)\widehat{\chi}(\alpha),$$

where $\lambda(\alpha) = (\alpha/|\alpha|)$. Because λ and $\widehat{\chi}$ are multiplicative this inner sum equals

$$\sum_{(\alpha,\mathfrak{q})=1}^{*} g(\mathrm{N}\alpha/x) H_L(\alpha)\lambda^m(\alpha)\widehat{\chi}(\alpha) \sum_{\varepsilon \text{ units}} \lambda^m(\varepsilon)\widehat{\chi}(\varepsilon), \tag{3.5}$$

where $\sum^{*}_{(\alpha,\mathfrak{q})=1}$ denotes a sum over unassociated integers. Yet

$$\sum_{\varepsilon \text{ units}} \lambda^m(\varepsilon)\widehat{\chi}(\varepsilon) = \begin{cases} \omega & \text{if } b \mid m \text{ and } \lambda^m(\varepsilon)\widehat{\chi}(\varepsilon) = 1 \text{ for all units,} \\ 0 & \text{otherwise,} \end{cases}$$

where $b = b(\mathfrak{n})$ is the number of units satisfying $\varepsilon \equiv 1 \bmod \mathfrak{n}$. Under these conditions $\lambda^m\widehat{\chi}$ is well defined on the principal ideals coprime to $\mathfrak{q}$.

The explicit method given in Section 3 of [19] to extend $\gamma(\alpha) = (\alpha/|\alpha|)^{\omega}$ from integers to ideals can be used here to extend $\lambda^m\widehat{\chi}$ to ideals. The principal ideals can be picked out by characters, φ, on the ideal class group. Thus (3.5) can be replaced, in turn, with $\omega h^{-1}\sum_{\varphi} \Psi_{\mathfrak{q}}(g, H_L, x, \lambda^m\widehat{\chi}\varphi)$ where

$$\Psi_{\mathfrak{q}}(g, H_L, x, \lambda^m\widehat{\chi}\varphi) = \sum_{(\mathfrak{a},\mathfrak{q})=1} g(\mathrm{N}\mathfrak{a}/x) H_L(\mathfrak{a})\lambda^m\widehat{\chi}\varphi(\mathfrak{a}), \tag{3.6}$$

the sum now being over ideals, hence the uppercase Ψ. Write $\chi = \widehat{\chi}\varphi$ and let $C(m,\mathfrak{n})$ be the set of all such products for which $\lambda^m\widehat{\chi}(\varepsilon) = 1$ for all units. Further, let $C^*(m,\mathfrak{n})$ be the subset with $\widehat{\chi}$ primitive $\bmod \mathfrak{n}$. So we now have

$$\sum_{\alpha} v(\alpha) H_{L,\beta,\mathfrak{q}}(\alpha) \ll_K \frac{a_0}{\phi(\mathfrak{q})} \sum_{\substack{\mathfrak{n}|\mathfrak{q}\\ \mathfrak{n}\ne(1)}} \sum_{\substack{|m|\le W\\ b(\mathfrak{n})|m}} \sum_{\chi\in C^*(m,\mathfrak{n})} |\Psi_{\mathfrak{q}}(g, H_L, x, \lambda^m\chi)|. \tag{3.7}$$

To deal with the condition $(\mathfrak{a},\mathfrak{q}) = 1$ in (3.6) we will need to introduce the weight

$$\prod_{\mathfrak{p}|\mathfrak{q}} \left(1 + \frac{1}{\sqrt{\mathrm{N}\mathfrak{p}}}\right)^2 = \varrho(\mathfrak{q}),$$

in the notation of [29]. Note that $\varrho(\mathfrak{q}) \ll_{\varepsilon} Q^{\varepsilon}$ for all $\mathrm{N}\mathfrak{q} \le Q$. As in [29], multiply the right hand side of (3.7) by $\varrho(\mathfrak{q})/\varrho(\mathfrak{q})$, sum over $\mathfrak{q}$, interchange with the sum over $\mathfrak{n}$ and use $\sum_{\mathrm{N}\mathfrak{q}\le Q} \varrho(\mathfrak{q})\phi^{-1}(\mathfrak{q}) \ll \mathcal{L}$. Then $B_L(Q,z,y) \ll \mathcal{L} \max_{\mathrm{N}\mathfrak{k}\le Q} B_{L,\mathfrak{k}}(Q,z,y)$ where

$$(3.8)\quad B_{L,\mathfrak{k}}(Q,z,y) = \frac{1}{\varrho(\mathfrak{k})} \sum_{1<\mathrm{N}\mathfrak{q}\le Q} \frac{y}{\phi(\mathfrak{q})} \sum_{\substack{|m|\le W\\ b(\mathfrak{q})|m}} \sum_{\chi\in C^*(m,\mathfrak{q})} \max_{\ell,x} |\Psi_{\mathfrak{k}}(g,H_L,x,\lambda^m\chi)|,$$

with the same ranges on ℓ and x as in (2.2).

4. Small Q

THEOREM 6. *For $n\ge 2$ set*

$$\gamma_n = \frac{24n^2-26n+12}{5n(n-1)}, \qquad \beta_n = 1-\frac{1}{2n}.$$

For all $A,C>0$ we have, with $F=\log^C z$,

$$(4.1)\qquad B_{L,\mathfrak{k}}(F,z,y) \ll \frac{y^2 z}{\log^{A+2} z},$$

provided we have Δy sufficiently small, $L\le z^{1-\delta}W^{-4}$ for some $\delta>0$ and one of

(i) $L\le zW^{-24/5}\mathcal{L}^{-c_0}$,
(ii) $zW^{-24/5}>W^{\varepsilon}$ *and* $L\le zW^{-108/25}\mathcal{L}^{-c_5}$,
(iii) $z>W^{\gamma_n}\mathcal{L}^{c_n}$ *and* $L\le zW^{-24\beta_{n-1}/5}\mathcal{L}^{-c_n}$, *for some* $2\le n\le 5$.

Proof. Unfolding the convolution gives

$$\begin{aligned}(4.2)\quad &\Psi_{\mathfrak{k}}(g,H_L,x,\lambda^m\chi)\\ &= \sum_{\substack{\mathrm{N}\mathfrak{a}<L\\ (\mathfrak{a},\mathfrak{k})=1}} h(\mathfrak{a})\lambda^m\chi(\mathfrak{a})\Psi_{\mathfrak{k}}(g,\Lambda,x/\mathrm{N}\mathfrak{a},\lambda^m\chi)\\ &= \sum_{\substack{\mathrm{N}\mathfrak{a}<L\\ (\mathfrak{a},\mathfrak{k})=1}} h(\mathfrak{a})\lambda^m\chi(\mathfrak{a})\Psi(g,\Lambda,x/\mathrm{N}\mathfrak{a},\lambda^m\chi) + O\Big(\mathcal{L}^2\sum_{\mathrm{N}\mathfrak{a}\le L}|h(\mathfrak{a})|\Big)\end{aligned}$$

where $\Psi=\Psi_{(1)}$. Because of (2.1) the error here is $\ll L\mathcal{L}^{c(\kappa)}$. To $B_{L,\mathfrak{k}}(F,z,y)$ this contributes $\ll LFWy\mathcal{L}^c$, which is $\ll y^2z\mathcal{L}^{-A-2}$ as long as $L\le zW^{-2}\mathcal{L}^{-c}$.

For the weight function $g=g_{\ell,\Delta}$ occurring in $\Psi(g,\Lambda,x/\mathrm{N}\mathfrak{a},\lambda^m\chi)$ we consider its Mellin transform

$$(4.3)\qquad \widehat{g}_{\ell,\Delta}(s) = \int_{-\infty}^{\infty} g_{\ell,\Delta}(w)w^{s-1}\,dw,$$

valid for $\mathrm{Re}\, s>1$. We have the trivial bound $\widehat{g}_{\ell,\Delta}(s)\ll\ell$ and, similarly to the derivation of (3.3), we can show for $-1\le\sigma\le 2$ that

$$(4.4)\qquad \widehat{g}_{\ell,\Delta}(\sigma+it)\ll|t|^{-100} \quad \text{for } |t|\ge\frac{c_1}{\Delta\ell}\log^2\left(\frac{1}{\Delta\ell}\right)$$

and $\Delta\ell$ sufficiently small.

Before replacing g by its Mellin transform we split the sum over $\mathfrak{a}$ in (4.2). Let $L_0 = zW^{-24/5}\mathcal{L}^{-c_0}$ and set $V_0 = 1$, $V_1 = \min(L, L_0)$ and $V_{j+1} = 2V_j$ for all $1 \le j \le J$ with some $J \ll \mathcal{L}$. Then each subsum with $V_j \le \mathrm{N}\mathfrak{a} < V_{j+1}$ of (4.2) can be replaced by

$$\frac{-1}{2\pi i}\int_{2-iW}^{2+iW} \widehat{g}(s)x^s h_{j,\mathfrak{k}}(s,\lambda^m\chi)\,\frac{L'}{L}(s,\lambda^m\chi)\,ds \tag{4.5}$$

with W as in (3.4). Here

$$h_{j,\mathfrak{k}}(s,\lambda^m\chi) = \sum_{\substack{V_j\le \mathrm{N}\mathfrak{a}<V_{j+1}\\ (\mathfrak{a},\mathfrak{k})=1}} \frac{h(\mathfrak{a})\lambda^m\chi(\mathfrak{a})}{\mathrm{N}\mathfrak{a}^s}. \tag{4.6}$$

Using Cauchy–Schwarz and (2.1) we can prove, for $j \ge 1$, that

$$|h_{j,\mathfrak{k}}(s,\lambda^m\chi)| \ll V_j^{1-\sigma}\log^{\kappa/2} V_j \tag{4.7}$$

for all s. When $j = 0$ we have

$$|h_{0,\mathfrak{k}}(s,\lambda^m\chi)| \ll V_1^{1-\sigma}\log^{\kappa/2} V_1, \tag{4.8}$$

but only for $\mathrm{Re}\, s < 1$. Also within (4.5) we have the Dirichlet series

$$L(s,\lambda^m\chi) = \sum_{\mathfrak{a}} \frac{\lambda^m\chi(\mathfrak{a})}{\mathrm{N}\mathfrak{a}^s}, \tag{4.9}$$

for $\mathrm{Re}\, s > 1$, where the sum is over all ideals of K. For $\chi \in C^*(m,\mathfrak{q})$ for which $b(\mathfrak{q}) \,|\, m$ this is a Hecke L-function of K with Grössencharacter $\lambda^m\chi$. These functions have an analytic continuation to the whole complex plane, with a pole at $s = 1$ when $m = 0$ and $\chi = \chi_0$, and they satisfy a functional equation. Details can be found in Hecke [9, 10], or [20], although an explanation more specific to the present case is given in Section 3 of [2]. In fact, as in [2], we actually truncate the integral in (4.5) at some $W/2 < \mathrm{Im}\, s = w < W$ for which the $L(\sigma + iw, \lambda^m\chi)$ with $|m| \le W$, $\chi \in C^*(m,\mathfrak{q})$ and $\sigma \le 2$ keep well away from their zeros. The function $\widehat{g}(s)$ decays so fast with $\mathrm{Im}\, s$ that when the line of integration in (4.5) is moved to the left the contribution from the horizontal lines of integration is negligible. As in Section 4 of [2] the contribution to $B_{L,\mathfrak{k}}$ from the new vertical line of integration, $\mathrm{Re}\, s = -1/2$, is $\ll y^2 z^{-1/2} L^{3/2} W^2 F \mathcal{L}^{c(\kappa)}$. This is sufficiently small if $L \le zW^{-4/3}\mathcal{L}^{-c}$. Thus from (4.5) the main contribution will be seen to come from the zeros, $\varrho_{m,\chi} = \beta_{m,\chi} + i\gamma_{m,\chi}$, of $L(s,\lambda^m\chi)$ with $|\gamma_{m,\chi}| \le W$. The contribution of these zeros to $B_{L,\mathfrak{k}}(F,z,y)$ is bounded by

$$\ll y^2 \max_{0\le\sigma<1} z^\sigma \sum_{1<\mathrm{N}\mathfrak{q}\le F}\ \sum_{\substack{|m|\le W\\ b|m}}\ \sum_{\chi\in C^*(m,\mathfrak{q})}\ \sum_{\substack{\beta_{m,\chi}\ge\sigma\\ |\gamma_{m,\chi}|\le W}} |h_{j,\mathfrak{k}}(\varrho_{m,\chi},\lambda^m\chi)|. \tag{4.10}$$

Let $N(\sigma, F, W) = \sum_{F,W} 1$, being the four-fold summation of (4.10). In [2], estimates for

$$N_{\mathfrak{q}}(\sigma, W) = \sum_{\substack{|m|\le W\\ b|m}} \sum_{\chi\in C^*(m,\mathfrak{q})} \sum_{\substack{\beta_{m,\chi}\ge\sigma\\ |\gamma_{m,\chi}|\le W}} 1$$

were given, with the dependence on $\mathfrak{q}$ being left implicit. The same results from there go through for $N(\sigma, F, W)$ subject to the following two observations.

Firstly, from Theorem 2 of [3], for all $\mathrm{N}\mathfrak{q} \le F$, $|m| \le W$ for which $b \mid m$, and $\chi \in C^*(m, \mathfrak{q})$, there are no zeros satisfying $\beta_{m,\chi} \ge 1 - c(K)U^{-1}$ and $|\gamma_{m,\chi}| \le W$, where $U = \max(\log Q,\ \log^{2/3} W \log\log^{1/3} W)$, apart from (possibly) at most one real zero for each $\mathfrak{q}$. Yet Fogels [7] has a Siegel type result for such zeros, namely that their real parts are $\le 1 - c(\varepsilon)F^{-\varepsilon}$ for all $\varepsilon > 0$. It can thus be shown that the sum of the contributions to (4.10) of such zeros with $F = \log^C W$ is $\ll_{A,C} y^2 z\mathcal{L}^{-A-C-1}$. Hence the range of σ over which we maximise on the right hand side of (4.10) may be reduced to $1/2 \le \sigma \le 1 - cU^{-1}$.

Secondly, we need to either make the dependence on $\mathfrak{q}$ explicit in the results of [2] or introduce a summation over $\mathfrak{q}$ into the Mean and Large Value results used in that paper. This latter option is achieved in Section 7 and so we need use our Theorems 9, 11 and 12 in place of Theorem 6.2, Lemma 7.3 and Theorem 6.3 of [2]. In this way we can prove, for $F = \log^C W$, that $N(\sigma, F, W) \ll (W^2)^{f(\sigma)(1-\sigma)} \log^{c(C)} W$ where $f(\sigma) = 3/(2-\sigma)$ for $1/2 \le \sigma < 1$ and $f(\sigma) = 3/(3\sigma - 1) + \varepsilon$ for $3/4 \le \sigma < 1$ and all $\varepsilon > 0$. This latter result shows that the density hypothesis, $f(\sigma) \le 2$, holds to the left of 1, a result used in the proof of case (i). For the quality of the result in case (i) we note that we have the tools necessary to prove an analogue of a bound given as (1.7) of Huxley [15], namely $f(\sigma) = (5\sigma - 3)/(\sigma^2 + \sigma - 1)$ for $3/4 \le \sigma \le 1$. We can thus show the following, which improves slightly the results of [2] in that the exponent no longer contains ε.

LEMMA 7. *For all $C > 0$, $W \ge 1$ and $F = \log^C W$ we have*

$$N(\sigma, F, W) \ll W^{2g(\sigma)(1-\sigma)} \log^{b(C)} W,$$

where $g(\sigma) \le 12/5$ for $1/2 \le \sigma \le 1$ and $g(\sigma) \le 2$ for $\sigma \ge 5/6 + \varepsilon$.

For case (i) of Theorem 6, when $L \le L_0$, we have only the $j = 0$ case of (4.5). We bound the corresponding $j = 0$ case of (4.10) simply as

$$\ll y^2\mathcal{L}^c \max_{0\le\sigma<1-cU^{-1}} z^\sigma L^{1-\sigma} N(\sigma, F, W). \tag{4.11}$$

For $0 \le \sigma \le 1/2$ use $N(\sigma, F, W) \ll W^2 \log^{C+1} W$. For $1/2 \le \sigma \le 6/7$, where $6/7$ has been simply chosen as larger than $5/6$, use $\ll W^{24(1-\sigma)/5} \log^{b(C)} W$, in which case (4.11) is $\ll y^2 z\mathcal{L}^{-A-1}$ subject to $LW^{24/5}/z < 1/\mathcal{L}^{c(A,C)}$.

Finally, for $6/7 \le \sigma \le 1 - cU^{-1}$ use $\ll W^{4(1-\sigma)} \log^{b(C)} W$. All that is important here is that $LW^4/z \le 1/z^{\delta}$ for some $\delta > 0$.

For the remaining cases of Theorem 6 we need to consider $1 \le j \le J$. An application of Hölder's inequality and an appropriate version of Theorem 9 gives

$$\sum_{Q,W} |h_{j,\mathfrak{k}}(\varrho_{m,\chi}, \lambda^m\chi)| \ll N(\sigma, Q, W)^{\beta_n} (W^{1/n} V_j^{1/2-\sigma} + V_j^{1-\sigma}) \mathcal{L}^{c_n}$$

for all $n \ge 1$. This is sufficiently small within (4.10), when $0 \le \sigma \le 1 - cU^{-1}$, if we have both $W^{1/n} \le V_j^{1/2}$ and $V_j W^{24\beta_n/5}/z \le 1/\mathcal{L}^{c_n}$, that is, $V_j \in I_n := [W^{2/n}, zW^{-24\beta_n/5}\mathcal{L}^{-c_n}]$. One of these intervals will overlap with the region dealt with earlier, namely $[1, L_0]$, if there exists n_0 such that $W^{2/n_0} < L_0$. In cases (ii) and (iii) we have $L_0 > W^{\varepsilon}$ and $L_0 > W^{1/50}\mathcal{L}^{-c}$ and so such an n_0 can always be found, perhaps depending on ε. Further, for $2 \le n \le n_0$ we find that I_{n-1} overlaps I_n if $z > W^{\gamma_n}\mathcal{L}^{c_n}$. We note that $\gamma_n \le 24/5$ if, and only if, $n \ge 6$. So if $z > W^{24/5}\mathcal{L}^{c_6}$ then all the intervals up to I_5 overlap and (4.1) holds for $L \le zW^{-24\beta_5/5}\mathcal{L}^{-c_5}$, which is case (ii). Finally, $\{\gamma_n\}_{n\ge 2}$ is decreasing for $2 \le n \le 11$ while $\gamma_n > 24/5$ for $2 \le n \le 5$. Thus if $z > W^{\gamma_n}\mathcal{L}^{c_n}$ for some $2 \le n \le 5$ then all intervals up to I_{n-1} overlap and (4.1) holds for $L \le zW^{-24\beta_{n-1}/5}\mathcal{L}^{-c_{n-1}}$, which is case (iii). ∎

5. Large Q. For Q larger than a fixed power of $\mathcal{L}$ we let $F = \mathcal{L}^C$, with C to be chosen, and write

$$B_{L,\mathfrak{k}}(Q, z, y) \le B_{L,\mathfrak{k}}(F, z, y) + y \sum_{\substack{i=0 \\ P=2^i F}}^{M} \sum_{j=0}^{L} P^{-1} V_{j,\mathfrak{k}}(P, z, y) \tag{5.1}$$

where $2^{M-1}F < Q \le 2^M F$ and

$$\begin{aligned} V_{j,\mathfrak{k}}(P, z, y) &= \frac{1}{\varrho(\mathfrak{k})} \sum_{P < \mathrm{N}\mathfrak{q} \le 2P} \frac{\mathrm{N}\mathfrak{q}}{\phi(\mathfrak{q})} \\ &\times \sum_{|m| \le W} \sum_{\chi \in C^*(m,\mathfrak{q})} \max_{x,\ell} \Big| \sum_{V_j \le \mathrm{N}\mathfrak{a} < V_{j+1}} h(\mathfrak{a}) \lambda^m\chi(\mathfrak{a}) \Psi_{\mathfrak{k}}\Big(g, \Lambda, \frac{x}{\mathrm{N}\mathfrak{a}}, \lambda^m\chi\Big) \Big|. \end{aligned} \tag{5.2}$$

6. Reduction of $V_{j,\mathfrak{k}}(P, z, y)$ using Heath-Brown's identity. From (5.1) and Theorem 6 we see it suffices to show, subject to the conditions of Theorem 5, that for all $A > 0$ there exists $C > 0$ such that

$$\max_{F \le P \le Q} P^{-1} V_{j,\mathfrak{k}}(P, z, y) \ll yz\mathcal{L}^{-A-2}, \tag{6.1}$$

uniformly in j and $\mathfrak{k}$. We now apply a smoothed form of Heath-Brown's identity (see [8]) to von Mangoldt's function in $\Psi_{\mathfrak{k}}(g, \Lambda, x/\mathrm{N}\mathfrak{a}, \lambda^m\chi)$. The necessity of smoothing is discussed in Section 10.

Consider arithmetic functions on the ideals of K. Examples of such functions are $1(\mathfrak{a}) = 1$ for all $\mathfrak{a}$; $e(\mathfrak{a}) = 1$ if $\mathfrak{a} = \mathcal{O}_K$, zero otherwise; von Mangoldt's function, Λ, and the Möbius function, μ, in K. As in the rational case we have Möbius inversion, namely $1 * \mu = e$ along with $\Lambda * 1 = \log$, where $\log(\mathfrak{a}) = \log \mathrm{N}\mathfrak{a}$.

For $y \in \mathbb{R}$ write $\eta(y) = u(y, 0, 9/8, 1/9)$, where u is given in Lemma 3. For ideals $\mathfrak{a}$ and reals $w > 0$ define $\eta_w(\mathfrak{a}) = \eta(\mathrm{N}\mathfrak{a}/w)$. So $\eta_w(\mathfrak{a}) = 1$ for $\mathrm{N}\mathfrak{a} \le w$ and $\eta_w(\mathfrak{a}) = 0$ for $\mathrm{N}\mathfrak{a} \ge 5w/4$. Then our form of Heath-Brown's identity in K is derived from the binomial expansion

$$\Lambda * (e - (1 * \eta_X \mu))^k = \Lambda - \sum_{q=1}^{k} b_q (1^{q-1} * \log * (\eta_X \mu)^q), \tag{6.2}$$

where $b_q = (-1)^{q-1}\binom{k}{q}$, and which holds for all $X > 0$ and $k \ge 1$. Here a qth power represents a q-fold convolution. We shall assume $k \ge 2$.

The left hand side of (6.2) evaluated at an ideal $\mathfrak{b}$ is a sum over products $\mathfrak{b}_0\mathfrak{b}_1 \cdots \mathfrak{b}_k = \mathfrak{b}$. If $\mathrm{N}\mathfrak{b} \le X^k$ then $\mathrm{N}\mathfrak{b}_i \le X$ for some $1 \le i \le k$. For this ideal $(1 * \eta_X \mu)(\mathfrak{b}_i) = (1 * \mu)(\mathfrak{b}_i) = e(\mathfrak{b}_i)$, and so the left hand side of (6.2) will be zero. Thus $\Lambda(\mathfrak{b}) = \sum_{q=1}^{k} b_q (1^{q-1} * \log * (\eta_X \mu)^q)(\mathfrak{b})$, for $\mathrm{N}\mathfrak{b} \le X^k$. The only ideals counted in $\Psi_{\mathfrak{k}}(g, \Lambda, x/\mathrm{N}\mathfrak{a}, \lambda^m \chi)$ for $x \le z$ have $\mathrm{N}\mathfrak{b} \le z(1+\ell(1+\Delta))/V_j$. Since $\ell, \Delta \le 1/2$ we certainly have $\mathrm{N}\mathfrak{b} \le 2z/V_j = z_V$, say. So we choose $X = (z_V)^{1/k}$. But $\mathrm{N}\mathfrak{b} \le z_V$ also means that all divisors of $\mathfrak{b}$ have norm no greater than z_V. So, for $\mathrm{N}\mathfrak{b} \le z_V$,

$$\begin{aligned}
\Lambda(\mathfrak{b}) &= \sum_{q=1}^{k} b_q (\eta_{z_V}^{q-1} * \eta_{z_V} \log * (\eta_X \mu)^q)(\mathfrak{b}) \\
&= \sum_{q=1}^{k} b_q (\eta_{z_V}^{q-1} * \eta_1^{k-q} * \eta_{z_V} \log * (\eta_X \mu)^q * (\eta_1 \mu)^{k-q})(\mathfrak{b}),
\end{aligned}$$

written as such so that, for each $1 \le q \le k$, the summand has a fixed number of convolutions, namely $2k$. Continue, defining for each $1 \le j \le k$ vectors $(p_{i,q})_{1 \le i \le 2k}$ and $(g_i)_{1 \le i \le 2k}$ where

$$p_{i,q} = \begin{cases} z_V & \text{if } 1 \le i \le q-1 \text{ or } i = k, \\ 1 & \text{if } q \le i \le k-1 \text{ or } k+q+1 \le i \le 2k, \\ X & \text{if } k+1 \le i \le k+q, \end{cases}$$

and

$$g_i = \begin{cases} 1 & \text{if } 1 \le i \le k-1, \\ \log & \text{if } i = k, \\ \mu & \text{if } k+1 \le i \le 2k. \end{cases}$$

Then

$$\eta_{z_V}^{q-1} * \eta_1^{k-q} * \eta_{z_V} \log * (\eta_X \mu)^q * (\eta_1 \mu)^{k-q} = \mathop{*}_{i=1}^{2k} \eta_{p_{i,q}} g_i,$$

a $2k$-fold convolution of the functions $\eta_{p_{i,q}} g_i$, $1 \le i \le 2k$. Next define $\xi_w = \eta_w - \eta_{w/2}$, in which case

$$\eta_w(\mathfrak{a}) = \sum_{0 \le m \le L(w)} \xi_{w/2^m}(\mathfrak{a}),$$

where $L(w) = \log w / \log 2$. Then

$$\mathop{*}_{i=1}^{2k} \eta_{p_{i,q}} g_i = \mathop{*}_{i=1}^{2k} \sum_{0 \le m_i \le L(p_{i,q})} \xi_{p_{i,q}/2^{m_i}} g_i = \sum_{\boldsymbol{m}} \mathop{*}_{i=1}^{2k} \xi_{p_{i,q}/2^{m_i}} g_i,$$

for a set of integer $2k$-tuples $\boldsymbol{m}$ which are $\ll \mathcal{L}^{2k}$ in number. Thus

$$\varLambda(\mathfrak{b}) = \sum_{q=1}^{k} b_q \sum_{\boldsymbol{m}} \mathop{*}_{i=1}^{2k} \xi_{p_{i,q}/2^{m_i}} g_i(\mathfrak{b})$$

for all $\mathrm{N}\mathfrak{b} \le z_V$. If we substitute for $\varLambda$ within (5.2), the inner sum can be written as a sum over $2k+1$-tuples, $\ll k\mathcal{L}^{2k}$ in number, of sums of the type

$$\sum_{\mathfrak{c}} g(\mathrm{N}\mathfrak{c}/x) \lambda^m \chi(\mathfrak{c}) \mathop{*}_{i=1}^{2k+1} \xi_{N_i} g_i(\mathfrak{c}). \tag{6.3}$$

Here $N_{2k+1} = V_j$, $\xi_{N_{2k+1}}$ is the characteristic function of $[V_j, V_{j+1}]$ and $g_{2k+1} = h$. Define $\varPi_{\boldsymbol{N}} = \prod_{i=1}^{2k+1} N_i$ when $\boldsymbol{N} = (N_i)_{1 \le i \le 2k+1}$. Note that $g(\mathrm{N}\mathfrak{c}/x) *_{i=1}^{2k+1} \xi_{N_i} g_i(\mathfrak{c}) \ne 0$ only if $z\mathcal{L}^{-A-2} \ll_k \varPi_{\boldsymbol{N}} \ll_k z$. The collection of all $\boldsymbol{N}$, with all possible V_j, for which (6.3) has at least one non-zero summand will be labelled as $\mathcal{N}(k,z)$. Note that $|\mathcal{N}(k,z)| \ll \mathcal{L}^{2k+1}$. As in Section 9 we now replace the weight function $g(\mathrm{N}\mathfrak{c}/x)$ using (4.3), and truncate the resulting integral using (4.4). Thus for all $k \ge 2$, $z\mathcal{L}^{-B} < x \le z$, $y\mathcal{L}^{-B} \le \ell \le y$ and $0 \le \varTheta < 1$ we can replace the inner sum of (5.2) by

$$\frac{1}{2\pi i} \sum_{\boldsymbol{N} \in \mathcal{N}(k,z)} c_{\boldsymbol{N}} \int_{1/2-iW}^{1/2+iW} \widehat{g}(s) x^s \prod_{r=1}^{2k+1} f_{r,\mathfrak{k}}(s, \lambda^m \chi, \boldsymbol{N})\, ds \tag{6.4}$$

for some coefficients $c_{\boldsymbol{N}} \ll_k 1$, where W is given by (3.4) and

$$f_{r,\mathfrak{k}}(s, \lambda^m \chi, \boldsymbol{N}) = \sum_{(\mathfrak{a},\mathfrak{k})=1} \frac{\xi_{N_r} g_r(\mathfrak{a}) \lambda^m \chi(\mathfrak{a})}{\mathrm{N}\mathfrak{a}^s}. \tag{6.5}$$

So $f_{2k+1,\mathfrak{k}} = h_{j.\mathfrak{k}}$ as seen in (4.6). The introduction of the weight functions ξ_N will allow the comparison of the Dirichlet polynomials for $1 \le r \le k$ with the Hecke L-functions of (4.9). From (5.2), (6.4) and $g(s) \ll \ell$ we have, for any $k \geq 2$,

$$V_{j,\mathfrak{k}}(P,z,y) \ll_k yz^{1/2}\mathcal{L}^{2k+1} \max_{\boldsymbol{N}\in\mathcal{N}(k,z)} \frac{1}{\varrho(\mathfrak{k})} U_{\mathfrak{k}}(P,W,k,\boldsymbol{N}). \tag{6.6}$$

Here $U_{\mathfrak{k}}(P,W,k,\boldsymbol{N}) = \mathcal{D}_{P,W}\prod_{r=1}^{2k+1}|f_{r,\mathfrak{k}}(1/2+it,\lambda^m\chi,\boldsymbol{N})|\,dt$, where $\mathcal{D}_{P,W}$ is the operator

$$\mathcal{D}_{P,W} = \sum_{P<\mathrm{N}\mathfrak{q}\leq 2P} \frac{\mathrm{N}\mathfrak{q}}{\phi(\mathfrak{q})} \sum_{|m|\leq W} \sum_{\chi\in C^*(m,\mathfrak{q})} \int_{-W}^{W}.$$

To prove (6.1) it now suffices, because of (6.6), to choose F such that for every Q,y,z allowed by the conditions of Theorem 5 there exists $k\geq 2$ for which

$$\max_{F\leq P\leq Q} P^{-1}\mathcal{L}^{2k+1} \max_{\boldsymbol{N}\in\mathcal{N}(k,z)} U_{\mathfrak{k}}(P,W,k,\boldsymbol{N}) \ll_k z^{1/2}\mathcal{L}^{-A-2}\varrho(\mathfrak{k}). \tag{6.7}$$

7. Necessary results on Dirichlet polynomials

The large sieve. Let θ_1,θ_2 be an integral basis for K, so every integer $\alpha\in\mathcal{O}_K$ is representable as $\alpha = n_1\theta_1+n_2\theta_2$ for rational integers n_1 and n_2.

Lemma 8 (Huxley [12]). *For any set $\{c(\alpha)\}_{\alpha\in\mathcal{O}_K}$ of coefficients we have*

$$\sum_{\mathrm{N}\mathfrak{q}\leq Q} \frac{\mathrm{N}\mathfrak{q}}{\phi(\mathfrak{q})} \sum_{\widehat{\chi} \bmod \mathfrak{q}}^{*} \Big|\sum_{\alpha}{}' c(\alpha)\widehat{\chi}(\alpha)\Big|^2 \ll (N_1+Q)(N_2+Q)\sum_{\alpha}{}' |c(\alpha)|^2,$$

where $\sum_{\alpha}'$ is a sum over $\alpha = n_1\theta_1+n_2\theta_2$ from the rectangle $M_i < n_i \leq M_i+N_i$, $i=1$ and 2.

For sums over ideals the ideas within the proof of Theorem 6.2 of [2] give

$$\sum_{\mathrm{N}\mathfrak{q}\leq Q} \frac{\mathrm{N}\mathfrak{q}}{\phi(\mathfrak{q})} \sum_{\substack{|m|\leq W\\ b|m}} \sum_{\chi\in C^*(m,\mathfrak{q})} \int_{-W}^{W} \Big|\sum_{\mathfrak{a}} \frac{c(\mathfrak{a})\lambda^m\chi(\mathfrak{a})}{\mathrm{N}\mathfrak{a}^{it}}\Big|^2 dt$$
$$\ll W^4 \int_0^{\infty}\int_{-1/2}^{1/2} \frac{h}{w^2}\sum_{H}\sum_{\mathrm{N}\mathfrak{q}\leq Q} \frac{\mathrm{N}\mathfrak{q}}{\phi(\mathfrak{q})} \sum_{\widehat{\chi} \bmod \mathfrak{q}}^{*} \Big|\sum_{\alpha\in\mathfrak{a}_H}{}'' c(\alpha/\mathfrak{a}_H)\widehat{\chi}(\alpha)\Big|^2 d\theta\,\frac{dy}{y}.$$

Here $\sum_H$ runs over the ideal classes H and, for each such class, $\mathfrak{a}_H$ is an ideal chosen from H^{-1}. The inner sum, $\sum''$, is over $\alpha\in\mathfrak{a}_H$ satisfying $y\mathrm{N}\mathfrak{a}_H < \mathrm{N}\alpha < \tau y\mathrm{N}\mathfrak{a}_H$, with $\tau = \exp(1/W)$, and $|(\arg\alpha)/2\pi - \theta| < 1/8W$. It is easy to see that if there are two integers α and α' satisfying these conditions then $|\alpha-\alpha'| \ll y^{1/2}/W$. So we can apply Lemma 8 with $N_1 = N_2 = cy^{1/2}/W$ to get

THEOREM 9. *For all $Q, W \geq 1$ we have*

$$\sum_{\mathrm{N}\mathfrak{q}\leq Q}\frac{\mathrm{N}\mathfrak{q}}{\phi(\mathfrak{q})}\sum_{\substack{|m|\leq W\\ b|m}}\sum_{\chi\in C^*(m,\mathfrak{q})}\int_{-W}^{W}\Big|\sum_{\mathfrak{a}}\frac{c(\mathfrak{a})\lambda^m\chi(\mathfrak{a})}{\mathrm{N}\mathfrak{a}^{it}}\Big|^2\,dt \ll \sum_{\mathfrak{a}}|c(\mathfrak{a})|^2(Q^2W^2+\mathrm{N}\mathfrak{a}), \tag{7.1}$$

provided the right hand side converges.

For each $\mathfrak{q}, m$ and χ on the left hand side of (7.1) we replace the integral by a sum over well-spaced points. Using Lemma 1.4 of Montgomery [23], due to Gallagher, we may derive the following from Theorem 9.

THEOREM 10. *Let Ω denote a set of quadruples $\omega = (\mathfrak{q}, m, \chi, n)$, with $\mathrm{N}\mathfrak{q} \leq Q$, $|m| \leq W$, $b(\mathfrak{q}) \mid m$, $\chi \in C^*(m, \mathfrak{q})$ and $n \in \mathbb{N}, n \leq 2W$. Assume that to each $\omega \in \Omega$ there is associated a real number t_ω. Further, assume that if $\omega = (\mathfrak{q}, m, \chi, n)$ and $\omega' = (\mathfrak{q}, m, \chi, n')$ are distinct then $|t_\omega - t_{\omega'}| \geq 1$, that is, they are well-spaced by 1. Then*

$$\sum_{\omega\in\Omega}\Big|\sum_{\mathfrak{a}}\frac{c(\mathfrak{a})\lambda^m\chi(\mathfrak{a})}{\mathrm{N}\mathfrak{a}^{it_\omega}}\Big|^2\,dt \ll \mathcal{L}\sum_{\mathfrak{a}}|c(\mathfrak{a})|^2(Q^2W^2+\mathrm{N}\mathfrak{a}),$$

provided the right hand side converges.

The large value estimate. The following result may be proved using a method identical to that in [2].

THEOREM 11. *Let Ω be a set of quadruples satisfying the conditions in Theorem 10. Suppose there is a number V such that*

$$\Big|\sum_{N_0<\mathrm{N}\mathfrak{a}\leq N_0+N}\frac{c(\mathfrak{a})\lambda^m\chi(\mathfrak{a})}{\mathrm{N}\mathfrak{a}^{it_\omega}}\Big| \geq V$$

for all $\omega \in \Omega$. Then

$$\#\Omega \ll \frac{GN}{V^2}\bigg(1+\frac{Q^2W^2G^2\mathcal{L}^2}{V^4}\bigg),$$

where $G = \sum_{N_0<\mathrm{N}\mathfrak{a}\leq N_0+N}|c(\mathfrak{a})|^2$.

The fourth power estimate. Using the method of Ramachandra [26] with Theorem 9 and the functional equation for $L(s, \lambda^m\chi)$ we can show the following result.

THEOREM 12. *For all $Q, W \geq 2$ and $|\delta| \ll \log^{-1}(QW)$ we have*

$$\sum_{\mathrm{N}\mathfrak{q}\leq Q}\frac{\mathrm{N}\mathfrak{q}}{\phi(\mathfrak{q})}\sum_{\substack{|m|\leq W\\ b|m}}\sum_{\chi\in C^*(m,\mathfrak{q})}\int_{-W}^{W}|L(1/2+\delta+it,\lambda^m\chi)|^4\,dt \ll Q^2W^2\log^c(QW).$$

The constant c is independent of K.

In [5, Lemma 10], the method of Ramachandra was used to give second power moments with both fixed m and fixed t along with explicit dependences. A proof of the present result can be based on (7.11) of that paper. This method has also been used by Johnson [17], along with an approximate functional equation for $L(s,\lambda^m\chi)$, to give the fourth power moment for fixed m with an explicit dependence on m. Previously, Huxley [13] used an approximate functional equation to give a result for fixed m, without an explicit dependence. Maknys [22] used an approximate functional equation to give a result for fixed $\mathfrak{q}$.

Using Theorem 10 in place of Theorem 9 we may deduce a discrete version of Theorem 12, namely $\sum_{\omega\in\Omega}|L(1/2+\delta+it_\omega,\lambda^m\chi)|^4 \ll Q^2W^2\log^c(QW)$.

The inclusion of δ means that, using $f'(s)=(2\pi i)^{-1}\int f(z)(z-s)^{-2}\,dz$ with the path of integration a circle with centre $1/2+it$ and radius $1/\log(W+|T|)$, we can deduce analogues of these results for $L'(s,\lambda^m\chi)$.

For an application of Theorem 12 recall the definition of $f_{r,\mathfrak{k}}(s,\lambda^m\chi,\boldsymbol{N})$ from (6.5).

LEMMA 13. *Let $k\ge 2$, $\boldsymbol{N}\in\mathcal{N}(k,z)$, and $1\le r\le k$ be given. Then*

$$\mathcal{D}_{P,W}|f_{r,\mathfrak{k}}(1/2+it,\lambda^m\chi,\boldsymbol{N})|^4\,dt \ll \varrho^2(\mathfrak{k})P^2W^2\mathcal{L}^{c(r)}, \tag{7.2}$$

uniformly in $\mathrm{N}\mathfrak{k}\ll z$.

Proof. From the definition in Section 6, $\xi_w(\mathfrak{a})\ne 0$ only when $w/2\le \mathrm{N}\mathfrak{a}\le w$. Thus the Mellin transform, $\widehat{\xi_w}(v)=\widehat{\eta}(v)w^v\mathrm{N}\mathfrak{a}^{-v}(1-2^{-v})$, is well defined for all $v\in\mathbb{C}$. Assume first that $1\le r<k$ when $g_r\equiv 1$. Using arguments that previously led to (4.5) we find that $f_{r,\mathfrak{k}}(s,\lambda^m\chi,\boldsymbol{N})$ can be replaced by

$$\frac{1}{2\pi i}\int_{2-iV}^{2+iV}\widehat{\eta}(v)N_r^v\left(1-\frac{1}{2^v}\right)\prod_{\mathfrak{p}|\mathfrak{k}}\left(1-\frac{\lambda^m\chi(\mathfrak{p})}{\mathrm{N}\mathfrak{p}^{s+v}}\right)L(s+v,\lambda^m\chi)\,dv, \tag{7.3}$$

where $V=c\mathcal{L}^2$ for some constant c. The value for V comes from the lower bound for $|t|$ in (4.4) with $\ell=9/8$ and $\Delta=1/9$. We move the integral back to $\operatorname{Re}v=0$. No pole is crossed since $\widehat{\chi}$ primitive mod $\mathfrak{q}$ with $\mathfrak{q}\ne(1)$ means $\widehat{\chi}$ is non-principal. Hölder's inequality gives

$$\mathcal{D}_{P,W}|f_r(1/2+it,\lambda^m\chi,\boldsymbol{N})|^4\,dt \ll \varrho^2(\mathfrak{k})\mathcal{L}^c\mathcal{D}_{P,W}\int_{|u|\le V}|L(1/2+it+iu,\lambda^m\chi)|^4\,du\,dt.$$

Now we apply Theorem 12, taking $W+V$ to be the W in that theorem. Note that $W+V\ll W$, since $W\gg\mathcal{L}^2$.

Now assume $r=k$. In this case we get (7.3) but with the integrand differentiated with respect to s. The additional factor that arises from the differentiation of the Euler product can be bounded by $\ll\varrho(\mathfrak{k})\log\mathrm{N}\mathfrak{k}$. ∎

Using the discrete version of Theorem 12 we can deduce a version of (7.2) with $\sum_{\Omega}$ in place of $\mathcal{D}_{P,W}$.

8. Proof of Theorem 5

Estimates of $U_{\mathfrak{k}}(P,W,k,\boldsymbol{N})$ *for all* $\boldsymbol{N}\in\mathcal{N}(k,z)$. For any non-empty set $\boldsymbol{\alpha}\subseteq\{1,\dots,2k+1\}$ define $\Pi_{\boldsymbol{N}}(\boldsymbol{\alpha})=\prod_{i\in\boldsymbol{\alpha}}N_i$, and set $\Pi_{\boldsymbol{N}}(\emptyset)=1$. Recall that we have earlier defined $\Pi_{\boldsymbol{N}}=\prod_{i=1}^{2k+1}N_i$. In the following results the exponents of $\mathcal{L}$, possibly different at each occurrence, will be functions of k, as will the constants implied within $\ll$.

LEMMA 14. *Let $k\geq 2$ and $\boldsymbol{N}\in\mathcal{N}(k,z)$ be given. Then for any disjoint $\boldsymbol{\alpha}\cup\boldsymbol{\beta}=\{1,\dots,2k+1\}$ we have*

$$U_{\mathfrak{k}}(P,W,k,\boldsymbol{N})\ll\mathcal{L}^c(P^2W^2+PW\Pi_{\boldsymbol{N}}(\boldsymbol{\alpha})^{1/2}+PW\Pi_{\boldsymbol{N}}(\boldsymbol{\beta})^{1/2}+\Pi_{\boldsymbol{N}}^{1/2})$$

uniformly in $\mathfrak{k}$.

Proof. We have, by Cauchy's inequality,

$$\begin{aligned}U_{\mathfrak{k}}(P,W,k,\boldsymbol{N})\leq{}&\Big(\mathcal{D}_{P,W}\prod_{r\in\boldsymbol{\alpha}}|f_{\mathfrak{k},r}(1/2+it,\lambda^m\chi,\boldsymbol{N})|^2\,dt\Big)^{1/2}\\&\times\Big(\mathcal{D}_{P,W}\prod_{r\in\boldsymbol{\beta}}|f_{\mathfrak{k},r}(1/2+it,\lambda^m\chi,\boldsymbol{N})|^2\,dt\Big)^{1/2}.\end{aligned}$$

Theorem 9 can now be applied to both parts, to get

$$U_{\mathfrak{k}}(P,W,k,\boldsymbol{N})\ll\mathcal{L}^c(P^2W^2+\Pi_{\boldsymbol{N}}(\boldsymbol{\alpha}))^{1/2}(P^2W^2+\Pi_{\boldsymbol{N}}(\boldsymbol{\beta}))^{1/2},$$

which gives the result since $\Pi_{\boldsymbol{N}}(\boldsymbol{\alpha})\Pi_{\boldsymbol{N}}(\boldsymbol{\beta})=\Pi_{\boldsymbol{N}}$. ∎

DEFINITION. Let $k\geq 2$ and $\boldsymbol{N}\in\mathcal{N}(k,z)$ be given. For $0\leq a<b\leq k$ set $S_{ab}=N_aN_b$ with the convention $N_0=1$.

LEMMA 15. *We have, for all $k\geq 2$, $\boldsymbol{N}\in\mathcal{N}(k,z)$ and $0\leq a<b\leq k$,*

$$U_{\mathfrak{k}}(P,W,k,\boldsymbol{N})\ll\mathcal{L}^cPW(PW+\Pi_{\boldsymbol{N}}^{1/2}S_{ab}^{-1/2})\varrho(\mathfrak{k})$$

uniformly in $\mathfrak{k}$.

Proof. For $1\leq a<b\leq k$ use Hölder's inequality to get

$$\begin{aligned}&U_{\mathfrak{k}}(P,W,k,\boldsymbol{N})\\&\ll\Big(\mathcal{D}_{P,W}\prod_{1\leq r\leq 2k,\,r\neq a,b}|f_{\mathfrak{k},r}(1/2+it,\lambda^m\chi,\boldsymbol{N})|^2\,dt\Big)^{1/2}\\&\quad\times(\mathcal{D}_{P,W}|f_{\mathfrak{k},a}(1/2+it,\lambda^m\chi)|^4\,dt)^{1/4}(\mathcal{D}_{P,W}|f_{\mathfrak{k},b}(1/2+it,\lambda^m\chi)|^4\,dt)^{1/4}.\end{aligned}\tag{8.1}$$

Theorem 9 is used to bound the first term on the right hand side, while the other two terms may be bounded using Lemma 13, giving the stated result. If $a=0$ the proof still holds with the convention that $f_0\equiv 1$. ∎

Further estimates of $U_{\mathfrak{k}}(P,W,k,\boldsymbol{N})$. Let $k \ge 2$, $\boldsymbol{N} \in \mathcal{N}(k,z)$ and $\boldsymbol{\alpha} \subseteq \{1,\dots,2k+1\}$ be given. With $\boldsymbol{\beta} = \{1,\dots,2k+1\} \setminus \boldsymbol{\alpha}$ define

$$f_{\mathfrak{k},0}(s,\lambda^m\chi,\boldsymbol{N},\boldsymbol{\beta}) = \prod_{r\in\boldsymbol{\beta}} f_{\mathfrak{k},r}(s,\lambda^m\chi,\boldsymbol{N}), \tag{8.2}$$

a Dirichlet polynomial of length $N_0 = \Pi(\boldsymbol{\beta})$. We therefore have

$$U_{\mathfrak{k}}(P,W,k,\boldsymbol{N}) = \mathcal{D}_{P,W} |f_{\mathfrak{k},0}(1/2+it,\lambda^m\chi,\boldsymbol{N},\boldsymbol{\beta})| \prod_{r\in\boldsymbol{\alpha}} |f_{\mathfrak{k},r}(1/2+it,\lambda^m\chi,\boldsymbol{N})|\,dt.$$

The integrand is clearly bounded throughout the range of integration, and for each $(\mathfrak{q},m,\chi)$ we may bound the integral by a sum over well-spaced points. That is, we can find a set Ω with properties as seen in Theorem 10 such that

$$U_{\mathfrak{k}}(P,W,k,\boldsymbol{N}) \ll \mathcal{L}_2 \sum_{\omega\in\Omega} |f_{\mathfrak{k},0}(1/2+it_\omega,\lambda^m\chi,\boldsymbol{N},\boldsymbol{\beta})| \prod_{r\in\boldsymbol{\alpha}} |f_{\mathfrak{k},r}(1/2+it_\omega,\lambda^m\chi)|,$$

where the $\mathcal{L}_2 = \log\mathcal{L}$ factor comes from bounding $\mathrm{N}\mathfrak{q}/\phi(\mathfrak{q})$.

We partition Ω according to the size of the Dirichlet polynomials. Let $\boldsymbol{\alpha}^0 = \boldsymbol{\alpha} \cup \{0\}$ and set $p = |\boldsymbol{\alpha}^0|$. For each $r \in \boldsymbol{\alpha}^0$ the polynomial given either by (6.5) or (8.2) satisfies $|f_{\mathfrak{k},r}(1/2+it,\lambda^m\chi)| \le (c_rN_r)^{1/2}\mathcal{L}^{\kappa_r}$ for some constant c_r and where

$$\kappa_r = \begin{cases} 1 & \text{if either } r=0 \text{ and } k\in\boldsymbol{\beta} \text{ or } r=k\in\boldsymbol{\alpha}, \\ 0 & \text{otherwise.} \end{cases}$$

For $r \in \boldsymbol{\alpha}^0$ define the numbers $\sigma(\omega,r)$ by $|f_{\mathfrak{k},r}(1/2+it_\omega,\lambda^m\chi,\boldsymbol{N})| = (c_rN_r)^{\sigma(\omega,r)}\mathcal{L}^{\kappa_r}$, so $\sigma(\omega,r) \le 1/2$. Define the intervals

$$J(0) = (-\infty,0], \qquad J(u) = \left(\frac{u-1}{\mathcal{L}}, \frac{u}{\mathcal{L}}\right] \quad \text{for } u = 1,\dots,\left[\frac{\mathcal{L}}{2}\right]+1.$$

For each quadruple $\omega \in \Omega$ there is a corresponding vector $\boldsymbol{n} = (n_r)_{r\in\boldsymbol{\alpha}^0} \in \mathbb{Z}^p$ such that $\sigma(\omega,r) \in J(n_r)$ for all $r \in \boldsymbol{\alpha}^0$. Let $\mathcal{M}(\boldsymbol{\alpha})$ be the set of all $\boldsymbol{n}$ which have at least one associated quadruple; note that $|\mathcal{M}(\boldsymbol{\alpha})| \ll \mathcal{L}^p$. For each $\boldsymbol{n} \in \mathcal{M}(\boldsymbol{\alpha})$ define $\Omega(\boldsymbol{n})$ to be the set of those quadruples in Ω which are associated with $\boldsymbol{n}$. Thus, for each $\boldsymbol{\alpha}$, we have the bound

$$U_{\mathfrak{k}}(P,W,k,\boldsymbol{N}) \ll \mathcal{L}^{p+1} \max_{\boldsymbol{n}\in\mathcal{M}(\boldsymbol{\alpha})} U_{\mathfrak{k},\boldsymbol{n}}(P,W,k,\boldsymbol{N}), \tag{8.3}$$

where

$$\begin{aligned} &U_{\mathfrak{k},\boldsymbol{n}}(P,W,k,\boldsymbol{N}) \\ &\quad = \sum_{\omega\in\Omega(\boldsymbol{n})} |f_{\mathfrak{k},0}(1/2+it_\omega,\lambda^m\chi,\boldsymbol{N},\boldsymbol{\beta})| \prod_{r\in\boldsymbol{\alpha}} |f_{\mathfrak{k},r}(1/2+it_\omega,\lambda^m\chi,\boldsymbol{N})|. \end{aligned} \tag{8.4}$$

For each $\boldsymbol{n}$ define $\sigma(\boldsymbol{n},\boldsymbol{\alpha}) = \mathcal{L}^{-1}\max_{r\in\boldsymbol{\alpha}^0} n_r$.

LEMMA 16. *Let $k \ge 2$, $\boldsymbol{N} \in \mathcal{N}(k,z)$ and $\boldsymbol{\alpha} \subseteq \{1,\dots,2k+1\}$ be given. Let $\boldsymbol{n} \in \mathcal{M}(\boldsymbol{\alpha})$, and $j = j(\boldsymbol{n})$ be any subscript such that $n_j = \max_{r\in\boldsymbol{\alpha}^0} n_r$. Then for all integers $g \ge 1$ there exists $c = c(g,k)$ such that*

$$U_{\mathfrak{k},\boldsymbol{n}}(P,W,k,\boldsymbol{N}) \ll \mathcal{L}^c \Pi_{\boldsymbol{N}}^{\sigma(\boldsymbol{n},\boldsymbol{\alpha})}(P^2W^2N_j^{g(1-6\sigma(\boldsymbol{n},\boldsymbol{\alpha}))} + N_j^{g(1-2\sigma(\boldsymbol{n},\boldsymbol{\alpha}))}) \tag{8.5}$$

uniformly in $\mathfrak{k}$.

Proof. We begin with the simple observation that $U_{\mathfrak{k},\boldsymbol{n}}(P,W,k,\boldsymbol{N}) \ll \mathcal{L}\Pi_{\boldsymbol{N}}^{\sigma(\boldsymbol{n},\boldsymbol{\alpha})}|\Omega(\boldsymbol{n})|$. Apply Theorem 11 to the Dirichlet polynomial $f_{\mathfrak{k},j}(1/2+it,\lambda^m\chi,\boldsymbol{N})^g$, which is bounded below by $V = N_j^{g\sigma(\boldsymbol{n},\boldsymbol{\alpha})}$. In the notation of Theorem 11 we have $N = N_j^g$. ∎

LEMMA 17. *Let $k \ge 2$, $\boldsymbol{N} \in \mathcal{N}(k,z)$ and $\boldsymbol{\alpha} \subseteq \{1,\dots,2k+1\}$ be given. Let $\boldsymbol{n} \in \mathcal{M}(\boldsymbol{\alpha})$, and $j = j(\boldsymbol{n})$ be any subscript such that $n_j = \max_{r\in\boldsymbol{\alpha}^0} n_r$. Then if $j \in \boldsymbol{\alpha}\cap\{1,\dots,k\}$ we have*

$$U_{\mathfrak{k},\boldsymbol{n}}(P,W,k,\boldsymbol{N}) \ll \mathcal{L}^c P^2W^2\Pi_{\boldsymbol{N}}^{\sigma(\boldsymbol{n},\boldsymbol{\alpha})}N_j^{-4\sigma(\boldsymbol{n},\boldsymbol{\alpha})}\varrho^2(\mathfrak{k})$$

uniformly in $\mathfrak{k}$.

Proof. For all $\omega \in \Omega(\boldsymbol{n})$ we have $|f_{\mathfrak{k},j}(1/2+it_\omega,\lambda^m\chi,\boldsymbol{N})| \ge cN_j^{\sigma(\boldsymbol{n},\boldsymbol{\alpha})}\mathcal{L}^{\kappa_j}$ for some constant c, since $j = j(\boldsymbol{n})$. Therefore we may write

$$\begin{aligned}U_{\mathfrak{k},\boldsymbol{n}}(P,W,k,\boldsymbol{N}) &\ll \mathcal{L}\Pi_{\boldsymbol{N}}^{\sigma(\boldsymbol{n},\boldsymbol{\alpha})}|\Omega(\boldsymbol{n})| \\ &\ll \mathcal{L}^{1-4k_j}\Pi_{\boldsymbol{N}}^{\sigma(\boldsymbol{n},\boldsymbol{\alpha})}N_j^{-4\sigma(\boldsymbol{n},\boldsymbol{\alpha})}\sum_{\Omega(\boldsymbol{n})}|f_{\mathfrak{k},j}(1/2+it_\omega,\lambda^m\chi,\boldsymbol{N})|^4.\end{aligned}$$

The conditions on j ensure that the discrete version of Lemma 13 can be applied. Thus we obtain $\sum_{\Omega(\boldsymbol{n})}|f_{\mathfrak{k},j}(1/2+it_\omega,\lambda^m\chi,\boldsymbol{N})|^4 \ll \varrho^2(\mathfrak{k})P^2W^2\mathcal{L}^c$, and hence the stated result. ∎

DEFINITION. For $\boldsymbol{N} \in \mathcal{N}(k,z)$ let

$$R(\boldsymbol{N}) = \min\{\Pi_{\boldsymbol{N}}(\boldsymbol{\alpha}) : \boldsymbol{\alpha} \subseteq \{1,\dots,2k+1\},\ \Pi_{\boldsymbol{N}}(\boldsymbol{\alpha}) \ge \Pi_{\boldsymbol{N}}^{1/2}\}.$$

Further, let $\mathcal{P}(\boldsymbol{N}) = \{\boldsymbol{\alpha} \subseteq \{1,\dots,2k+1\} : \Pi_{\boldsymbol{N}}(\boldsymbol{\alpha}) = R(\boldsymbol{N})\}$.

Immediately we see from Lemma 14 applied to any $\boldsymbol{\alpha} \in \mathcal{P}(\boldsymbol{N})$ that

$$U_{\mathfrak{k}}(P,W,k,\boldsymbol{N}) \ll \mathcal{L}^c(P^2W^2 + PWR(\boldsymbol{N})^{1/2} + \Pi_{\boldsymbol{N}}^{1/2}). \tag{8.6}$$

LEMMA 18. *Let $k \ge 2$, $\boldsymbol{N} \in \mathcal{N}(k,z)$ and $\boldsymbol{\alpha} \in \mathcal{P}(\boldsymbol{N})$ be given. Then for any $a \in \boldsymbol{\alpha}$ we have $N_a \ge R(\boldsymbol{N})^2/\Pi_{\boldsymbol{N}}$.*

Proof. Let $a \in \boldsymbol{\alpha}$. For any set $\boldsymbol{\delta} \subseteq \{1,\dots,2k+1\}$ we have, by the definition of $R(\boldsymbol{N})$, either $\Pi_{\boldsymbol{N}}(\boldsymbol{\delta}) \ge R(\boldsymbol{N})$ or $\Pi_{\boldsymbol{N}}(\boldsymbol{\delta}) \le \Pi_{\boldsymbol{N}}/R(\boldsymbol{N})$. Let $\boldsymbol{\delta} = \boldsymbol{\alpha}\backslash\{a\}$. Since $\Pi_{\boldsymbol{N}}(\boldsymbol{\alpha}) = R(\boldsymbol{N})$ we have $\Pi_{\boldsymbol{N}}(\boldsymbol{\delta}) = R(\boldsymbol{N})/N_a$, which is $< R(\boldsymbol{N})$. Thus from the choice above we must have $\Pi_{\boldsymbol{N}}(\boldsymbol{\delta}) \le \Pi_{\boldsymbol{N}}/R(\boldsymbol{N})$, which leads to the stated result. ∎

COROLLARY 19. *Let $k \geq 2$, $\boldsymbol{N} \in \mathcal{N}(k,z)$ and $\boldsymbol{\alpha} \in \mathcal{P}(\boldsymbol{N})$ be given. Then*

$$|\boldsymbol{\alpha}| \leq \left(2 - \frac{\log \Pi_{\boldsymbol{N}}}{\log R(\boldsymbol{N})}\right)^{-1}. \tag{8.7}$$

In particular, if $R(\boldsymbol{N}) > \Pi_{\boldsymbol{N}}^{3/5}$ then $|\boldsymbol{\alpha}| \leq 2$.

Proof. Lemma 18 immediately gives (8.7). If $R(\boldsymbol{N}) > \Pi_{\boldsymbol{N}}^{3/5}$ then (8.7) gives $|\boldsymbol{\alpha}| < 3$. Yet $|\boldsymbol{\alpha}|$ is an integer. ∎

LEMMA 20. *Let $k \geq 2$, $\boldsymbol{N} \in \mathcal{N}(k,z)$ and $\boldsymbol{\alpha} \in \mathcal{P}(\boldsymbol{N})$ be given. Then for all $\boldsymbol{n} \in \mathcal{M}(\boldsymbol{\alpha})$ and $r \in \boldsymbol{\alpha}$ we have*

$$U_{\mathfrak{k},\boldsymbol{n}}(P,W,k,\boldsymbol{N}) \ll \mathcal{L}^c(P^2W^2N_r^{\sigma(\boldsymbol{n},\boldsymbol{\alpha})} + \Pi_{\boldsymbol{N}}^{1/2}). \tag{8.8}$$

Proof. Let $r \in \boldsymbol{\alpha}$ and $\boldsymbol{n} \in \mathcal{M}(k)$ be given. Then from (8.4) we have

$$\begin{aligned} U_{\mathfrak{k},\boldsymbol{n}}(P,W,k,\boldsymbol{N}) &\ll N_r^{n_r/\mathcal{L}} \sum_{\Omega(\boldsymbol{n})} |f_{\mathfrak{k},0}(1/2+it_\omega, \lambda^m\chi, \boldsymbol{N}, \boldsymbol{\beta})| \\ &\qquad \times \prod_{j\in\boldsymbol{\alpha}\setminus\{r\}} |f_{\mathfrak{k},j}(1/2+it_\omega, \lambda^m\chi, \boldsymbol{N})| \\ &\ll N_r^{n_r/\mathcal{L}} \Big(\sum_{\Omega(\boldsymbol{n})} |f_{\mathfrak{k},0}(1/2+it_\omega, \lambda^m\chi, \boldsymbol{N}, \boldsymbol{\beta})|^2\Big)^{1/2} \\ &\qquad \times \Big(\sum_{\Omega(\boldsymbol{n})} \prod_{j\in\boldsymbol{\alpha}\setminus\{r\}} |f_{\mathfrak{k},j}(1/2+it_\omega, \lambda^m\chi, \boldsymbol{N})|^2\Big)^{1/2}. \end{aligned}$$

Now $f_{\mathfrak{k},0}$ has length $N_0 = \Pi_{\boldsymbol{N}}/R(\boldsymbol{N})$, and the product of the other factors has length $R(\boldsymbol{N})/N_r$, which, by Lemma 18, is $\leq N_0$. We apply Theorem 9 twice, obtaining

$$U_{\mathfrak{k},\boldsymbol{n}}(P,W,k,\boldsymbol{N}) \ll \mathcal{L}^c N_r^{n_r/\mathcal{L}}(P^2W^2 + PW(\Pi_{\boldsymbol{N}}/R(\boldsymbol{N}))^{1/2} + (\Pi_{\boldsymbol{N}}/N_r)^{1/2}).$$

Since $n_r/\mathcal{L} \leq \sigma(\boldsymbol{n},\boldsymbol{\alpha}) \leq 1/2$ we may therefore bound $U_{\mathfrak{k},\boldsymbol{n}}(P,W,k,\boldsymbol{N})$ by

$$\ll \mathcal{L}^c\left(P^2W^2N_r^{\sigma(\boldsymbol{n},\boldsymbol{\alpha})} + \frac{PW\Pi_{\boldsymbol{N}}^{1/2}N_r^{\sigma(\boldsymbol{n},\boldsymbol{\alpha})}}{R(\boldsymbol{N})^{1/2}} + \Pi_{\boldsymbol{N}}^{1/2}\right). \tag{8.9}$$

If $R(\boldsymbol{N}) > (PW)^{-2}\Pi_{\boldsymbol{N}}$ then (8.8) follows from (8.9). Otherwise Lemma 14 is sufficient to show that $U_{\mathfrak{k},\boldsymbol{n}}(P,W,k,\boldsymbol{N}) \ll \mathcal{L}^c(P^2W^2 + \Pi_{\boldsymbol{N}}^{1/2})$. ∎

Completion of the proof of Theorem 5

THEOREM 21. *Let $G > 0$ be given.*

(i) *Assume $k \geq 6$ and $\boldsymbol{N} \in \mathcal{N}(k,z)$. Then if $W\mathcal{L}^G \leq \Pi_{\boldsymbol{N}}^{1/5}$ and $LW^2\mathcal{L}^{2G} < \Pi_{\boldsymbol{N}}$ there exists a constant $c(k)$ such that*

$$U_{\mathfrak{k}}(P,W,k,\boldsymbol{N}) \ll \mathcal{L}^c(P^2W^2 + P\Pi_{\boldsymbol{N}}^{1/2}\mathcal{L}^{-G} + \Pi_{\boldsymbol{N}}^{1/2})\varrho(\mathfrak{k}) \tag{8.10}$$

uniformly in $\mathfrak{k}$.

(ii) *Assume* $k \geq 11$ *and* $\boldsymbol{N} \in \mathcal{N}(k,z)$. *Then, if* $\Pi_{\boldsymbol{N}}^{1/5} \leq W\mathcal{L}^G \leq \Pi_{\boldsymbol{N}}^{9/40}$, $LW^8\mathcal{L}^{8G} \leq \Pi_{\boldsymbol{N}}^2$ *and* $L < \Pi_{\boldsymbol{N}}^{1/4}$, *there exists a constant* $c(k,\kappa)$ *such that*

$$U_{\mathfrak{k}}(P,W,k,\boldsymbol{N}) \ll \mathcal{L}^c(P^2W^2\Pi_{\boldsymbol{N}}^{1/20} + P\Pi_{\boldsymbol{N}}^{1/2}\mathcal{L}^{-G} + \Pi_{\boldsymbol{N}}^{1/2})\varrho(\mathfrak{k}) \tag{8.11}$$

for $\varrho(\mathfrak{k}) \leq \Pi_{\boldsymbol{N}}^{1/20}$.

Proof. Assume $\boldsymbol{N}$ satisfies the conditions of either case and there exists $\boldsymbol{\alpha} \in \mathcal{P}(\boldsymbol{N})$ with $|\boldsymbol{\alpha}| \leq 2$. We will show that (8.10) holds for such $\boldsymbol{N}$.

In both cases we have $LW^2\mathcal{L}^{2G} < \Pi_{\boldsymbol{N}}$, in which case Lemma 14 applied with $\boldsymbol{\alpha} = \{2k+1\}$ gives (8.10) subject to the additional constraint $N_{2k+1} \geq W^2\mathcal{L}^{2G}$.

Assume $N_{2k+1} < W^2\mathcal{L}^{2G}$. If $R(\boldsymbol{N})$ or any other sub-product lies in

$$\left[W^2\mathcal{L}^{2G}, \frac{\Pi_{\boldsymbol{N}}}{W^2\mathcal{L}^{2G}}\right] \tag{8.12}$$

then Lemma 14 again gives (8.10).

So we may assume that $R(\boldsymbol{N}) > \Pi_{\boldsymbol{N}}/W^2\mathcal{L}^{2G}$. Take a set $\boldsymbol{\alpha} \in \mathcal{P}(\boldsymbol{N})$ with $|\boldsymbol{\alpha}| \leq 2$. Again, by minimality, if $a \in \boldsymbol{\alpha}$ then

$$\begin{aligned} N_a &\geq \frac{R(\boldsymbol{N})^2}{\Pi_{\boldsymbol{N}}} \geq \frac{\Pi_{\boldsymbol{N}}}{W^4\mathcal{L}^{4G}} \\ &\geq \begin{cases} \Pi_{\boldsymbol{N}}^{1/5} > (2z)^{1/k} & \text{for } k \geq 6 \text{ in case (i)}, \\ \Pi_{\boldsymbol{N}}^{1/10} > (2z)^{1/k} & \text{for } k \geq 11 \text{ in case (ii)}. \end{cases} \end{aligned} \tag{8.13}$$

Thus $\boldsymbol{\alpha} \subseteq \{1,\dots,k,2k+1\}$ for k appropriate to cases (i) and (ii).

If $|\boldsymbol{\alpha}| = 1$ then $LW^2\mathcal{L}^{2G} < \Pi_{\boldsymbol{N}}$ implies the centre inequality in the chain $R(\boldsymbol{N}) \geq \Pi_{\boldsymbol{N}}/W^2\mathcal{L}^{2G} > L \geq N_{2k+1}$. Hence $R(\boldsymbol{N}) \neq N_{2k+1}$ and so $R(\boldsymbol{N}) = N_a$ for some $1 \leq a \leq k$ and we can apply Lemma 15 to get

$$U_{\mathfrak{k}}(P,W,k,N) \ll \mathcal{L}^cPW(PW + W\mathcal{L}^G)\varrho(\mathfrak{k}). \tag{8.14}$$

This bound is again dominated by that in (8.10).

Assume $|\boldsymbol{\alpha}| = 2$. If $2k+1 \notin \boldsymbol{\alpha}$ then $\boldsymbol{\alpha} \subseteq \{1,\dots,k\}$ and we can again apply Lemma 15 with $S_{ab} = R(\boldsymbol{N})$ to get (8.14).

If $2k+1 \in \boldsymbol{\alpha}$ then $R(\boldsymbol{N}) = N_{2k+1}N_i$ for some $1 \leq i \leq k$. Relabelling if necessary assume that $R(\boldsymbol{N}) = N_{2k+1}N_1$.

If there exists $2 \leq a \leq k$ such that $N_a \geq N_{2k+1}$ then apply Lemma 15 with $S_{ab} = N_aN_1 \geq R(\boldsymbol{N})$.

Otherwise, assume that for all $2 \leq a \leq k$ we have $N_a < N_{2k+1}$, in which case $N_aN_1 < N_{2k+1}N_1 = R(\boldsymbol{N})$. By the minimality of $R(\boldsymbol{N})$ we must have

$N_aN_1 < W^2\mathcal{L}^{2G}$. Then, for $2 \le a \le k$ we have

$$N_a < \frac{W^2\mathcal{L}^{2G}}{N_1} = \frac{W^2\mathcal{L}^{2G}N_{2k+1}}{R(\boldsymbol{N})} < \frac{W^4\mathcal{L}^{4G}N_{2k+1}}{\varPi_{\boldsymbol{N}}}$$
$$\le \begin{cases} W^6\mathcal{L}^{6G}/\varPi_{\boldsymbol{N}} \le \varPi_{\boldsymbol{N}}/W^4\mathcal{L}^{4G} & \text{in case (i), using } N_{2k+1} \le W^2\mathcal{L}^{2G}, \\ W^4\mathcal{L}^{4G}L/\varPi_{\boldsymbol{N}} \le \varPi_{\boldsymbol{N}}/W^4\mathcal{L}^{4G} & \text{in case (ii).} \end{cases}$$

As seen in (8.13) we also have, in both cases and for appropriate k,

$$N_a \le (2z)^{1/k} \le \frac{\varPi_{\boldsymbol{N}}}{W^4\mathcal{L}^{4G}} \tag{8.15}$$

for all $k+1 \le a \le 2k$. Hence the bound in (8.15) holds for all $2 \le a \le 2k$. Starting with $N_1N_2 \le W^2\mathcal{L}^{2G}$ we find that

$$N_1N_2N_3 \le (W^2\mathcal{L}^{2G})\,\frac{\varPi_{\boldsymbol{N}}}{W^4\mathcal{L}^{4G}} \le \frac{\varPi_{\boldsymbol{N}}}{W^2\mathcal{L}^{2G}}. \tag{8.16}$$

So either $N_1N_2N_3$ lies in the interval (8.12) or $N_1N_2N_3 \le W^2\mathcal{L}^{2G}$. In the latter case repeat the process by looking at $N_1N_2N_3N_4$, *et cetera*. Since the product $\prod_{i=1}^{2k} N_i = \varPi_{\boldsymbol{N}}/N_{2k+1} \ge \varPi_{\boldsymbol{N}}/W^2\mathcal{L}^{2G}$ lies to the right of the interval (8.12), some sub-product must lie within the interval. To this we can apply Lemma 14 to complete the proof that (8.10) holds for $\boldsymbol{N}$ for which there exists $\boldsymbol{\alpha} \in \mathcal{P}(\boldsymbol{N})$ with $|\boldsymbol{\alpha}| \le 2$.

This ends the proof of case (i) since $R(\boldsymbol{N}) > \varPi_{\boldsymbol{N}}/W^2\mathcal{L}^{2G}$ along with $W\mathcal{L}^G \le \varPi_{\boldsymbol{N}}^{1/5}$ imply $R(\boldsymbol{N}) > \varPi_{\boldsymbol{N}}^{3/5}$, and thus $|\boldsymbol{\alpha}| \le 2$ for all $\boldsymbol{\alpha} \in \mathcal{P}(\boldsymbol{N})$, by Corollary 19.

Finally, consider those $\boldsymbol{N}$ satisfying case (ii) and for which all $\boldsymbol{\alpha} \in \mathcal{P}(\boldsymbol{N})$ satisfy $|\boldsymbol{\alpha}| \ge 3$. From Corollary 19 we deduce that $R(\boldsymbol{N}) \le \varPi_{\boldsymbol{N}}^{3/5}$. If $R(\boldsymbol{N}) \le \varPi_{\boldsymbol{N}}^{11/20}$ use (8.6) along with the observation that

$$P^2W^2 + PW\varPi_{\boldsymbol{N}}^{11/40} + \varPi_{\boldsymbol{N}}^{1/2} \ll P^2W^2\varPi_{\boldsymbol{N}}^{1/20} + \varPi_{\boldsymbol{N}}^{1/2}.$$

Thus we may assume

$$\varPi_{\boldsymbol{N}}^{11/20} < R(\boldsymbol{N}) \le \varPi_{\boldsymbol{N}}^{3/5}. \tag{8.17}$$

Recalling (8.3) we see that it suffices to prove

$$U_{\mathfrak{k},\boldsymbol{n}}(P,W,k,\boldsymbol{N}) \ll \mathcal{L}^{c(k)}(P^2W^2\varPi_{\boldsymbol{N}}^{1/20} + \varPi_{\boldsymbol{N}}^{1/2})\varrho(\mathfrak{k}) \tag{8.18}$$

for all $\boldsymbol{n} \in \mathcal{M}(\boldsymbol{\alpha})$, subject to (8.17) and $|\boldsymbol{\alpha}| \ge 3$.

We assume to begin with that $\boldsymbol{n}$ and $\boldsymbol{\alpha}$ satisfy $\sigma(\boldsymbol{n},\boldsymbol{\alpha}) \ge 1/4$. We have Lemma 16 available but since we cannot choose the value of $j(\boldsymbol{n})$ we shall consider each possible value in turn.

First, assume that $j(\boldsymbol{n}) = 0$ when $N_0 = \varPi_{\boldsymbol{N}}/R(\boldsymbol{N})$. We take $g = 1$ in Lemma 16, which with the lower bound in (8.17) and $\sigma \le 1/2$ gives

$$U_{\mathfrak{k},\boldsymbol{n}}(P,W,k,\boldsymbol{N}) \ll \mathcal{L}^{c(k)}(P^2W^2R(\boldsymbol{N})^{6\sigma(\boldsymbol{n},\boldsymbol{\alpha})-1}\varPi_{\boldsymbol{N}}^{1-5\sigma(\boldsymbol{n},\boldsymbol{\alpha})} + \varPi_{\boldsymbol{N}}^{1/2}). \tag{8.19}$$

The first term on the right in (8.19) is no more than $P^2W^2\Pi_{\boldsymbol{N}}^{1/20}$ when $\sigma(\boldsymbol{n},\boldsymbol{\alpha}) \geq 1/4$, because of the upper bound on $R(\boldsymbol{N})$ in (8.17). So (8.18) follows.

Second, assume that $j(\boldsymbol{n}) \in \boldsymbol{\alpha}$ in Lemma 16. If $N_{j(\boldsymbol{n})} > \Pi_{\boldsymbol{N}}^{1/4}$ then $j(\boldsymbol{n}) \neq 2k+1$ and so Lemma 17 suffices to give a bound of $\ll \mathcal{L}^c P^2W^2\varrho^2(\mathfrak{k}) \ll \mathcal{L}^c P^2W^2\Pi_{\boldsymbol{N}}^{1/20}\varrho(\mathfrak{k})$. If $N_{j(\boldsymbol{n})} \leq \Pi_{\boldsymbol{N}}^{1/(20\sigma(\boldsymbol{n},\boldsymbol{\alpha}))}$ then Lemma 20 with $r = j(\boldsymbol{n})$ suffices to give (8.18). Therefore we may assume $\Pi_{\boldsymbol{N}}^{1/(20\sigma(\boldsymbol{n},\boldsymbol{\alpha}))} \leq N_{j(\boldsymbol{n})} \leq \Pi_{\boldsymbol{N}}^{1/4}$. We deal with this range in two ways. Firstly, we may choose an integer $g \geq 2$ such that $\Pi_{\boldsymbol{N}}^{1/3} \leq N_{j(\boldsymbol{n})}^g \leq \Pi_{\boldsymbol{N}}^{1/2}$. All such g satisfy $g/20\sigma(\boldsymbol{n},\boldsymbol{\alpha}) \leq 1/2$ and so $g \leq 5$. Thus (8.5) gives

$$U_{\mathfrak{k},\boldsymbol{n}}(P,W,k,\boldsymbol{N}) \ll \mathcal{L}^{c(k)}(P^2W^2\Pi_{\boldsymbol{N}}^{1/3-\sigma(\boldsymbol{n},\boldsymbol{\alpha})} + \Pi_{\boldsymbol{N}}^{1/2}). \tag{8.20}$$

Secondly, applying Lemma 16 with $g = 2$ and using $N_{j(\boldsymbol{n})} \geq \Pi_{\boldsymbol{N}}^{1/(20\sigma(\boldsymbol{n},\boldsymbol{\alpha}))}$ gives

$$U_{\mathfrak{k},\boldsymbol{n}}(P,W,k,\boldsymbol{N}) \ll \mathcal{L}^{c(k)}(P^2W^2\Pi_{\boldsymbol{N}}^{\sigma(\boldsymbol{n},\boldsymbol{\alpha})+1/10\sigma(\boldsymbol{n},\boldsymbol{\alpha})-3/5} + \Pi_{\boldsymbol{N}}^{1/2}). \tag{8.21}$$

The bound (8.20) gives (8.18) for $17/60 \leq \sigma(\boldsymbol{n},\boldsymbol{\alpha}) \leq 1/2$, while (8.21) gives (8.18) for $1/4 \leq \sigma(\boldsymbol{n},\boldsymbol{\alpha}) < 17/60$.

Assume now that $\boldsymbol{n}$ and $\boldsymbol{\alpha}$ satisfy $\sigma(\boldsymbol{n},\boldsymbol{\alpha}) \leq 1/4$. We note that, since $|\boldsymbol{\alpha}| \geq 3$, there exists at least one $a \in \boldsymbol{\alpha}$ such that $N_a \leq R(\boldsymbol{N})^{1/3}$, which is $\leq \Pi_{\boldsymbol{N}}^{1/5}$ by (8.17). We apply Lemma 20 with $r = a$ and observe $N_a^{\sigma(\boldsymbol{n},\boldsymbol{\alpha})} \ll \Pi_{\boldsymbol{N}}^{1/20}$ to deduce (8.18).

Hence, for all $\boldsymbol{n} \in \mathcal{M}(\boldsymbol{\alpha})$ we have (8.18). ∎

Proof of Theorem 4. We might just note here that case (a) of Theorem 4 follows from Theorem 21(i) and Theorem 6(iii) with $n = 2$, while case (b) uses $n = 3$. Case (c) follows from Theorem 21(ii) along with Theorem 6(ii), while case (d) uses part (i) of Theorem 6. ∎

9. Proof of Theorem 5; small values of ℓ and x. We first estimate the number of integers $\mu \equiv \nu \bmod \mathfrak{q}$ with $\mu \in \mathcal{S}(x,\Theta,\ell)$.

LEMMA 22. *For all $\nu \in \mathcal{O}_K$ with $(\nu,\mathfrak{q}) = 1$ we have*

$$\sum_{\substack{\mu \equiv \nu \bmod \mathfrak{q} \\ \mu \in \mathcal{S}(x,\Theta,\ell)}} 1 = \frac{2\pi}{|d|^{1/2}}\,\frac{4\ell^2 x}{\mathrm{N}\mathfrak{q}} + O\Big(\frac{\ell x^{1/2}}{\mathrm{N}\mathfrak{q}^{1/2}} + 1\Big) \tag{9.1}$$

where d is the discriminant of K.

Proof. The algebraic integers in $\mathfrak{q}$ form a lattice $L \subseteq \mathbb{C}$. Take a $\mathbb{Z}$-basis $\{\alpha_1,\beta\}$ with $|\alpha_1|$ minimum over all non-zero elements of the lattice. Next choose $n \in \mathbb{Z}$ to minimise $\mathrm{Re}\,\overline{\alpha}_1(\beta+n\alpha_1)$. With $\alpha_2 = \beta+n\alpha_1$ we get a basis

$\{\alpha_1, \alpha_2\}$ in which the angle between α_1 and α_2 lies between $\pi/3$ and $\pi/2$. (See the proof of Lemma 4 in [3] for more details.) In particular,

$$A \le |\alpha_1|\,|\alpha_2| \le 2A/3^{1/2}, \tag{9.2}$$

where A is the area of a fundamental region of the lattice, in this case $2^{-1}\sqrt{d}\,\mathrm{N}\mathfrak{q}$. Since $\alpha_i \in \mathfrak{q}$ we have $|\alpha_i| \ge \mathrm{N}\mathfrak{q}^{1/2}$, $i = 1, 2$. Combine these observations with (9.2) to get

$$\mathrm{N}\mathfrak{q}^{1/2} \le |\alpha_i| \le (d/3)^{1/2}\mathrm{N}\mathfrak{q}^{1/2} \tag{9.3}$$

for $i = 1, 2$. We now get the asymptotic result stated by estimating the number of translates of a fundamental region of the lattice that either lie totally within, or have a non-empty intersection with $T(x, \Theta, \ell) = \{z : \big||z|^2 - x\big| < \ell x,\ -\ell \le \langle(\arg z)/2\pi - \Theta\rangle \le \ell\}$. Yet because of (9.3), the relevant translates of a fundamental region of the lattice subtend an angle $\ll \mathrm{N}\mathfrak{q}^{1/2}/x^{1/2}$ at the origin. Thus

$$\sum_{\substack{\mu \equiv \nu \bmod \mathfrak{q} \\ \mu \in \mathcal{S}(x,\Theta,\ell)}} 1 \lessgtr \frac{|T(x, \Theta, \ell \pm c\mathrm{N}\mathfrak{q}^{1/2}/x^{1/2})|}{2^{-1}d^{1/2}\mathrm{N}\mathfrak{q}}$$

for some $c > 0$. From $|T(x, \Theta, \ell)| = 4\pi\ell^2 x$ we get the stated result. ∎

If $\alpha \in \mathcal{S}(x, \Theta, \ell(1 - \Delta))$ then $v_{x,\Theta,\ell,\Delta}(\alpha) = 1$, while if $v_{x,\Theta,\ell,\Delta}(\alpha) \ne 0$ then $\alpha \in \mathcal{S}(x, \Theta, \ell(1 + \Delta))$. These observations along with Lemma 22 give $\sum_{\alpha \equiv \beta \bmod \mathfrak{q}} v_{x,\Theta,\ell,\Delta}(\alpha) \ll \ell^2 x\mathrm{N}\mathfrak{q}^{-1} + 1$. Thus

$$\begin{aligned} &\Big|\sum_{\alpha} v_{x,\Theta,\ell,\Delta}(\alpha)\Lambda_{\beta,\mathfrak{q}}(\alpha)\Big| \\ &\quad\le \Big(\sum_{\alpha \equiv \beta \bmod \mathfrak{q}} v_{x,\Theta,\ell,\Delta}(\alpha) + \frac{1}{\phi(\mathfrak{q})}\sum_{\alpha} v_{x,\Theta,\ell,\Delta}(\alpha)\Big)\mathcal{L} \ll \ell^2 x\phi(\mathfrak{q})^{-1}\mathcal{L} + \mathcal{L}. \end{aligned} \tag{9.4}$$

If we sum (9.4) over all $\mathrm{N}\mathfrak{q} \le Q$, the contribution is $\ll \ell^2 x\mathcal{L}^2 + Q\mathcal{L}$, which is sufficiently small if $\ell^2 x \le y^2 z\mathcal{L}^{-A-2}$. Note that $\ell^2 x > y^2 z\mathcal{L}^{-A-2}$ along with $x \le z$ and $\ell \le y$ imply $\ell \ge y\mathcal{L}^{-A/2-1}$ and $x > z\mathcal{L}^{-A-2}$. Let $B(Q, z, y)$ denote the expression on the left hand side of (2.3). We have, by the above, $B(Q, z, y) \ll y^2 z\mathcal{L}^{-A} + B^*(Q, z, y)$ where

$$\begin{aligned} &B^*(Q, z, y) \\ &\quad= \sum_{\mathrm{N}\mathfrak{q} \le Q} \max_{(\beta,\mathfrak{q})=1} \max_{\substack{z\mathcal{L}^{-A-2} \le x \le z \\ 0 \le \Theta < 1}} \max_{y\mathcal{L}^{-A/2} \le \ell \le y} \Big|\sum_{\alpha} v_{x,\Theta,\ell,\Delta}(\alpha)\Lambda_{\beta,\mathfrak{q}}(\alpha)\Big|. \end{aligned} \tag{9.5}$$

An application of Theorem 4 completes the proof of Theorem 5. ∎

10. Stripping weights and the necessity of weights

Derivation of (1.2) *in Theorem 1.* For this section only let $1_{x,\Theta,\ell}$ be the characteristic function of integers $\alpha \in \mathcal{S}(x,\Theta,\ell)$. Then, by an observation in Section 9,

$$|1_{x,\Theta,\ell} - v_{x,\Theta,\ell,\Delta}| \le 1_{x,\Theta,\ell(1+\Delta)} - 1_{x,\Theta,\ell(1-\Delta)} = 1^*_{x,\Theta,\ell,\Delta},$$

say. Thus

$$(10.1)\quad \Big|\sum_{\alpha\in\mathcal{S}(x,\Theta,\ell)} \Lambda_{\beta,\mathfrak{q}}(\alpha) - \sum_{\alpha} v_{x,\Theta,\ell,\Delta}(\alpha)\Lambda_{\beta,\mathfrak{q}}(\alpha)\Big|$$

$$\le \sum_{\alpha\equiv\beta \bmod \mathfrak{q}} 1^*(\alpha)\Lambda(\alpha) + \frac{1}{\phi(\mathfrak{q})}\sum_{\alpha} 1^*(\alpha)\Lambda(\alpha) \ll \bigg(\frac{\ell^2\Delta x}{\phi(\mathfrak{q})} + \frac{\ell x^{1/2}}{\phi(\mathfrak{q})^{1/2}} + 1\bigg)\mathcal{L},$$

by Lemma 22. Taking $\Delta = \mathcal{L}^{-A-2}$ and summing (10.1) over $\mathrm{N}\mathfrak{q} \le Q$ gives a contribution $\ll \ell^2 x\mathcal{L}^{-A}$, assuming $Q \le \ell^2 x\mathcal{L}^{-2A-2}$. Hence (1.2) follows from Theorem 5. ∎

Necessity of weights. To deal with the condition $-\ell < \langle(\arg\alpha)/2\pi - \Theta\rangle < \ell$ we have to use a weight function and it may as well have the best possible properties. We could examine the condition $\big||\alpha|^2 - x\big| < x\ell$ without introducing weights when we would use Perron's Theorem to get an expression like (4.5) but with truncation at some T far larger than W. Thus we would have to count zeros with $|m| \le W$, $|\gamma_{m\chi}| \le T$. As explained in Section 4 this is done using Mean Value results. The version of Theorem 9 with an integral over $[-T,T]$ has a factor $W^2 + T^2$ in place of W^2 on the right hand side of (7.1). This is already seen in Theorem 6.2 of [2] and arises from counting lattice points in rectangular regions. It transpires that since the dependence is a *square* of T, we require a weighted integrand in (4.5) decaying faster than the $(1/|t|)dt$ arising from Perron's Theorem. Thus we have to put a weight function on the norms whose Mellin transform decays sufficiently fast. It simplifies matters to derive such a weight from the same function as used to give the weight on the arguments.

Similarly, if we used Perron's Theorem back in the proof of Lemma 13 to relate $f_{r,\mathfrak{k}}(s,\lambda^m\chi,\boldsymbol{N})$ to $L(s,\lambda^m\chi)$ for $1 \le r \le k$, the length of integration in (7.3) would have been longer than W. For instance, in [29, p. 323], the integration is up to $T_0 = z^{100}$. The method of [26] used to prove Theorem 12 uses the functional equation for our Hecke L-functions and this is of the form $L(s,\lambda^m\chi) = G(s,\lambda^m\chi)L(1-s,\overline{\lambda}^m\overline{\chi})$. Here the gamma factors satisfy $G(\sigma+it,\lambda^m\chi) \ll (m^2+t^2)^{1/2-\sigma}$. Again, because the dependence is a *square* of t, to deal with the range $W < |t| \le T_0$ in any mean value of L^4 would require a weighted integrand decaying faster than the $(1/|t|)dt$ arising from Perron's Theorem. Thus we see the necessity of introducing some weight function, such as ξ_w, into the $f_{r,\mathfrak{k}}$. Hence the reason for taking the time in Section 6 to insert the weight function $\eta_w = \sum_i \xi_{w/2^i}$ into Heath-Brown's identity for Λ.

11. A Bombieri–Vinogradov Theorem for homogeneously expanding domains in $\mathbb{Q}(i)$

Proof of Theorem 2. We shall use $|\cdot|$ to denote either the area, length or cardinality of a set, and it should be obvious from the context which is meant. So, for example, we have $|\mathcal{R}_{w,v}| = |v|^2|\mathcal{R}_{0,1}|$ and $|\partial\mathcal{R}_{w,v}| = |v|\,|\partial\mathcal{R}_{0,1}|$. Further, from Theorem 2 on p. 128 of [21] we have, for all $w \in \mathbb{C}$,

$$|\mathbb{Z}[i] \cap \mathcal{R}_{w,v}| = |v|^2|\mathcal{R}_{0,1}| + O(|v|) \quad \text{ as } |v| \to \infty.$$

First observe that for all $z \in \mathbb{C}$,

$$\begin{aligned}\int_0^\infty\!\!\int_0^1 v_{r,\Theta,\ell,\Delta}(z)\,\frac{dr}{r}\,d\Theta &= \int_0^\infty g_{\ell,\Delta}\Big(\frac{|z|^2}{r}\Big)\frac{dr}{r}\int_{-1/2}^{1/2} f_{\ell,\Delta}\Big(\frac{\arg z}{2\pi}-\Theta\Big)\,d\Theta\\ &= \int_0^\infty g_{\ell,\Delta}(w)\,\frac{dw}{w}\int_{-1/2}^{1/2} f_{\ell,\Delta}(\Theta)\,d\Theta = 4\ell^2(1+O(\Delta)),\end{aligned}$$

having used the periodicity of f. Importantly the double integral is independent of z. Label it as c_ℓ. Assume that $v, w \in \mathbb{C}$ are given. With ℓ and Δ to be chosen we have

$$\begin{aligned}\sum_{\alpha\in\mathcal{R}_{w,v}} \Lambda_{\beta,\mathfrak{q}}(\alpha) &= \frac{1}{c_\ell}\sum_{\alpha\in\mathcal{R}_{w,v}} \Lambda_{\beta,\mathfrak{q}}(\alpha)\int_0^\infty\!\!\int_0^1 v_{r,\Theta,\ell,\Delta}(\alpha)\,\frac{dr}{r}\,d\Theta\\ &= \frac{1}{c_\ell}\iint_{J_0}\sum_{\alpha\in\mathcal{R}_{w,v}} v_{r,\Theta,\ell,\Delta}(\alpha)\Lambda_{\beta,\mathfrak{q}}(\alpha)\,\frac{dr}{r}\,d\Theta,\end{aligned} \tag{11.1}$$

where $J_0 = \{(r,\Theta) : (\exists\alpha \in \mathbb{Z}(i))\ \alpha \in \mathcal{R}_{w,v} \wedge v_{r,\Theta,\ell,\Delta}(\alpha) \neq 0\}$.

Firstly, $\alpha \in \mathcal{R}_{w,v}$ means that $|\alpha - w| < |v|$ and so $|\alpha| > |w| - |v| \geq |w|/2$ as well as $|\alpha| < 3|w|/2$.

Secondly, $v_{r,\Theta,\ell,\Delta}(\alpha) \neq 0$ implies both $\big||\alpha|^2 - r\big| < r\ell'$ and $|\arg\alpha - 2\pi\Theta| < 2\pi\ell'$ where $\ell' = \ell(1+\Delta)$. The first inequality along with our bounds on α give $r \asymp |w|^2$. We can also justify the steps in the following:

$$\begin{aligned}|r^{1/2}e^{2\pi\Theta i} - \alpha| &= |r^{1/2}e^{2\pi\Theta i} - |\alpha|e^{2\pi\Theta i} + |\alpha|e^{2\pi\Theta i} - \alpha|\\ &\leq \big|r^{1/2} - |\alpha|\big| + |\alpha|\,|e^{2\pi\Theta i} - e^{i\arg\alpha}| \leq c_1|w|\ell.\end{aligned}$$

Next, define

$$J_1 = \{(r,\Theta) : (\forall\alpha \in \mathbb{Z}[i])\ v_{r,\Theta,\ell,\Delta}(\alpha) \neq 0 \Rightarrow \alpha \in \mathcal{R}_{w,v}\}.$$

Then $J_1 \subseteq J_0$ and, for $(r,\Theta) \in J_0 \setminus J_1$, we can find $\alpha, \beta \in \mathbb{Z}[i]$ for which $\alpha \in \mathcal{R}_{w,v}, v_{r,\Theta,\ell,\Delta}(\alpha) \neq 0$, $v_{r,\Theta,\ell,\Delta}(\beta) \neq 0$ but $\beta \notin \mathcal{R}_{w,v}$. This means that the region of $\mathbb{C}$ on which $v_{r,\Theta,\ell,\Delta}$ is non-zero for such (r,Θ) cuts the boundary of $\mathcal{R}_{w,v}$. Also, $r^{1/2}e^{2\pi\Theta i}$ is within a distance $c_1|w|\ell$ of both α and β and so of some point on the boundary. If we define

$$\mathcal{E}_{w,v} = \{(r,\Theta) : (\exists z \in \partial\mathcal{R}_{w,v})\ |r^{1/2}e^{2\pi\Theta i} - z| \leq c_1|w|\ell\},$$

we have shown that $J_0 \setminus J_1 \subseteq \mathcal{E}_{w,v}$. Recall that $\mathcal{R}$ has a boundary that is Lipschitz parametrisable. As seen in Chapter VI, §2 of [21] this means that $\partial\mathcal{R}_{w,v}$ intersects $\leq c_2|v|$ translates of the fundamental region of a lattice such as $\mathbb{Z}[i]$. If $|v|/|w|\ell$ is sufficiently large we can cover the boundary by a union of $\ll |v|/|w|\ell$ discs of radius $c_3|w|\ell$. Here, c_3 can be chosen sufficiently large so that the union of the discs contains all points within a distance $c_1|w|\ell$ of the boundary. Hence $|\mathcal{E}_{w,v}| \ll |v|\,|w|\ell$.

We replace the region J_0 in (11.1) by J_1; the condition $\alpha \in \mathcal{R}_{w,v}$ can then be removed from the inner sum. We go further by replacing J_1 by $J_{w,v} = \{(r,\Theta) : r^{1/2}e^{2\pi\Theta i} \in \mathcal{R}_{w,v}\}$, a region over which it is easier to integrate. But we first note that if $(r,\Theta) \in J_1 \setminus J_{w,v}$ then $r^{1/2}e^{2\pi\Theta i} \notin \mathcal{R}_{w,v}$ but is within $c_1|w|\ell$ of some $\alpha \in \mathbb{Z}[i] \cap \mathcal{R}_{w,v}$, while if $(r,\Theta) \in J_{w,v} \setminus J_1$ then $r^{1/2}e^{2\pi\Theta i} \in \mathcal{R}_{w,v}$ and is within $c_1|w|\ell$ of some $\alpha \in \mathbb{Z}[i]$ with $\alpha \notin \mathcal{R}_{w,v}$. In both cases $(r,\Theta) \in \mathcal{E}_{w,v}$ and so $J_1 \triangle J_{w,v} \subseteq \mathcal{E}_{w,v}$.

Thus

$$\sum_{\alpha\in\mathcal{R}_{w,v}} \Lambda_{\beta,\mathfrak{q}}(\alpha) = \frac{1}{c_\ell} \iint\limits_{J_{w,v}} \sum_{\alpha} v_{r,\Theta,\ell,\Delta}(\alpha)\Lambda_{\beta,\mathfrak{q}}(\alpha)\,\frac{dr}{r}\,d\Theta + E_{\beta,\mathfrak{q}} \tag{11.2}$$

where

$$E_{\beta,\mathfrak{q}} \ll \frac{1}{c_\ell} \iint\limits_{\mathcal{E}_{w,v}} \sum_{\alpha\in\mathcal{R}_{w,v}} |v_{r,\Theta,\ell,\Delta}(\alpha)\Lambda_{\beta,\mathfrak{q}}(\alpha)|\,\frac{dr}{r}\,d\Theta. \tag{11.3}$$

For the error term we need

$$\iint\limits_{\mathcal{E}_{w,v}} \frac{dr}{r}\,d\Theta \ll \frac{1}{|w|^2} \iint\limits_{\mathcal{E}_{w,v}} dr\,d\Theta \ll \frac{|v|}{|w|}\,\ell. \tag{11.4}$$

Note that r here is the square of the usual radial variable in polar coordinates. Thus

$$E_{\beta,\mathfrak{q}} \ll \mathcal{L}\left(\frac{|w|^2\ell^2}{\mathrm{N}\mathfrak{q}} + 1\right)\frac{1}{c_\ell} \iint\limits_{\mathcal{E}_{w,v}} \frac{dr}{r}\,d\Theta \ll \mathcal{L}\left(\frac{|v|\,|w|\ell}{\mathrm{N}\mathfrak{q}} + \frac{|v|}{|w|\ell}\right).$$

We now see that it is appropriate to set $\ell = |v|\,|w|^{-1}\mathcal{L}^{-A-2}$ when

$$E_{\beta,\mathfrak{q}} \ll \mathcal{L}\left(\frac{|v|^2}{\mathrm{N}\mathfrak{q}\mathcal{L}^{A+2}} + \mathcal{L}^{A+2}\right).$$

We can now introduce the maximums over v and w along with the summation over $\mathrm{N}\mathfrak{q} \le Q$ to get a contribution to (1.3) of $O(h^2 X\mathcal{L}^{-A} + Q\mathcal{L}^{A+3})$.

For the main term we need

$$\iint\limits_{J_{w,v}} \frac{dr}{r}\, d\Theta = \frac{4|v|^2}{\pi} \iint\limits_{s+it\in\mathcal{R}} \frac{ds\, dt}{|w+v(s+it)|^2} \ll \frac{|v|^2}{|w|^2},$$

since $|v| \le |w|/2$.

The result (11.2) can be used to give asymptotic results but, since we require only upper bounds, it suffices to choose $\Delta = 1/2$. An upper bound for the first term from (11.2) is

$$\ll \max_{(r,\Theta)\in J_{w,v}} \Big| \sum_{\alpha} v_{r,\Theta,\ell,1/2}(\alpha)\Lambda_{\beta,\mathfrak{q}}(\alpha)\Big| \frac{1}{c_\ell} \iint\limits_{J_{w,v}} \frac{dr}{r}\, d\Theta$$

$$\ll \max_{(r,\Theta)\in J_{w,v}} \Big| \sum_{\alpha} v_{r,\Theta,\ell,1/2}(\alpha)\Lambda_{\beta,\mathfrak{q}}(\alpha)\Big| \mathcal{L}^{2A+4}$$

by (11.4) and the choice of ℓ above. Next, $(r,\Theta) \in J_{w,v}$ along with $|w|^2 \le X$ and $|v| \le h|w|$ imply the existence of a constant $c_2 > 1$ such that $r < c_2 X$. So

$$\max_{X\mathcal{L}^{-C(A)}\le |w|^2\le X}\ \max_{|v|<h|w|}\ \max_{(r,\Theta)\in J_{w,v}} |\cdot| \le \max_{\substack{r<c_2X\\ 0\le\Theta<1}}\ \max_{\ell<h\mathcal{L}^{-A-2}} |\cdot|.$$

We can now feed in Theorem 5 to get a contribution to (1.3) of $\ll h^2 X\mathcal{L}^{-A}$. ∎

Finally, note that Lemma 7 can be used to give an asymptotic result for $\sum_\alpha v_{x,\Theta,\ell,\Delta}(\alpha)\Lambda(\alpha)$ and, for $\mathrm{N}\mathfrak{q} \le \log^C x$, an upper bound for $\sum_\alpha v_{x,\Theta,\ell,\Delta}(\alpha)\Lambda_{\beta,\mathfrak{q}}(\alpha)$. We leave it to the reader to check that (11.2) can be used with Δ sufficiently small to strip out the weights v from these two results, replacing it with the condition $\alpha \in \mathcal{R}_{w,v}$. The two results so found can be combined as (1.4).

References

[1] E. Bombieri, J. B. Friedlander, and H. Iwaniec, *Primes in arithmetic progressions to large moduli. II*, Math. Ann. 277 (1987), 361–393.

[2] M. D. Coleman, *The distribution of points at which binary quadratic forms are prime*, Proc. London Math. Soc. (3) 61 (1990), 433–456.

[3] —, *A zero-free region for the Hecke L-functions*, Mathematika 37 (1990), 287–304.

[4] —, *The distribution of points at which norm-forms are prime*, J. Number Theory 41 (1992), 359–378.

[5] —, *The Rosser–Iwaniec sieve in number fields, with an application*, Acta Arith. 65 (1993), 53–83.

[6] —, *The normal density of prime ideals in small regions*, Monatsh. Math. 125 (1998), 111–126.

[7] E. Fogels, *Über die Ausnahmenullstelle der Heckeschen L-Funktionen*, Acta Arith. 8 (1963), 307–309.

[8] D. R. Heath-Brown, *Prime numbers in short intervals and a generalized Vaughan identity*, Canad. J. Math. 34 (1982), 1365–1377.

[9] E. Hecke, *Eine neue Art von Zetafunktionen und ihre Beziehungen zur Verteilung der Primzahlen*, *I*, Math. Z. 1 (1918), 357–376.

[10] —, *Eine neue Art von Zetafunktionen und ihre Beziehungen zur Verteilung der Primzahlen*, *II*, ibid. 6 (1920), 11–51.

[11] J. G. Hinz, *A generalization of Bombieri's prime number theorem to algebraic number fields*, Acta Arith. 51 (1988), 173–193.

[12] M. N. Huxley, *The large sieve inequality for algebraic number fields*, Mathematika 15 (1968), 178–187.

[13] —, *The large sieve inequality for algebraic number fields*, *II*: *Means of moments of Hecke zeta-functions*, Proc. London Math. Soc. (3) 21 (1970), 108–128.

[14] —, *The large sieve inequality for algebraic number fields*, *III*: *Zero-density results*, J. London Math. Soc. (2) 3 (1971), 233–240.

[15] —, *On the difference between consecutive primes*, Invent. Math. 15 (1972), 164–170.

[16] M. N. Huxley and H. Iwaniec, *Bombieri's theorem in short intervals*, Mathematika 22 (1975), 188–194.

[17] D. Johnson, *Mean values of Hecke L-functions*, J. Reine Angew. Math. 305 (1979), 195–205.

[18] M. Jutila, *A statistical density theorem for L-functions with applications*, Acta Arith. 16 (1969), 207–216.

[19] S. Knapowski, *On a theorem of Hecke*, J. Number Theory 1 (1969), 235–251.

[20] E. Landau, *Über Ideale und Primideale in Idealklassen*, Math. Z. 2 (1918), 52–154.

[21] S. Lang, *Algebraic Number Theory*, 2nd ed., Grad. Texts in Math. 110, Springer, New York, 1994.

[22] M. Maknys, *Zeros of Hecke Z-functions and the distribution of primes of an imaginary quadratic field*, Litovsk. Mat. Sb. 15 (1975), 173–184 (in Russian); English transl.: Lithuanian Math. J. 15 (1975), 140–149.

[23] H. L. Montgomery, *Topics in Multiplicative Number Theory*, Lecture Notes in Math. 227, Springer, Berlin, 1971.

[24] A. Perelli, J. Pintz, and S. Salerno, *Bombieri's theorem in short intervals*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 11 (1984), 529–539.

[25] —, —, —, *Bombieri's theorem in short intervals*, *II*, Invent. Math. 79 (1985), 1–9.

[26] K. Ramachandra, *A simple proof of the mean fourth power estimate for $\zeta(1/2+it)$ and $L(1/2+it,\chi)$*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 1 (1974), 81–97.

[27] S. J. Ricci, *Mean-value theorems for primes in short intervals*, Proc. London Math. Soc. (3) 37 (1978), 230–242.

[28] G. J. Rieger, *Verallgemeinerung der Siebmethode von A. Selberg auf algebraische Zahlkörper*, *III*, J. Reine Angew. Math. 208 (1961), 79–90.

[29] N. M. Timofeev, *Distribution in the mean of arithmetic functions in short intervals in progressions*, Izv. Akad. Nauk SSSR Ser. Mat. 51 (1987), 341–362 (in Russian); English transl.: Math. USSR-Izv. 30 (1988), 315–335.

[30] R. J. Wilson, *The large sieve in algebraic number fields*, Mathematika 16 (1969), 189–204.

[31] J. Wu, *Théorèmes généralisés de Bombieri–Vinogradov dans les petits intervalles*, Quart. J. Math. Oxford Ser. (2) 44 (1993), 109–128.

[32] T. Zhan, *Bombieri's theorem in short intervals*, Acta Math. Sinica (N.S.) 5 (1989), 37–47.

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