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Variation of the number of lattice points in large balls

by

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1. Introduction. In the present paper we address an aspect of the classical question of counting lattice points in balls of large radius. We are concerned with the lower bounds on the variation of the number of lattice points when the centre of the ball varies.

We introduce some notation. Let $\Gamma \subset \mathbb{R}^d$ be a lattice in the d-dimensional Euclidean space. For any bounded set $\mathcal{C} \subset \mathbb{R}^d$ we denote by $\mathcal{N}[\mathcal{C}]$ the number of lattice points in $\mathcal{C}$, that is,

$$\mathcal{N}[\mathcal{C}] = \#\{\boldsymbol{\gamma} \in \Gamma : \boldsymbol{\gamma} \in \mathcal{C}\}.$$

We omit the dependence on Γ in the notation as the lattice is always fixed. Denote by

$$B(r; \mathbf{k}) = \{\boldsymbol{\xi} : |\boldsymbol{\xi} - \mathbf{k}| < r\}$$

the open ball of radius $r > 0$ centred at the point $\mathbf{k} \in \mathbb{R}^d$. Obviously, the function $\mathcal{N}[B(r; \mathbf{k})]$ is a periodic function of the variable $\mathbf{k}$ with period lattice Γ, and hence it is bounded. We are interested in the variation of the quantity $\mathcal{N}[B(r; \mathbf{k})]$ as a function of $\mathbf{k}$. Define, for all $r > 0$,

$$\mathcal{N}^+(r) = \max_{\mathbf{k}} \mathcal{N}[B(r; \mathbf{k})], \quad \mathcal{N}^-(r) = \min_{\mathbf{k}} \mathcal{N}[B(r; \mathbf{k})], \tag{1.1}$$

and introduce

Definition 1.1. For given real numbers $\lambda \geq 0$ and $\delta \in [0, \lambda]$ the δ-*variation* is defined by

$$V(\lambda, \delta) = \mathcal{N}^+(\sqrt{\lambda - \delta}) - \mathcal{N}^-(\sqrt{\lambda + \delta}). \tag{1.2}$$

Our objective is to find out when the δ-variation is non-negative and to obtain lower bounds for $V(\lambda, \delta)$ for small δ and large λ. Let us first review the

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results known in the literature. Define the functions $\mathcal{R}^{\pm}(r)$ by the formula

$$\mathcal{N}^{\pm}(r) = \frac{\mathrm{w}_d}{\mu_\Gamma}\, r^d + \mathcal{R}^{\pm}(r), \tag{1.3}$$

where w_d is the volume of the unit ball in $\mathbb{R}^d$, and μ_Γ is the volume of the fundamental domain $\mathbb{R}^d/\Gamma$. It is clear that $\mathcal{R}^{\pm}(r) = o(r^d)$ as $r \to \infty$; more precise estimates will be stated later on.

Suppose that $\delta = o(\lambda)$ as $\lambda \to \infty$. Substituting (1.3) into (1.2), we get

$$\begin{aligned} V(\lambda, \delta) &\\ = -\frac{d\,\mathrm{w}_d}{\mu_\Gamma}\, \delta\lambda^{(d-2)/2} &+ O(\delta^2\lambda^{(d-4)/2}) + \mathcal{R}^+(\sqrt{\lambda-\delta}) - \mathcal{R}^-(\sqrt{\lambda+\delta}). \end{aligned} \tag{1.4}$$

Clearly the r.h.s. is non-negative if the difference $\mathcal{R}^+ - \mathcal{R}^-$ is large in comparison with the other two terms. The next proposition provides appropriate lower bounds for $\mathcal{R}^{\pm}$.

Proposition 1.2. *For an arbitrary lattice Γ and $r \geq 1$ we have the bounds*

$$\mathcal{R}^+(r) > r^{(d-1)/2}\varphi(r), \qquad \mathcal{R}^-(r) < -r^{(d-1)/2}\varphi(r), \tag{1.5}$$

with

$$\varphi(r) = \begin{cases} c_\Gamma & \text{if } d \neq 1 \pmod 4, \\ c_\Gamma \exp(-a_\Gamma (\ln\ln r)^4) & \text{if } d = 1 \pmod 4. \end{cases} \tag{1.6}$$

Here c_Γ and a_Γ are some positive constants independent of r.

As in the above proposition, throughout the paper we denote by C or c, with or without indices, various positive constants whose value is of no importance.

For $d \neq 1 \pmod 4$ the bounds (1.5) were established in [1] (see also [3], [6]). The more delicate case $d = 1 \pmod 4$ was handled first in [11], where (1.5) was proved for $\varphi(r) = r^{-\varepsilon}$ with an arbitrarily small $\varepsilon > 0$. The improved φ in (1.6) was obtained in [9].

Using Proposition 1.2 and the relation (1.4) we arrive at the following result.

Corollary 1.3. *Let $\Gamma \subset \mathbb{R}^d$ with $d \geq 2$ be an arbitrary lattice. Then for all sufficiently large $\lambda \geq \lambda_0(\Gamma)$ and all $\delta \in [0, \delta_0(\lambda)]$, where*

$$\delta_0(\lambda) = \frac{\mu}{2d\,\mathrm{w}_d}\, \lambda^{(3-d)/4}\varphi(\sqrt{\lambda}),$$

we have the bound

$$V(\lambda, \delta) > \lambda^{(d-1)/4}\varphi(\sqrt{\lambda}). \tag{1.7}$$

The function $\varphi(\cdot)$ is given by (1.6).

For $d = 2, 3$ the bounds (1.5) are consistent with the natural conjecture that

$$\mathcal{R}^{\pm}(r) = O(r^{1/2+\varepsilon}), \quad d = 2, \quad \text{and} \quad \mathcal{R}^{\pm}(r) = O(r^{1+\varepsilon}), \quad d = 3,$$

with an arbitrarily small $\varepsilon > 0$. When one increases the dimension, one begins to observe the dependence of the error terms $\mathcal{R}^{\pm}(r)$, and hence of the variation $V(\lambda, \delta)$, on the arithmetic properties of the lattice Γ.

DEFINITION 1.4. A lattice $\Gamma \subset \mathbb{R}^d$ is said to be *rational* if for any two vectors $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \in \Gamma$ their inner product satisfies the relation

$$\langle \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \rangle = \beta_\Gamma r_{12}, \tag{1.8}$$

where $\beta_\Gamma \neq 0$ is a real-valued constant independent of $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2$, and $r_{12} = r_{21}$ is an integer. Otherwise the lattice is called *irrational.*

It is clear that in order to check the rationality of Γ it suffices to verify (1.8) only for the basis vectors of Γ. Without loss of generality one may assume that $\beta_\Gamma = 1$, since rescaling the lattice will affect $\mathcal{R}^{\pm}$ and $V(\lambda, \delta)$ in the obvious (controllable) way.

It was proved in [4] for $d \geq 5$ that

$$\mathcal{R}^{\pm}(r) = O(r^{d-2}), \tag{1.9}$$

and that the remainder can be replaced with $o(r^{d-2})$ if and only if the lattice Γ is irrational. Furthermore, in [7] examples of irrational lattices $\Gamma \subset \mathbb{R}^d$ with $d \geq 5$ were constructed for which

$$\mathcal{N}[B(r; \mathbf{0})] = \frac{\mathrm{w}_d}{\mu_\Gamma} r^d + O(r^{d/2+\varepsilon}), \qquad r \to \infty,$$

with an arbitrarily small $\varepsilon > 0$. This suggests that (1.7) is quite sharp for irrational lattices. Our aim is to obtain sharp lower bounds for $V(\lambda, \delta)$ in the case of *rational* lattices. The main results of the paper are contained in Theorems 1.5–1.7.

THEOREM 1.5. *Let $\Gamma \subset \mathbb{R}^d$ be a rational lattice and let $d \geq 5$. Then there are three positive constants $\delta_0 = \delta_0(\Gamma)$, $\lambda_0 = \lambda_0(\Gamma)$ and c_Γ such that for all $\delta \in [0, \delta_0]$ and all $\lambda \geq \lambda_0$, we have*

$$V(\lambda, \delta) \geq c_\Gamma \lambda^{(d-2)/2}. \tag{1.10}$$

Comparing (1.10) with the bound (1.9) we see that (1.10) is sharp. The next theorem deals with the four-dimensional case:

THEOREM 1.6. *Let $\Gamma \subset \mathbb{R}^4$ be a rational lattice. Then there are three positive constants $\delta_0 = \delta_0(\Gamma)$, $\lambda_0 = \lambda_0(\Gamma)$ and c_Γ such that for all $\delta \in [0, \delta_0]$ and all $\lambda \geq \lambda_0$, we have*

$$V(\lambda, \delta) \geq c_\Gamma \lambda (\ln \ln \lambda)^{-1}. \tag{1.11}$$

It is not yet clear whether one can get rid of the ln ln factor in (1.11) for general rational lattices. However, for the case of a cubic lattice Γ, this can be done:

THEOREM 1.7. *Let* $\Gamma = \mathbb{Z}^4$. *Then for each* $\delta \in [0, 2^{-15}]$, *all sufficiently large* $\lambda \geq \lambda_0 > 0$ *and some* $c > 0$ *one has the bound*

$$V(\lambda, \delta) > c\lambda. \tag{1.12}$$

The above theorems are proved simultaneously in a single proof. A more specific argument allows one to prove the lower bound (1.12) for all $\delta \in [0, \delta_0]$ with $\delta_0 = 40^{-1}$ instead of 2^{-15}. However, we are not concerned with a possible optimisation of the constants.

The authors' interest in the quantity $V(\lambda, \delta)$ comes from the link with the spectral theory of periodic operators. The lower bounds for the δ-variation allow one to justify the *Bethe–Sommerfeld conjecture* for the periodic Schrödinger operator, that is, to prove that the number of spectral gaps is finite (see [3], [15], [16], [11], [12] and references therein). More precisely, Proposition 1.2 is instrumental in the proof of the conjecture in dimensions $d = 2, 3, 4$ for arbitrary lattices, whereas Theorems 1.5–1.7 can be used to handle the rational lattices in dimensions $d \geq 4$.

Some facts on the lattice points counting, close in spirit to the Main Theorems in the present paper, were found in [15]. In fact, our proofs rely on the idea put forward in [15]: they use the classical results on representation of integers by integer quadratic forms and some arguments from the geometry of numbers.

The paper is organised as follows. The necessary facts about integer quadratic forms are collected in Sect. 2. In Sect. 3 these are used to study lattice points on spheres. Spherical shells containing no lattice points (the *empty shells*) are described in Sect. 4. These empty shells are crucial for the derivation of the Main Theorems 1.5–1.7 from the Key Lemma 4.3. Finally, the proof of the Key Lemma is given in Sect. 5.

The reader will notice that some facts about integer forms and geometry of lattices are discussed in more detail than might be necessary for a number-theoretic audience. The reason is that the paper is addressed not only to number theorists, but also to analysts interested in applications of number theory to spectral problems.

2. Integer quadratic forms

2.1. *Representation of integers by quadratic forms.* Rational lattices are closely related to integer quadratic forms. Let $\Gamma \subset \mathbb{R}^d$ with $d \geq 2$ be a rational lattice. Recall that we assume without loss of generality that $\beta_\Gamma = 1$ in (1.8). Let $\gamma_1, \dots, \gamma_d$ be a basis of the lattice Γ. Representing each vector

$\boldsymbol{\gamma} \in \Gamma$ in the form $\boldsymbol{\gamma} = \sum_{j=1}^{d} x_j \boldsymbol{\gamma}_j$ with $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{Z}^d$, we introduce the quadratic form

$$f(\mathbf{x}) = |\boldsymbol{\gamma}|^2 = \sum_{j,l=1}^{d} f_{jl} x_j x_l, \qquad f_{jl} = \langle \boldsymbol{\gamma}_j, \boldsymbol{\gamma}_l \rangle,\ j,l = 1, \ldots, d. \tag{2.1}$$

By (1.8) the coefficients f_{jl} are integer numbers, and the determinant

$$D = \det\{f_{jl}\}$$

is a positive integer.

Let us recall some results on the representation of integers by the form (2.1). For details we refer to [8], [17], [14] and [10]. Here we follow mainly [10]. A (positive) integer M is said to be *representable by the form* (2.1) if for some $\mathbf{x} \in \mathbb{Z}^d$ one has $f(\mathbf{x}) = M$. Denote by

$$R(M) = R_f(M) = \#\{\mathbf{x} \in \mathbb{Z}^d : f(\mathbf{x}) = M\} \tag{2.2}$$

the number of representations of M by the form (2.1). For $d \geq 4$ the Hardy–Littlewood method (see [8], [10], [13]) gives the formula

$$R(M) = \frac{(2\pi)^{d/2}}{D^{1/2}\Gamma(d/2)} M^{d/2-1} \sigma(M) + O(M^{(d-1)/4+\varepsilon}) \tag{2.3}$$

as $M \to \infty$, where $\varepsilon > 0$ is arbitrarily small, and $\sigma(M)$ is the so-called *singular series*, which can be explicitly found in the following way. Write the canonical expansion of M as a product of primes,

$$M = \prod_p p^{\alpha(p)}, \tag{2.4}$$

and define

$$\lambda(2) = \alpha(2) + 3, \qquad \lambda(p) = \alpha(p) + 1, \qquad p > 2. \tag{2.5}$$

Now denote by $\nu(p)$ the number of solutions $\mathbf{x} \in \mathbb{Z}^d \pmod{p^{\lambda(p)}}$ of the following congruence:

$$f(\mathbf{x}) = M \pmod{p^{\lambda(p)}}. \tag{2.6}$$

Then the singular series $\sigma(M)$ is given by

$$\sigma(M) = \prod_p \chi(p), \qquad \chi(p) = p^{-(d-1)\lambda(p)} \nu(p).$$

Clearly, the formula (2.3) is non-trivial only if the number $\sigma(M)$ is not very small, that is, the first term in (2.3) is larger than the remainder. The following result gives a lower bound for $\sigma(M)$ (see [10, Chapter III, Theorem 2 and Remark 7]):

PROPOSITION 2.1. *Suppose that the congruence* (2.6) *is solvable for each prime p dividing $2D$. Then*

$$\sigma(M) > c \quad \text{if } d \geq 5, \tag{2.7}$$

$$\sigma(M) > c(\ln\ln M)^{-1}\Big(\prod_{p|2D} p^{\alpha(p)}\Big)^{-1} \quad \text{if } d = 4. \tag{2.8}$$

Here c denotes some positive constant independent of M.

In combination with (2.3), the above proposition shows that the first term on the r.h.s. of (2.3) is dominant for $d \geq 5$, and hence the solvability of the congruence (2.6) suffices for a large number M to be representable by the form f. On the contrary, for $d = 4$, in order to guarantee the representability of M one needs to assume also that M does not contain large powers of primes p dividing $2D$. In fact, an example shows (see [2, Chapter 11, beginning of Sect. 9]) that the latter condition is essential for quaternary forms.

We use Proposition 2.1 for the numbers M defined in the following special way. Choose any integer M_0 representable by the form f. For instance, one can take $M_0 = f_{jj}$ for some $j = 1, \dots, d$. Consider the arithmetic progression

$$M_s = M_0 + Ts, \qquad s = 0, 1, 2, \dots, \tag{2.9}$$

where

$$T = \prod_{p|2D} p^{\lambda(p)}, \tag{2.10}$$

$\lambda(p)$ being determined as in (2.5) from the representation (2.4) for M_0. The next lemma provides a lower bound for $R(M_s)$ which is more practically usable than Proposition 2.1.

LEMMA 2.2. *For all sufficiently large $s > s_0(f, M_0)$ the integers M_s in* (2.9) *are representable by the form f, and the number of representations satisfies the bound*

$$R(M_s) > cM_s^{(d-2)/2} \quad \text{if } d \geq 5, \tag{2.11}$$

$$R(M_s) > c(\ln\ln M_s)^{-1} M_s \quad \text{if } d = 4. \tag{2.12}$$

Here $c = c_f > 0$ is independent of s.

Proof. From the definitions (2.9) and (2.10) we conclude that

$$M_s = M_0 \pmod{p^{\lambda(p)}} \quad \text{for all } p \mid 2D.$$

Therefore for each $p \mid 2D$ the congruence (2.6) with $M = M_s$ is solvable, as M_0 is defined to be representable by f. Now the bound (2.11) follows from (2.7) and (2.3).

In order to prove (2.12), note that by the definitions (2.9) and (2.10) the number M_s has the representation

$$M_s = \prod_{p|2D} p^{\alpha(p)} M_s',$$

where M_s' and $2D$ are prime to one another. Since the exponents $\alpha(p)$ are taken from the representation (2.4) for M_0, they are independent of s. This implies that the product over $p \mid 2D$ in the bound (2.8) does not depend on s, and therefore (2.8) implies (2.12). ∎

For the sum of four squares the bound (2.12) can be improved. Let

$$f_0(\mathbf{x}) = x_1^2 + x_2^2 + x_3^2 + x_4^2, \tag{2.13}$$

so that $D = 1$. Take $M_0 = 1$, which is clearly representable by f_0. Then $T = 8$ by (2.10), and the progression (2.9) takes the form

$$M_s = 1 + 8s, \qquad s = 1, 2, \dots. \tag{2.14}$$

LEMMA 2.3. *The integers* (2.14) *are representable by the form* (2.13) *and the number $R_{f_0}(M)$ satisfies the bound*

$$R_{f_0}(M_s) > 8M_s. \tag{2.15}$$

Proof. By Jacobi's Theorem (see [5, Theorem 386]), for every positive integer M we have

$$R_{f_0}(M) = 8 \sum_{\substack{q|M \\ q \not\equiv 0 \ (\mathrm{mod}\, 4)}} q. \tag{2.16}$$

Since $M_s = 1 \pmod 4$ by assumption, this formula yields (2.15). ∎

Let us rewrite the estimates obtained in Lemmas 2.2 and 2.3 as one formula, using the notation

$$\psi_d(t) = \begin{cases} t^{d-2}, & d \geq 5, \\ t^2(\ln\ln t)^{-1}, & d = 4, \\ t^2, & \Gamma = \mathbb{Z}^4, \end{cases} \tag{2.17}$$

with $t > 0$. Now we can put together the estimates (2.11), (2.12) and (2.15):

LEMMA 2.4. *Let $d \geq 4$, and let M_s be given either by* (2.9) *or* (2.14) *(for $\Gamma = \mathbb{Z}^4$). Then for sufficiently large $s > s_0(f, M_0)$ the numbers M_s are representable by the form f and*

$$R(M_s) > c\psi_d(\sqrt{M_s}), \tag{2.18}$$

with a constant $c = c(\Gamma, f) > 0$ independent of s.

2.2. *Special sequences of integers.* In this subsection we derive a version of Lemma 2.4 for classes of lattice points with fixed residues. Consider the following arithmetic progressions labelled by $q = 1, 2, \dots$:

$$M_s^{(q)} = M_0 + 2sT^q, \qquad s = 0, 1, \dots. \tag{2.19}$$

Obviously, for each q the numbers $M_s^{(q)}$ form a subprogression of (2.9), and hence the estimate (2.18) holds for sufficiently large $M_s^{(q)}$. In order to proceed we need the following remark.

REMARK 2.5. Suppose that

$$\mathbf{x} = \mathbf{y} \pmod{A}$$

with A even. Then

$$f(\mathbf{x}) = f(\mathbf{y}) \pmod{2A}.$$

Indeed, let $\phi(\mathbf{a}, \mathbf{b}) = \langle \mathbf{a}, \mathbf{b} \rangle$ be the symmetric bilinear form associated with the quadratic form f, so that $\phi(\mathbf{a}, \mathbf{a}) = f(\mathbf{a})$. For the vector $\mathbf{x} = \mathbf{y} + A\mathbf{z}$ we get

$$f(\mathbf{x}) = f(\mathbf{y}) + 2A\phi(\mathbf{y}, \mathbf{z}) + A^2 f(\mathbf{z}).$$

As A is even we have $2A \mid A^2$, whence the claim.

We now introduce classes of lattice points with fixed residues, important for our argument in what follows. Denote by $\mathcal{F}^{(q)}$ the set of all solutions $\mathbf{y} \in \mathbb{Z}^d \cap [0, T^q)^d$ of the congruence

$$f(\mathbf{y}) = M_0 \pmod{2T^q}. \tag{2.20}$$

Since the number T, defined in (2.10), is even, by Remark 2.5 one can conclude that any solution $\mathbf{x} \in \mathbb{Z}^d$ of the above congruence has the form $\mathbf{x} = \mathbf{y} \pmod{T^q}$ with some $\mathbf{y} \in \mathcal{F}^{(q)}$. In particular, this implies that the set $\mathcal{F}^{(q)}$ is not empty, as M_0 is assumed to be representable by the form f. Denote by $R_f^{(q)}(M_s^{(q)}, \mathbf{y}), \mathbf{y} \in \mathcal{F}^{(q)}$, the number of vectors $\mathbf{x} \in \mathbb{Z}^d$ such that

$$f(\mathbf{x}) = M_s^{(q)} \quad \text{and} \quad \mathbf{x} = \mathbf{y} \pmod{T^q}. \tag{2.21}$$

Observe that

$$R_f(M_s^{(q)}) = \sum_{\mathbf{y} \in \mathcal{F}^{(q)}} R_f^{(q)}(M_s^{(q)}, \mathbf{y}). \tag{2.22}$$

LEMMA 2.6. *Let $d \geq 4$. For all sufficiently large $s > s_0(f, M_0)$, there exists a vector $\mathbf{y}_s^{(q)} \in \mathcal{F}^{(q)}$ such that*

$$R_f^{(q)}(M_s^{(q)}, \mathbf{y}_s^{(q)}) > c\psi_d(\sqrt{M_s^{(q)}}).$$

Here $c = c(\Gamma, f, q) > 0$ is independent of s.

Proof. It follows from (2.22) by the Dirichlet principle that there is at least one $\mathbf{y} = \mathbf{y}_s^{(q)} \in \mathcal{F}^{(q)}$ such that

$$R_f^{(q)}(M_s^{(q)}, \mathbf{y}) \ge \frac{1}{T^{qd}} R_f(M_s^{(q)}),$$

since the number of lattice points $\mathbf{y} \in \mathbb{Z}^d \cap [0, T^q)^d$ is precisely T^{qd}. This number is independent of s. Now the proclaimed result follows from the bound (2.18). ∎

3. Geometry of rational lattices. Now we apply the results of the previous section to study the distribution of lattice points on spheres.

Let, as before, $\Gamma \subset \mathbb{R}^d$ with $d \ge 4$ be a rational lattice, and f be the quadratic form (2.1). We assume that a basis $\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_d$ of the lattice Γ is fixed. For the case $\Gamma = \mathbb{Z}^4$ we take the standard orthonormal basis. Denote by

$$S(r; \mathbf{k}) = \{\boldsymbol{\xi} : |\boldsymbol{\xi} - \mathbf{k}| = r\},$$

the sphere of radius $r > 0$ centred at $\mathbf{k} \in \mathbb{R}^d$. The radius r is said to be *admissible* for $\mathbf{k}$ if $S(r; \mathbf{k})$ contains at least one point $\boldsymbol{\gamma} \in \Gamma$. Define a special collection $\mathcal{Q}^{(q)}, q = 1, 2, \dots,$ of centres $\mathbf{k}$ by the formula

$$\mathbf{k} \in \mathcal{Q}^{(q)} \quad \text{if and only if} \quad \mathbf{k} = \frac{1}{T^q} \sum_{j=1}^{d} y_j \boldsymbol{\gamma}_j, \ \mathbf{y} = (y_1, \dots, y_d) \in \mathcal{F}^{(q)}. \tag{3.1}$$

Since $\mathcal{F}^{(q)} \subset [0, T^q)^d$, the set $\mathcal{Q}^{(q)}$ is contained entirely in the fundamental parallelepiped spanned by the basis vectors of the lattice Γ. For the elements $\mathbf{k}$ of $\mathcal{Q}^{(q)}$ one can describe the admissible radii:

LEMMA 3.1. *Consider the sequence of positive numbers $r_s^{(q)} > 0$, $s = 0, 1, \dots,$ defined by the relation*

$$(r_s^{(q)})^2 = \frac{M_s^{(q)}}{T^{2q}}, \tag{3.2}$$

where $M_s^{(q)}$ are the integers introduced in (2.19). *Then*

(i) *The admissible radii for each $\mathbf{k} \in \mathcal{Q}^{(q)}$ form a subsequence of* (3.2).

(ii) *For each sufficiently large integer $M_s^{(q)}$ in the progression* (2.19) *there exists $\mathbf{k}_s^{(q)} \in \mathcal{Q}^{(q)}$ such that $r_s^{(q)}$ is an admissible radius for $\mathbf{k}_s^{(q)}$ and*

$$\mathcal{N}[S(r_s^{(q)}; \mathbf{k}_s^{(q)})] > c\psi_d(r_s^{(q)}). \tag{3.3}$$

Here $c = c(\Gamma, q)$ is independent of s.

Proof. (i) By definition (3.1), for any

$$\mathbf{k} = \frac{1}{T^q}\sum_{j=1}^d y_j\boldsymbol{\gamma}_j \in \mathcal{Q}^{(q)},\ \mathbf{y}\in\mathcal{F}^{(q)}, \quad \text{and} \quad \boldsymbol{\gamma} = -\sum_{j=1}^d z_j\boldsymbol{\gamma}_j \in \Gamma,\ \mathbf{z}\in\mathbb{Z}^d,$$

we have

$$|\boldsymbol{\gamma}-\mathbf{k}|^2 = \frac{1}{T^{2q}} f(\mathbf{y}+T^q\mathbf{z}). \tag{3.4}$$

Thus by (2.19) and (2.20) one can conclude that

$$|\boldsymbol{\gamma}-\mathbf{k}|^2 = \frac{M_s^{(q)}}{T^{2q}}$$

with some $M_s^{(q)}$ from the progression (2.19).

(ii) Now suppose that $M_s^{(q)}$ is sufficiently large. Then by Lemma 2.6 the system (2.21) is solvable with some $\mathbf{x}\in\mathbb{Z}^d$ and $\mathbf{y}_s^{(q)}\in\mathcal{F}^{(q)}$. Therefore, for some $\mathbf{z}\in\mathbb{Z}^d$ we have

$$\frac{M_s^{(q)}}{T^{2q}} = \frac{1}{T^{2q}} f(\mathbf{y}_s^{(q)}+T^q\mathbf{z}) = |\boldsymbol{\gamma}-\mathbf{k}_s^{(q)}|^2, \tag{3.5}$$

where $\mathbf{k}_s^{(q)}$, $\mathbf{y}_s^{(q)}$ and $\boldsymbol{\gamma}$, $\mathbf{z}$ are related as in (3.4). The relation (3.5) and the definition of the number $R_f^{(q)}(M_s^{(q)},\mathbf{y})$ (see (2.21)) imply that

$$\mathcal{N}[S(r_s^{(q)};\mathbf{k}_s^{(q)})] = R_f^{(q)}(M_s^{(q)},\mathbf{y}_s^{(q)}).$$

Now the proclaimed lower bounds follow from Lemma 2.6. ∎

Observe an elementary property of the radii $r_s^{(q)}$, which follows from (3.2) and (2.19):

$$(r_{s+1}^{(q)})^2 - (r_s^{(q)})^2 = \frac{2}{T^q}. \tag{3.6}$$

3.1. *Spherical caps.* Let $r>0$, $\theta\in[0,\pi/2]$, $\mathbf{k}\in\mathbb{R}^d$ and $\mathbf{e}\in\mathbb{R}^d$, $|\mathbf{e}|=1$. The set defined by the formula

$$K(r,\theta;\mathbf{k},\mathbf{e}) = \{\boldsymbol{\xi}\in S(r;\mathbf{k}) : \langle\boldsymbol{\xi}-\mathbf{k},\mathbf{e}\rangle > r\cos\theta\} \tag{3.7}$$

is called a *spherical cap of radius* r *and angle* θ, *centred at* $\mathbf{k}\in\mathbb{R}^d$. The unit vector $\mathbf{e}$ determines the *orientation* of the spherical cap.

Consider the spherical caps $K(r_s^{(q)},\theta;\mathbf{k}_s^{(q)},\mathbf{e})$, where the radius $r_s^{(q)}$ is defined in (3.2) and the point $\mathbf{k}_s^{(q)}$ is chosen as in Lemma 3.1, so that $r_s^{(q)}$ is an admissible radius for $\mathbf{k}_s^{(q)}$ and the bound (3.3) holds.

LEMMA 3.2. *For all sufficiently large* $s>s_0$ *there exists a unit vector* $\mathbf{e}_s$ *such that*

$$\mathcal{N}[K(r_s^{(q)},\theta;\mathbf{k}_s^{(q)},\mathbf{e}_s)] > c\psi_d(r_s^{(q)}). \tag{3.8}$$

Here $c=C(\Gamma,q,\theta)$ *is positive for all* $\theta\in(0,\pi/2]$, *and it is independent of* s.

Proof. Consider an arbitrary finite covering of the unit sphere $S(1;\mathbf{k})$ by spherical caps of angle θ:

$$S(1;\mathbf{k}) = \bigcup_{j=1}^{n(\theta)} K(1,\theta;\mathbf{k},\mathbf{e}_j), \quad \text{so that} \quad S(r;\mathbf{k}) = \bigcup_{j=1}^{n(\theta)} K(r,\theta;\mathbf{k},\mathbf{e}_j),$$

with the same number $n(\theta)$ of spherical caps. By the Dirichlet principle we conclude that there exists at least one spherical cap in the above covering such that

$$\mathcal{N}[K(r_s^{(q)},\theta;\mathbf{k}_s^{(q)},\mathbf{e})] \geq \frac{1}{n(\theta)}\,\mathcal{N}[S(r_s^{(q)};\mathbf{k}_s^{(q)})].$$

It remains to use Lemma 3.1(ii) and relabel the vector $\mathbf{e}$ as $\mathbf{e}_s$. ∎

Certainly, the bound (3.8) can be deduced from the equidistribution of lattice points on the spheres $S(r_s^{(q)};\mathbf{k}_s^{(q)})$ (see [10], [13]). One can even use a more advanced result from [10] which states the above equidistribution for spheres centred at arbitrary rational points (with respect to the lattice Γ). On the other hand, our strategy is to use the most elementary number-theoretic information available, and hence we content ourselves with a more elementary proof.

4. Empty spherical shells

4.1. *Empty shells.* Denote by

$$L(r_1,r_2;\mathbf{k}) = \{\boldsymbol{\xi} : r_1 < |\boldsymbol{\xi}-\mathbf{k}| < r_2\}, \quad 0 < r_1 < r_2,$$

the open spherical shell with radii r_1, r_2, centred at $\mathbf{k}$. The boundary of $L(r_1,r_2;\mathbf{k})$ consists of two spheres $S(r_1;\mathbf{k})$ and $S(r_2;\mathbf{k})$. We call them the interior and exterior boundary of the shell respectively. We call the shell $L(r_1,r_2;\mathbf{k})$ *empty* if it does not contain any points of the lattice Γ. For a given rational lattice Γ introduce the following spherical shells:

$$L_{1,s}^{(q)} = L(r_s^{(q)}, r_{s+1}^{(q)};\mathbf{k}_s^{(q)}), \quad L_{2,s}^{(q)} = L(r_s^{(q)}, r_{s+1}^{(q)};\mathbf{k}_{s+1}^{(q)}), \tag{4.1}$$

where the radii $r_s^{(q)}$ are defined by (3.2) and the centres $\mathbf{k}_s^{(q)}$ are defined as in Lemma 3.1(ii). By Lemma 3.1 the shells (4.1) are empty. Furthermore, by the same lemma the shell $L_{1,s}^{(q)}$ has lattice points on its interior boundary, whereas $L_{2,s}^{(q)}$ has lattice points on its exterior boundary.

Now define the following closed intervals associated with the shells (4.1):

$$\begin{aligned} \Delta_{q,s}^{-} &= \left[(r_s^{(q)})^2,\ \frac{(r_s^{(q)})^2 + (r_{s+1}^{(q)})^2}{2}\right], \\ \Delta_{q,s}^{+} &= \left[\frac{(r_s^{(q)})^2 + (r_{s+1}^{(q)})^2}{2},\ (r_{s+1}^{(q)})^2\right]. \end{aligned} \tag{4.2}$$

One can immediately state a few obvious properties of these intervals. First,

$$\Delta^-_{q,s} \cup \Delta^+_{q,s} = [(r_s^{(q)})^2, (r_{s+1}^{(q)})^2], \qquad \bigcup_{s\ge 0} (\Delta^-_{q,s} \cup \Delta^+_{q,s}) = [(r_0^{(q)})^2, \infty), \tag{4.3}$$

for each $q \ge 1$. Furthermore, in view of (3.6) the intervals (4.2) have constant length T^{-q}. The intervals with different q may overlap. In order to characterise this overlap, we introduce the function which is naturally called the *overlap length*.

Let $\mathfrak{M} = \{\Delta_l\}$, $l \ge 1, 2, \dots$, be a collection (finite or infinite) of closed intervals such that for each $R > 0$ only finitely many intersections $\Delta_l \cap [-R, R]$ are not empty. For each $\lambda \in \mathbb{R}$ define

$$Z(\lambda) = \max_l \max\{t : [\lambda - t, \lambda + t] \subset \Delta_l\}.$$

The function $2Z(\lambda)$ gives the length of the maximal closed interval centred at λ which fits in one of the closed intervals Δ_l. In other words, the inequality $Z(\lambda) > 0$ means that there is a $\Delta_l \in \mathfrak{M}$ such that $[\lambda - Z(\lambda), \lambda + Z(\lambda)] \subset \Delta_l$. Clearly, the function $Z(\lambda)$ is continuous and piecewise linear. If $Z(\lambda)$ is positive on a subset $E \subset \mathbb{R}$, then the intervals Δ_l entirely cover E. To indicate the dependence of $Z(\lambda)$ on the collection $\mathfrak{M} = \{\Delta_l\}$ we use the notation $Z(\lambda; \mathfrak{M})$.

Lemma 4.1. *Let $q > 1$ be an integer such that*

$$T^{q-1} > 4M_0. \tag{4.4}$$

Consider the following collection of closed intervals:

$$\mathfrak{M} = \{\Delta^-_{q,s}, \Delta^+_{q,s}, \Delta^-_{q+1,s}, \Delta^+_{q+1,s} : s = 1, 2, \dots\}. \tag{4.5}$$

Then for all $\lambda \ge 1$ we have

$$Z(\lambda; \mathfrak{M}) \ge \zeta \quad \textit{with} \quad \zeta = \frac{M_0(T^2 - 1)}{2T^{2q+2}}.$$

In particular, $\zeta > 2^{-2-6q}$ if $\Gamma = \mathbb{Z}^4$.

Proof. By definitions (3.2) and (2.19) the endpoints of the intervals $\Delta^\pm_{q,s}$ and $\Delta^\pm_{q+1,s}$ are of the form

$$\frac{M_0}{T^{2q}} + \frac{m}{T^q}, \quad m \in \mathbb{N} \cup \{0\}, \quad \text{and} \quad \frac{M_0}{T^{2(q+1)}} + \frac{n}{T^{q+1}}, \quad n \in \mathbb{N} \cup \{0\},$$

respectively. By the condition (4.4),

$$0 < \frac{M_0}{T^{2q+2}} < \frac{M_0}{T^{2q}} \le \frac{1}{4T^{q+1}}.$$

Consequently,

$$2Z(\lambda) \ge 2\zeta = \frac{M_0}{T^{2q}} - \frac{M_0}{T^{2q+2}} > 0.$$

To prove the estimate for $\Gamma = \mathbb{Z}^4$ one recalls (see the line before (2.14)) that in this case $M_0 = 1$ and $T = 8$. ∎

LEMMA 4.2.

(i) *If λ is an interior point of the closed interval $\varDelta^-_{q,s} \cup \varDelta^+_{q,s}$, then $S(\sqrt{\lambda}; \mathbf{k}_s^{(q)}) \subset L_{1,s}^{(q)}$ and $S(\sqrt{\lambda}; \mathbf{k}_{s+1}^{(q)}) \subset L_{2,s}^{(q)}$.*

(ii) *Let q satisfy* (4.4), *and let $0 \le \delta < \zeta$. Then for all sufficiently large λ one can find an $s = 1, 2, \ldots$ and a $j = q, q+1$ such that the empty shell $L_{1,s}^{(j)}$ contains the closure of the shell $L(\sqrt{\lambda - \delta}, \sqrt{\lambda + \delta}; \mathbf{k}_s^{(j)})$, and the empty shell $L_{2,s}^{(j)}$ contains the closure of the shell $L(\sqrt{\lambda - \delta}, \sqrt{\lambda + \delta}; \mathbf{k}_{s+1}^{(j)})$. Moreover,*

$$\mathcal{N}[B(\sqrt{\lambda - \delta}; \mathbf{k}_s^{(j)})] = \mathcal{N}[B(\sqrt{\lambda + \delta}; \mathbf{k}_s^{(j)})] = \mathcal{N}[B(r_{s+1}^{(j)}; \mathbf{k}_s^{(j)})], \tag{4.6}$$

$$\mathcal{N}[B(\sqrt{\lambda - \delta}; \mathbf{k}_{s+1}^{(j)})] = \mathcal{N}[B(\sqrt{\lambda + \delta}; \mathbf{k}_{s+1}^{(j)})] = \mathcal{N}[B(r_{s+1}^{(j)}; \mathbf{k}_{s+1}^{(j)})]. \tag{4.7}$$

Proof. Part (i) follows directly from the definitions (4.2) (see also (4.3)).

(ii) By Lemma 4.1, for $\lambda \ge 1$ one can find an interval $\varDelta$ from the collection (4.5) such that $[\lambda - \delta, \lambda + \delta] \subset \varDelta$. Let $\varDelta = \varDelta^+_{j,s}$ or $\varDelta^-_{j,s}$ with some $s = 1, 2, \ldots$ and $j = q$ or $q + 1$. Then by (i) the closures of the shells $L(\sqrt{\lambda - \delta}, \sqrt{\lambda + \delta}; \mathbf{k}_s^{(j)})$ and $L(\sqrt{\lambda - \delta}, \sqrt{\lambda + \delta}; \mathbf{k}_{s+1}^{(j)})$ belong to the shells $L_{1,s}^{(j)}$ and $L_{2,s}^{(j)}$ respectively. Since these shells are empty, the relations (4.6) and (4.7) follow at once. ∎

4.2. *The Key Lemma. Proof of the Main Theorems.* Recall that the numbers $\psi_d(t)$ and $\mathcal{N}^{\pm}$ are defined by (2.17) and (1.1) respectively, and ζ is defined in Lemma 4.1.

LEMMA 4.3. *Let $\Gamma \subset \mathbb{R}^d$ be a rational lattice*, *and let q be such that* (4.4) *is satisfied. Let ζ be the positive constant defined in Lemma* 4.1. *Suppose that $j = q$ or $q + 1$. Then for all sufficiently large $r > r_0(\Gamma)$ the following bounds hold.*

(i) *If for some $s \ge 1$ and $j = q$ or $q + 1$,*

$$[r^2, r^2 + \zeta/100] \subset \varDelta^-_{j,s}, \tag{4.8}$$

then

$$\mathcal{N}[B(r; \mathbf{k}_s^{(j)})] - \mathcal{N}^-(r) \ge c_\Gamma \psi_d(r_s^{(j)}). \tag{4.9}$$

(ii) *If for some $s \ge 1$ and $j = q$ or $q + 1$,*

$$[r^2 - \zeta/100, r^2] \subset \varDelta^+_{j,s}, \tag{4.10}$$

then

$$\mathcal{N}^+(r) - \mathcal{N}[B(r; \mathbf{k}_{s+1}^{(j)})] \ge c_\Gamma \psi_d(r_{s+1}^{(j)}). \tag{4.11}$$

Let us now deduce the Main Theorems from the above Key Lemma:

Proof of Theorems 1.5–1.7. By Lemma 4.1 for all sufficiently large λ there is an interval $\mathit{\Delta}$ in the collection (4.5) such that $[\lambda-\zeta, \lambda+\zeta] \subset \mathit{\Delta}$. Setting

$$\delta_0 = \frac{99}{100}\,\zeta, \tag{4.12}$$

we see that for every $\mu \in [\lambda-\delta, \lambda+\delta]$ with $0 \le \delta \le \delta_0$,

$$[\mu - \zeta/100, \mu + \zeta/100] \subset [\lambda-\delta, \lambda+\delta] \subset \mathit{\Delta},$$

which means that $r = \sqrt{\mu}$ satisfies at least one of the conditions (4.8) or (4.10) with suitable $j = q, q+1$ and $s = 1, 2, \dots$.

Suppose that $\mathit{\Delta} = \mathit{\Delta}^-_{j,s}$, i.e. that (4.8) holds. Then, using (4.9) with $r^2 = \lambda+\delta$, $0 \le \delta \le \delta_0$ and (4.6), we find that

$$\begin{aligned} \mathcal{N}^+(\sqrt{\lambda-\delta}) - \mathcal{N}^-(\sqrt{\lambda+\delta}) &\ge \mathcal{N}[B(\sqrt{\lambda-\delta}; \mathbf{k}_s^{(j)})] - \mathcal{N}^-(\sqrt{\lambda+\delta}) \\ &= \mathcal{N}[B(\sqrt{\lambda+\delta}; \mathbf{k}_s^{(j)})] - \mathcal{N}^-(\sqrt{\lambda+\delta}) \\ &> c_{\mathsf{\Gamma}}\psi_d(r_s^{(j)}). \end{aligned}$$

By definition (4.2) and (3.6) the number $\lambda \in \mathit{\Delta}^-_{j,s}$ satisfies the estimate $\lambda \ge (r_s^{(j)})^2 \ge \lambda - 1$, which leads to (1.10)–(1.12) in view of the definition (2.17).

Similarly, if $\mathit{\Delta} = \mathit{\Delta}^+_{j,s}$, i.e. if (4.10) holds, then using (4.11) with $r^2 = \lambda-\delta$ and (4.7), we find that

$$\begin{aligned} \mathcal{N}^+(\sqrt{\lambda-\delta}) - \mathcal{N}^-(\sqrt{\lambda+\delta}) &\ge \mathcal{N}^+(\sqrt{\lambda-\delta}) - \mathcal{N}[B(\sqrt{\lambda+\delta}; \mathbf{k}_{s+1}^{(j)})] \\ &= \mathcal{N}^+(\sqrt{\lambda-\delta}) - \mathcal{N}[B(\sqrt{\lambda-\delta}; \mathbf{k}_{s+1}^{(j)})] \\ &> c_{\mathsf{\Gamma}}\psi_d(r_{s+1}^{(j)}). \end{aligned}$$

By definition (4.2) and (3.6) the number $\lambda \in \mathit{\Delta}^+_{j,s}$ satisfies the estimate $\lambda + 1 \ge (r_{s+1}^{(j)})^2 \ge \lambda$, which again leads to (1.10)–(1.12) in view of (2.17).

To obtain the value $\delta_0 = 2^{-15}$ in Theorem 1.7 one notes that for $\mathsf{\Gamma} = \mathbb{Z}^4$ the relation (4.4) holds with $M_0 = 1, T = 8$ and $q = 2$, so that $\zeta > 2^{-14}$, and hence (4.12) yields the bound $\delta_0 > 2^{-15}$, which completes the proof if one redefines $\delta_0 = 2^{-15}$. ∎

5. Proof of Lemma 4.3

5.1. *Idea of the proof.* Our approach uses a simple idea which eventually reduces the problem to an elementary question of planar geometry. Suppose that a number r satisfies (4.8) and $r > r_s^{(j)}$. In order to prove (4.9) we find a point $\mathbf{k}'$ such that $\mathcal{N}[B(r, \mathbf{k}_s^{(j)})] - \mathcal{N}[B(r; \mathbf{k}')] > c\psi_d(r_s^{(j)})$.

The process of finding $\mathbf{k}'$ is based on a simple geometrical observation. If $\mathbf{k}' = \mathbf{k}_s^{(j)}$, then, by Lemma 4.2(i), the sphere $S(r; \mathbf{k}')$ lies in the empty shell $L_{1,s}^{(j)} = L(r_s^{(j)}, r_{s+1}^{(j)}; \mathbf{k}_s^{(j)})$ and hence the ball $B(r; \mathbf{k}')$ contains the same number of lattice points as the exterior ball $B(r_{s+1}^{(j)}; \mathbf{k}_s^{(j)})$. If one moves $\mathbf{k}'$ from $\mathbf{k}_s^{(j)}$, then the number of lattice points does not change, as long as the difference $\mathbf{k}' - \mathbf{k}_s^{(j)}$ remains sufficiently small. However, if one keeps moving $\mathbf{k}'$ until the sphere $S(r; \mathbf{k}')$ crosses the interior sphere $S(r_s^{(j)}; \mathbf{k}_s^{(j)})$, remaining entirely inside the exterior one, the ball $B(r; \mathbf{k}')$ does not acquire new lattice points, but loses some. For instance, the lattice points on the spherical cap cut off from $S(r_s^{(j)}; \mathbf{k}_s^{(j)})$ by $S(r; \mathbf{k}')$ are outside $B(r; \mathbf{k}')$. Thus the number $\mathcal{N}[B(r; \mathbf{k}')]$ decreases, and this drop can be estimated with the help of Lemma 3.2. Thus it remains to find a new suitable position of the centre, i.e. $\mathbf{k}'$.

The proof of (4.11) uses a similar idea. Namely, we seek a sphere $S(r; \mathbf{k}'')$ which encloses the interior boundary of the empty shell $L_{2,s}^{(j)}$ and crosses the exterior one. When the centre moves from $\mathbf{k}_{s+1}^{(j)}$ to $\mathbf{k}''$, the ball $B(r; \mathbf{k}'')$ does not lose any lattice points, but acquires the points $\boldsymbol{\gamma} \in \Gamma$ that lie on the spherical cap cut out from $S(r_{s+1}^{(j)}; \mathbf{k}_{s+1}^{(j)})$ by $S(r; \mathbf{k}'')$. The increase in the number of points is again estimated by Lemma 3.2.

5.2. *Two elementary geometrical problems.* In order to implement the simple idea described above, we need to solve two problems of elementary geometry concerned with arbitrary shells of the form $L(r_1, r_2; \mathbf{k})$ with $0 < r_1 < r_2$ in $\mathbb{R}^d$. The first of the following two lemmas provides the maximal radius r of a sphere $S(r; \mathbf{k}')$ which is placed inside the closure of the shell $L(r_1, r_2; \mathbf{k})$ in such a way that it cuts off a spherical cap of some specified angle θ from the interior sphere $S(r_1; \mathbf{k})$.

LEMMA 5.1. *For a given spherical shell $L(r_1, r_2; \mathbf{k})$ with $0 \le r_1 < r_2$ and for a given angle $\theta \in [0, \pi/2]$, let $r_- = r_-(r_1, r_2, \theta) > 0$ and $\mathbf{k}_- = \mathbf{k}_-(r_1, r_2, \theta) \in \mathbb{R}^d$ be such that $S(r_-; \mathbf{k}_-) \subset \overline{B(r_2; \mathbf{k})}$ and the sphere $S(r_-; \mathbf{k}_-)$ satisfies the following requirements*:

(1) *it touches the exterior sphere $S(r_2; \mathbf{k})$*,
(2) *$S(r_1; \mathbf{k}) \setminus \overline{B(r_-; \mathbf{k}_-)}$ is a spherical cap of angle θ centred at $\mathbf{k}$.*

Then

$$t_-(r_1, r_2, \theta) := |\mathbf{k} - \mathbf{k}_-| = \frac{r_2^2 - r_1^2}{2(r_1 \cos\theta + r_2)},$$

$$r_- = r_-(r_1, r_2, \theta) = r_2 - t_-(r_1, r_2, \theta). \tag{5.1}$$

The next lemma provides the minimal radius r of a sphere $S(r;\mathbf{k}'')$ which encloses the interior boundary $S(r_1;\mathbf{k})$ of the shell $L(r_1,r_2;\mathbf{k})$, and cuts out from the exterior boundary a spherical cap of a specified angle θ.

Lemma 5.2. *For a given spherical shell $L(r_1,r_2;\mathbf{k})$ with $0\le r_1<r_2$ and for a given angle $\theta\in[0,\pi/2]$, let $r_+=r_+(r_1,r_2,\theta)>0$ and $\mathbf{k}_+=\mathbf{k}_+(r_1,r_2,\theta)\in\mathbb{R}^d$ be such that $S(r_1;\mathbf{k})\subset\overline{B(r_+;\mathbf{k}_+)}$ and the sphere $S(r_+;\mathbf{k}_+)$ satisfies the following requirements*:

(1) *it touches the interior sphere $S(r_1;\mathbf{k})$*,
(2) *$S(r_2;\mathbf{k})\cap B(r_+;\mathbf{k}_+)$ is a spherical cap of angle θ centred at $\mathbf{k}$.*

Then

$$t_+(r_1,r_2,\theta):=|\mathbf{k}-\mathbf{k}_+|=\frac{r_2^2-r_1^2}{2(r_2\cos\theta+r_1)},$$

$$r_+=r_+(r_1,r_2,\theta)=r_1+t_+(r_1,r_2,\theta). \tag{5.2}$$

Remark 5.3. Due to the rotational symmetry of the problem, along with the sphere $S(r_\pm;\mathbf{k}_\pm)$, any sphere $S(r_\pm;\widetilde{\mathbf{k}})$ with $|\widetilde{\mathbf{k}}-\mathbf{k}|=|\mathbf{k}_\pm-\mathbf{k}|$ satisfies conditions (1) or (2) of Lemma 5.1 or 5.2 respectively. Thus we may assume that $\mathbf{k}_\pm-\mathbf{k}=|\mathbf{k}_\pm-\mathbf{k}|\mathbf{e}_1$, where $\mathbf{e}_1$ is the first vector of the canonical basis in $\mathbb{R}^d$.

Moreover, in view of axial symmetry, it suffices to prove Lemmas 5.1 and 5.2 for $d=2$ only.

Proof of Lemma 5.1. Referring to Remark 5.3 we consider only the case $d=2$. Furthermore, without loss of generality we may assume that $\mathbf{k}=0$ and $\mathbf{k}_-$ lies on the x-axis, so that $t_-=|\mathbf{k}_-|$. Let x,y be the coordinates of the point in the upper half plane where $S(r_-;\mathbf{k}_-)$ and $S(r_1;\mathbf{0})$ intersect (see Figure 1). Write the following simple identities:

$$\begin{cases} x^2+y^2=r_1^2,\\ (x-t_-)^2+y^2=r_-^2,\\ t_-+r_-=r_2,\\ x=-r_1\cos\theta. \end{cases}$$

From the first two relations we find that

$$-2t_-x+t_-^2=r_-^2-r_1^2.$$

From the third equality we get

$$r_-^2-r_1^2=(r_2-t_-)^2-r_1^2=r_2^2-2r_2t_-+t_-^2-r_1^2,$$

which together with the previous formulae implies that

$$-2xt_-=r_2^2-2r_2t_--r_1^2.$$

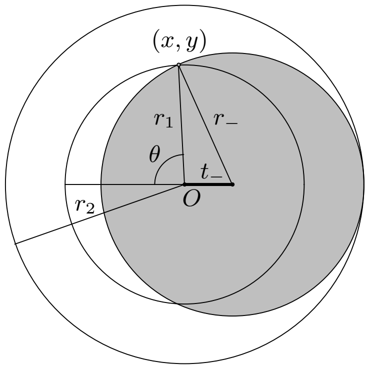

Fig. 1

Inserting x from the fourth relation, we get

$$2r_1 \cos\theta t_- + 2r_2 t_- = r_2^2 - r_1^2,$$

which leads to

$$t_- = \frac{r_2^2 - r_1^2}{2(r_1 \cos\theta + r_2)}.$$

The required formula for r_- follows from the third relation. ∎

Proof of Lemma 5.2. As in the previous proof, we consider only the case $d = 2$. We also assume that $\mathbf{k} = 0$ and $\mathbf{k}_+$ lies on the x-axis, so that $t_+ = |\mathbf{k}_+|$. Let x, y be the coordinates of the point in the upper half plane where $S(r_+; \mathbf{k}_+)$ and $S(r_2; \mathbf{0})$ intersect (see Figure 2). Write the following simple identities:

$$\begin{cases} x^2 + y^2 = r_2^2, \\ (x - t_+)^2 + y^2 = r_+^2, \\ r_+ - t_+ = r_1, \\ x = r_2 \cos\theta. \end{cases}$$

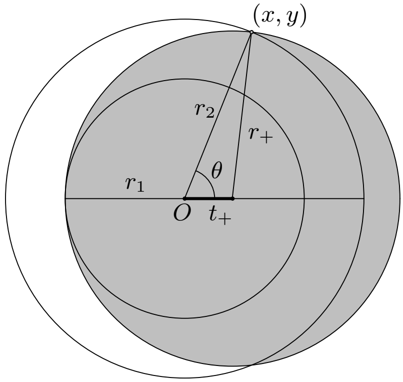

Fig. 2

From the first two relations we find that

$$-2t_+x + t_+^2 = r_+^2 - r_2^2.$$

From the third equality we get

$$r^2 - r_2^2 = (r_1 - t_+)^2 - r_2^2 = r_1^2 - 2r_1t_+ + t_+^2 - r_2^2,$$

which together with the previous formulae implies that

$$-2xt_+ = r_1^2 - 2r_1t_+ - r_2^2.$$

Inserting x from the fourth relation, we get

$$2r_2\cos\theta t_+ + 2r_1t_+ = r_2^2 - r_1^2,$$

which leads to

$$t_+ = \frac{r_2^2 - r_1^2}{2(r_2\cos\theta + r_1)}.$$

The required formula for r_+ follows from the third relation. ∎

Let us now investigate the behaviour of the radii $r_\pm$ as $r_1, r_2 \to \infty$:

Lemma 5.4. *Suppose that $r_1 \to \infty$ and*

$$\sup(r_2^2 - r_1^2) \le A < \infty, \tag{5.3}$$

with some $A > 0$. For a given $\omega \in (0, A)$ let $\theta_0 = \theta_0(A, \omega) \in (0, \pi/2)$ be the angle such that

$$\cos\theta_0 = \frac{A - \omega}{A + \omega}. \tag{5.4}$$

Then for all sufficiently large r_1 one has the inequalities

$$r_-^2(r_1, r_2, \theta_0) > \frac{r_1^2 + r_2^2}{2} - \omega, \qquad r_+^2(r_1, r_2, \theta_0) < \frac{r_1^2 + r_2^2}{2} + \omega. \tag{5.5}$$

Proof. The condition (5.3) implies that

$$r_2 = r_1 + O(r_1^{-1}), \qquad r_1 = r_2 + O(r_2^{-1}),$$

as $r_1 \to \infty$. Consequently, it follows from Lemmas 5.1 and 5.2 that

$$r_-(\theta) = r_2 - \frac{r_2^2 - r_1^2}{2r_2(\cos\theta + 1 + O(r_2^{-2}))} = r_2 - \frac{r_2^2 - r_1^2}{2r_2(\cos\theta + 1)} + O(r_2^{-3}),$$

$$r_+(\theta) = r_1 + \frac{r_2^2 - r_1^2}{2r_1(\cos\theta + 1 + O(r_1^{-2}))} = r_1 + \frac{r_2^2 - r_1^2}{2r_1(\cos\theta + 1)} + O(r_1^{-3}).$$

Here for the sake of brevity we omit the dependence on r_1, r_2 from the notation of $r_\pm$. Squaring these equalities we get

$$r_-^2(\theta) = r_2^2 - \frac{r_2^2 - r_1^2}{\cos\theta + 1} + O(r_2^{-2}),$$

$$r_+^2(\theta) = r_1^2 + \frac{r_2^2 - r_1^2}{\cos\theta + 1} + O(r_1^{-2}).$$

In particular,

$$r_-^2(0) = r_2^2 - \frac{r_2^2 - r_1^2}{2} + O(r_2^{-2}) = \frac{r_1^2 + r_2^2}{2} + O(r_2^{-2}),$$
$$r_+^2(0) = r_1^2 + \frac{r_2^2 - r_1^2}{2} + O(r_1^{-2}) = \frac{r_1^2 + r_2^2}{2} + O(r_1^{-2}).$$

Consider the differences:

$$r_-^2(0) - r_-^2(\theta) = (r_2^2 - r_1^2)g(\theta) + O(r_2^{-2}),$$
$$r_+^2(\theta) - r_+^2(0) = (r_2^2 - r_1^2)g(\theta) + O(r_1^{-2}),$$

with

$$g(\theta) = \frac{1}{\cos\theta + 1} - \frac{1}{2}.$$

Note that $g(\theta)$ varies from 0 to 1/2 for θ in $[0, \pi/2]$. Define the angle $\theta_0 = \theta_0(A, \omega)$ by the requirement

$$g(\theta_0) = \frac{\omega}{2A}.$$

This condition is equivalent to (5.4). Then

$$r_-^2(0) - r_-^2(\theta_0) \le \frac{\omega}{2} + O(r_2^{-2}), \qquad r_+^2(\theta_0) - r_+^2(0) \le \frac{\omega}{2} + O(r_1^{-2}),$$

and thus for sufficiently large r_1 the inequalities (5.5) hold. ∎

Relying on Lemma 5.4 we shall now find intervals of values r for which the sphere $S(r; \mathbf{k}')$ cuts out a spherical cap either from the interior or from the exterior boundary of the shell $L(r_1, r_2; \mathbf{k})$.

Lemma 5.5. *Let r_1 be sufficiently large and suppose the condition* (5.3) *holds with some $A > 0$. Suppose also that*

(5.6) $$r_2^2 - r_1^2 \ge 9\omega$$

for some $\omega \in (0, A/9)$. Let $\theta_0 \in (0, \pi/2)$ be the angle uniquely defined by (5.4).

(i) *If*

(5.7) $$r_1^2 \le r^2 \le \frac{r_1^2 + r_2^2}{2} - \omega,$$

then there is $\mathbf{k}' \in B(r_2, \mathbf{k})$ such that $S(r_1; \mathbf{k}) \setminus \overline{B(r; \mathbf{k}')}$ is a spherical cap of radius r_1 and angle θ_0 centred at $\mathbf{k}$, and $S(r; \mathbf{k}') \subset B(r_2; \mathbf{k})$.

(ii) *If*

(5.8) $$\frac{r_1^2 + r_2^2}{2} + \omega \le r^2 \le r_2^2,$$

then there is $\mathbf{k}'' \in \mathbb{R}^d$ such that $S(r_2; \mathbf{k}) \cap B(r; \mathbf{k}'')$ is a spherical cap of radius r_2 and angle θ_0 centred at $\mathbf{k}$, and $S(r_1; \mathbf{k}) \subset B(r; \mathbf{k}'')$.

Proof. Let $\theta = \theta_0(A,\omega)$ be as constructed in Lemma 5.4, and let r_1 be a sufficiently large fixed number.

(i) Consider the radius r_- in (5.1) as a function of the radius r_2 and write it as $r(\varrho)$:

$$r(\varrho) = \varrho - \frac{\varrho^2 - r_1^2}{2(r_1\cos\theta_0 + \varrho)} = \frac{1}{2}(r_1\cos\theta_0 + \varrho) + \frac{r_1^2\sin^2\theta_0}{2(r_1\cos\theta_0 + \varrho)}.$$

Here we assume that $\varrho \in [r_1, r_2]$. Clearly, $r(r_1) = r_1$ and $r(r_2) = r_- = r_-(r_1, r_2, \theta_0)$. It is an easy exercise to find that

$$2\partial_\varrho r(\varrho) = 1 - \frac{r_1^2\sin^2\theta_0}{(r_1\cos\theta_0 + \varrho)^2} > 0, \qquad \varrho \in [r_1, r_2].$$

Consequently, the function $r(\varrho)$ is strictly increasing on $[r_1, r_2]$, and thus the inverse function $\varrho(r)$ is strictly increasing on $[r_1, r_-]$. In view of Lemma 5.4,

$$r_-^2 > \frac{r_1^2 + r_2^2}{2} - \omega.$$

Consequently, for all r satisfying (5.7), we have $\varrho(r) < r_2$, and hence, the sphere

$$S(r; \mathbf{k} + t_-(r_1, \varrho, \theta_0)\mathbf{e}_1)$$

(see (5.1) for definition of t_-) satisfies all the requirements of (i).

Before proving (ii), note that in view of (5.4), the condition $\omega \in (0, A/9)$ ensures that $\cos\theta_0 \geq 4/5$, which implies that $\theta_0 \in (0, \pi/4)$, and hence $0 < \tan\theta_0 < 1$.

Consider the radius r_+ in (5.2) as a function of r_1 and write it as $r(\varrho)$:

$$r(\varrho) = \varrho + \frac{r_2^2 - \varrho^2}{2(r_2\cos\theta_0 + \varrho)} = \frac{1}{2}(r_2\cos\theta_0 + \varrho) + \frac{r_2^2\sin^2\theta_0}{2(r_2\cos\theta_0 + \varrho)}.$$

Here we assume that $\varrho \in [r_1, r_2]$. Clearly, $r(r_1) = r_+ = r_+(r_1, r_2, \theta_0)$ and $r(r_2) = r_2$. It is an easy exercise to find that

$$2\partial_\varrho r(\varrho) = 1 - \frac{r_2^2\sin^2\theta_0}{(r_2\cos\theta_0 + \varrho)^2}, \qquad \varrho \in [r_1, r_2].$$

Since $0 < \tan\theta_0 < 1$, this derivative is strictly positive for all $\varrho \in [r_1, r_2]$. Consequently, $r(\varrho)$ is strictly increasing on $[r_1, r_2]$, and thus $\varrho(r)$ is strictly increasing on $[r_+, r_2]$. In view of Lemma 5.4,

$$r_+^2 < \frac{r_1^2 + r_2^2}{2} + \omega.$$

Consequently, for all r satisfying (5.8), we have $\varrho(r) > r_1$, and hence, the sphere

$$S(r; \mathbf{k} + t_+(\varrho, r_2, \theta_0)\mathbf{e}_1)$$

(see (5.2) for definition of t_+) satisfies all the requirements of (ii). ∎

5.3. *Proof of Lemma 4.3.* Let q satisfy (4.4). Now we apply Lemma 5.5 with $r_1 = r_s^{(j)}$, $r_2 = r_{s+1}^{(j)}$ and $\mathbf{k} = \mathbf{k}_s^{(j)}$ or $\mathbf{k}_{s+1}^{(j)}$ with $s = 1, 2, \dots,\ j = q, q+1$. Note straightaway that the conditions (5.3) and (5.6) are fulfilled in view of (3.6). More precisely, (5.3) is satisfied with $A = 1$. Furthermore, by definition of ζ, we have

$$(r_{s+1}^{(j)})^2 - (r_s^{(j)})^2 \geq 2\zeta > \frac{\zeta}{100},$$

so that (5.6) is fulfilled with $\omega = \zeta/100 < A/9$.

Suppose that (4.8) holds for $j = q$ or $q+1$, that is,

$$(r_s^{(j)})^2 \leq r^2 \leq \frac{(r_s^{(j)})^2 + (r_{s+1}^{(j)})^2}{2} - \frac{\zeta}{100}$$

(see definition (4.2)). We use Lemma 5.5(i) with $r_1 = r_s^{(j)}$, $r_2 = r_{s+1}^{(j)}$ and $\mathbf{k} = \mathbf{k}_s^{(j)}$. Let $S(r; \mathbf{k}')$ and $K(r_s^{(j)}, \theta_0; \mathbf{k}_s^{(j)}, \mathbf{e}) = S(r_s^{(j)}; \mathbf{k}_s^{(j)}) \setminus \overline{B(r; \mathbf{k}')}$ with some unit vector $\mathbf{e}$ (see definition (3.7)) be the sphere and the spherical cap constructed in Lemma 5.5(i). Since $S(r; \mathbf{k}')$ is strictly inside $B(r_{s+1}^{(j)}; \mathbf{k}_s^{(j)})$ and the shell $L(r_s^{(j)}, r_{s+1}^{(j)}; \mathbf{k}_s^{(j)})$ is empty, one can write

$$\begin{aligned}\mathcal{N}[B(r; \mathbf{k}')] &\leq \mathcal{N}[B(r_{s+1}^{(j)}; \mathbf{k}_s^{(j)})] - \mathcal{N}[K(r_s^{(j)}, \theta_0; \mathbf{k}_s^{(j)}, \mathbf{e})] \\ &= \mathcal{N}[B(r; \mathbf{k}_s^{(j)})] - \mathcal{N}[K(r_s^{(j)}, \theta_0; \mathbf{k}_s^{(j)}, \mathbf{e})].\end{aligned}$$

By Remark 5.3 the direction $\mathbf{e}$ can be chosen arbitrarily. We choose it as in Lemma 3.2, that is, $\mathbf{e} = \mathbf{e}_s$. Consequently, by Lemma 3.2,

$$\begin{aligned}\mathcal{N}[B(r; \mathbf{k}_s^{(j)})] - \mathcal{N}^-(r) &\geq \mathcal{N}[B(r; \mathbf{k}_s^{(j)})] - \mathcal{N}[B(r; \mathbf{k}')] \\ &\geq \mathcal{N}[K(r_s^{(j)}, \theta_0; \mathbf{k}_s^{(j)}, \mathbf{e}_s)] \geq c(\theta_0)\psi_d(r_s^{(j)}),\end{aligned}$$

which coincides with (4.9).

Similarly one proves the bound (4.11). Precisely, suppose that (4.10) holds, that is,

$$\frac{(r_s^{(j)})^2 + (r_{s+1}^{(j)})^2}{2} + \frac{\zeta}{100} \leq r^2 \leq (r_{s+1}^{(j)})^2.$$

Now we use Lemma 5.5(ii) with $r_1 = r_s^{(j)}$, $r_2 = r_{s+1}^{(j)}$ and $\mathbf{k} = \mathbf{k}_{s+1}^{(j)}$. Let $S(r; \mathbf{k}'')$ and $K(r_{s+1}^{(j)}, \theta_0; \mathbf{k}_{s+1}^{(j)}, \mathbf{e}) = S(r_{s+1}^{(j)}; \mathbf{k}_{s+1}^{(j)}) \cap B(r; \mathbf{k}'')$ be the sphere and the spherical cap constructed in Lemma 5.5(ii). Since $S(r_s^{(j)}; \mathbf{k}_{s+1}^{(j)})$ is strictly inside $B(r; \mathbf{k}'')$ and the shell $L(r_s^{(j)}, r_{s+1}^{(j)}; \mathbf{k}_{s+1}^{(j)})$ is empty, one can write

$$\begin{aligned}\mathcal{N}[B(r; \mathbf{k}'')] &\geq \mathcal{N}[\overline{B(r_s^{(j)}; \mathbf{k}_{s+1}^{(j)})}] + \mathcal{N}[K(r_{s+1}^{(j)}, \theta_0; \mathbf{k}_{s+1}^{(j)}, \mathbf{e})] \\ &= \mathcal{N}[B(r; \mathbf{k}_{s+1}^{(j)})] + \mathcal{N}[K(r_{s+1}^{(j)}, \theta_0; \mathbf{k}_{s+1}^{(j)}, \mathbf{e})].\end{aligned}$$

Referring again to Remark 5.3 and using Lemma 3.2 again, we get

$$\begin{aligned}\mathcal{N}^+(r) - \mathcal{N}[B(r;\mathbf{k}_{s+1}^{(j)})] &\geq \mathcal{N}[B(r;\mathbf{k}'')] - \mathcal{N}[B(r;\mathbf{k}_{s+1}^{(j)})]\\ &\geq \mathcal{N}[K(r_{s+1}^{(j)},\theta_0;\mathbf{k}_{s+1}^{(j)},\mathbf{e}_s)] \geq c(\theta_0)\psi_d(r_{s+1}^{(j)}),\end{aligned}$$

which is (4.11). ∎

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