The greatest prime divisor of a product of consecutive integers

by

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1. Introduction. Let \( k \geq 2 \) and \( n \geq 1 \) be integers. We define

\[
\Delta(n, k) = n(n + 1) \cdots (n + k - 1).
\]

For an integer \( \nu > 1 \), we denote by \( \omega(\nu) \) and \( P(\nu) \) the number of distinct prime divisors of \( \nu \) and the greatest prime factor of \( \nu \), respectively, and we put \( \omega(1) = 0, \ P(1) = 1 \).

A well known theorem of Sylvester [7] states that

\[
P(\Delta(n, k)) > k \quad \text{if} \ n > k.
\]

We observe that \( P(\Delta(1, k)) \leq k \) and therefore the assumption \( n > k \) in (1) cannot be removed. For \( n > k \), Moser [5] sharpened (1) to \( P(\Delta(n, k)) > \frac{11}{10} k \) and Hanson [3] to \( P(\Delta(n, k)) > 1.5k \) unless \( (n, k) = (3, 2), (8, 2), (6, 5) \). Further Faulkner [2] proved that \( P(\Delta(n, k)) > 2k \) if \( n \) is greater than or equal to the least prime exceeding \( 2k \) and \( (n, k) \neq (8, 2), (8, 3) \).

In this paper, we sharpen the results of Hanson and Faulkner. We shall not use these results in the proofs of our improvements. We prove

**Theorem 1.** We have

(a) \[ P(\Delta(n, k)) > 2k \quad \text{for} \ n > \max\left(k + 13, \frac{279}{262} k\right). \]

(b) \[ P(\Delta(n, k)) > 1.97k \quad \text{for} \ n > k + 13. \]

We observe that 1.97 in (3) cannot be replaced by 2 since there are arbitrarily long chains of consecutive composite positive integers. The same reason implies that Theorem 1(a) is not valid under the assumption \( n > \]

\[ \]

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Further the assumption \( n > \frac{279}{262} k \) in Theorem 1(a) is necessary since 
\[ P(\Delta(279, 262)) \leq 2 \cdot 262. \]

Now we give a lower bound for \( P(\Delta(n, k)) \) which is valid for \( n > k > 2 \) except for an explicitly given finite set. For this, we need some notation. For a pair \((n, k)\) and a positive integer \(h\), we write \([n, k, h]\) for the set of all pairs \((n, k), \ldots, (n + h - 1, k)\) and we set \([n, k] = [n, k, 1] = \{(n, k)\}.\) Let
\[
A_{10} = \{58\}, \quad A_8 = A_{10} \cup \{59\}, \quad A_6 = A_8 \cup \{60\},
\]
\[
A_4 = A_6 \cup \{12, 16, 46, 61, 72, 93, 103, 109, 151, 163\},
\]
\[
A_2 = A_4 \cup \{4, 7, 10, 13, 17, 19, 25, 28, 32, 38, 43, 47, 62, 73, 94, 104, 110, 124, 152, 164, 269\}
\]
and \(A_{2i+1} = A_{2i}\) for \(1 \leq i \leq 5\). Further let
\[
A_1 = A_2 \cup \{3, 5, 6, 8, 9, 11, 14, 15, 18, 20, 23, 26, 29, 33, 35, 39, 41, 44, 48, 50, 53, 56, 63, 68, 74, 78, 81, 86, 89, 95, 105, 111, 125, 146, 153, 165, 173, 270\}.
\]
Finally, we set
\[
B = [8, 3] \cup [5, 4, 3] \cup [14, 13, 3] \cup \{(k + 1, k) | k = 3, 5, 8, 11, 14, 18, 63\}.
\]
Then

**Theorem 2.** We have
\[
P(\Delta(n, k)) > 1.95k \quad \text{for } n > k > 2
\]
except when \((n, k) \in [k + 1, k, h]\) for \(k \in A_h\) with \(1 \leq h \leq 22\) or \((n, k) = (8, 3)\).

If \(k = 2\), we observe (see Lemma 7) that \(P(\Delta(n, k)) > 2k\) unless \(n = 3, 8\) and that \(P(\Delta(3, 2)) = P(\Delta(8, 2)) = 3\). Thus the estimate (4) is valid for \(k = 2\) whenever \(n \neq 3, 8\). We observe that \(P(\Delta(k + 1, k)) \leq 2k\) and therefore 1.95 in (4) cannot be replaced by 2.

There are few exceptions if 1.95 is replaced by 1.8 in Theorem 2. We derive from Theorem 2 the following result.

**Corollary 1.** We have
\[
P(\Delta(n, k)) > 1.8k \quad \text{for } n > k > 2
\]
except when \((n, k) \in B\).

**2. Lemmas.** We begin with a well known result due to Levi ben Gerson on a particular case of the Catalan equation.

**Lemma 1.** The solutions of
\[
2^a - 3^b = \pm 1 \quad \text{in integers } a > 0, \ b > 0
\]
are given by \((a, b) = (1, 1), (2, 1), (3, 2)\).
Next we state a result of Saradha and Shorey [6] on a lower bound for \( \omega(\Delta(n, k)) \).

**Lemma 2.** For \( n > k > 2 \), we have

\[
\omega(\Delta(n, k)) \geq \pi(k) + \left[ \frac{1}{3} \pi(k) \right] + 2
\]

except when \( (n, k) \) belongs to the union of the sets

\[
\begin{align*}
\{[4, 3], [6, 3, 3], [16, 3], [6, 4], [6, 5, 4], [12, 5], [14, 5, 3], [23, 5, 2], \\
[7, 6, 2], [15, 6], [8, 7, 3], [12, 7], [14, 7, 2], [24, 7], [9, 8], [14, 8], \\
[14, 13, 3], [18, 13], [20, 13, 2], [24, 13], [15, 14], [20, 14], [20, 17].
\end{align*}
\]

We shall use Lemma 2 only when \( k = 3 \) or \( 5 \leq k \leq 8 \). Let \( p_i \) denote the \( i \)th prime number. Then

**Lemma 3.** We have

\[
p_{i+1} - p_i < \begin{cases} 35 & \text{for } p_i \leq 5591, \\ 15 & \text{for } p_i \leq 1123, \ p_i \neq 523, 887, 1069, \\ 21 & \text{for } p_i = 523, 887, 1069, \\ 9 & \text{for } p_i \leq 361, \\ p_i \neq 113, 139, 181, 199, 211, 241, 283, 293, 317, 337. \end{cases}
\]  

**Lemma 4.** Let \( \mathfrak{N} \) be a positive real number and \( k_0 \) a positive integer. Let

\( I(\mathfrak{N}, k_0) = \{i \mid p_{i+1} - p_i \geq k_0, p_i \leq \mathfrak{N}\} \). Then

\[
P(n(n + 1) \cdots (n + k - 1)) > 2k
\]

for \( 2k \leq n < \mathfrak{N} \) and \( k \geq k_0 \) except possibly when \( p_i < n < n + k - 1 < p_{i+1} \) for \( i \in I(\mathfrak{N}, k_0) \).

**Proof.** Let \( 2k \leq n < \mathfrak{N} \) and \( k > k_0 \). We may suppose that none of \( n, n + 1, \ldots, n + k - 1 \) is a prime, otherwise the result follows. Let \( p_i < n < n + k - 1 < p_{i+1} \). Then \( i = \pi(n) \) and \( p_{\pi(n)} < n < \mathfrak{N} \). For \( \pi(n) \notin I(\mathfrak{N}, k_0) \), we have

\[
k - 1 = n + k - 1 - n < p_{\pi(n)+1} - p_{\pi(n)} < k_0,
\]

which implies \( k - 1 < k_0 - 1 \), a contradiction. Hence the assertion. \( \blacksquare \)

The following result on the estimates for primes is due to Dusart [1, p. 14].

**Lemma 5.** For \( \nu > 1 \), we have

(i) \( \pi(\nu) \leq \frac{\nu}{\log \nu} \left( 1 + \frac{1.2762}{\log \nu} \right) \),

(ii) \( \pi(\nu) \geq \frac{\nu}{\log \nu - 1} \) for \( \nu \geq 5393 \).
Lemma 6. Let $X > 0$ and $0 < \theta < e - 1$ be real numbers. For $l \geq 0$, let
\[
X_0 = \max \left( \frac{5393}{1 + \theta} \exp \left( \frac{\log(1 + \theta) + 0.2762}{\theta} \right) \right),
\]
\[
X_{l+1} = \max \left( \frac{5393}{1 + \theta} \exp \left( \frac{\log(1 + \theta) + 0.2762}{\theta + \frac{1.2762(1 - \log(1 + \theta))}{\log^2 X_l}} \right) \right).
\]
Then
\[
\pi((1 + \theta)X) - \pi(X) > 0 \quad \text{for } X > X_l.
\]
Proof. Let $l \geq 0$ and $X > X_l$. Then $(1 + \theta)X \geq 5393$. By Lemma 5, we have
\[
\delta := \pi((1 + \theta)X) - \pi(X) \geq \frac{(1 + \theta)X}{\log(1 + \theta)X - 1} - \frac{X}{\log X} \left( 1 + \frac{1.2762}{\log X} \right)
\]
\[
\geq \frac{X}{\log(1 + \theta)X - 1} \left\{ 1 + \theta - \frac{\log(1 + \theta)X - 1}{\log X} \left( 1 + \frac{1.2762}{\log X} \right) \right\}
\]
\[
\geq \frac{X}{\log(1 + \theta)X - 1} \left\{ 1 + \theta - \left( 1 - \frac{1 - \log(1 + \theta)}{\log X} \right) \left( 1 + \frac{1.2762}{\log X} \right) \right\}
\]
\[
\geq \frac{X}{\log(1 + \theta)X - 1} (F(X) + G(X))
\]
where
\[
F(X) = \theta - \frac{\log(1 + \theta) + 0.2762}{\log X}, \quad G(X) = \frac{1.2762(1 - \log(1 + \theta))}{\log^2 X}.
\]
We see that $G(X) > 0$ and is decreasing since $0 < \theta < e - 1$. Further we observe that $\{X_l\}$ is a non-increasing sequence. We notice that $\delta > 0$ if $F(X) + G(X) > 0$. But $F(X) + G(X) > F(X) > 0$ for $X > X_0$ by the definition of $X_0$. Thus $\delta > 0$ for $X > X_0$.

Let now $X \leq X_0$. Then $F(X) + G(X) \geq F(X) + G(X_0)$ and $F(X) + G(X_0) > 0$ if $X > X_1$ by the definition of $X_1$. Hence $\delta > 0$ for $X > X_1$. Now we proceed inductively as above to see that $\delta > 0$ for $X > X_l$ with $l \geq 2$. $
$
Lemma 7. Let $n > k$ and $k \leq 16$. Then
\[
(7) \quad P(\Delta(n, k)) \leq 2k
\]
implies that $(n, k) \in \{(8, 2), (8, 3)\}$ or $(n, k) \in [k + 1, k]$ for $k \in \{2, 3, 5, 6, 8, 9, 11, 14, 15\}$ or $(n, k) \in [k + 1, k, 3]$ for $k \in \{4, 7, 10, 13\}$ or $(n, k) \in [k + 1, k, 5]$ for $k \in \{12, 16\}$.

Proof. We apply Lemma 1 to derive that (7) is possible only if $n = 3, 8$ when $k = 2$ and $n = 5, 6, 7$ when $k = 4$. For the latter assertion, we apply Lemma 1 after securing $P((n+i)(n+j)) \leq 3$ with $0 \leq i < j \leq 3$ by deleting the terms divisible by 5 and 7 in $n, n+1, n+2$ and $n+3$. For $k = 3$ and $5 \leq k \leq 8$, the assertion follows from Lemma 2.
Thus we may assume that $k \geq 9$. Assume that (7) holds. Then in the product $\Delta(n, k)$, there are at most $1 + [(k - 1)/p]$ terms divisible by the prime $p$. After removing all the terms divisible by $p \geq 7$, we are left with at least four terms only divisible by 2, 3 and 5. Further out of these terms, for each prime 2, 3 and 5, we remove a term in which the prime divides to a maximal power. Then we are left with a term $n + i$ such that $n \leq n + i \leq 8 \cdot 9 \cdot 5 = 360$.

Let $n \geq 2k$. We now apply Lemma 4 with $\mathfrak{R} = 361$, $k_0 = 9$ and (6) to get $P(\Delta(n, k)) > 2k$ for $k \geq 9$ except possibly when $p_i < n < n + k - 1$ and $p_i + 1$, $p_i = 113, 139, 181, 199, 211, 241, 283, 293, 317, 337$. For these values of $n$, we check that $P(\Delta(n, k)) > 2k$ is valid for $9 \leq k \leq 16$. Thus it suffices to consider $k < n < 2k$. We calculate $P(\Delta(n, k))$ for $(n, k)$ with $9 \leq k \leq 16$ and $k < n < 2k$. We find that (7) holds only if $(n, k)$ is as given in the statement of Lemma 7.

3. Proof of Theorem 1(a). Let $n > \max(k + 13, 279/262k)$. In view of Lemma 7, we may take $k \geq 17$ since $n \leq k + 5$ for the exceptions $(n, k)$ given in Lemma 7. It suffices to prove (2) for $k$ such that $2k - 1$ is prime. Let $k_1 < k_2$ be such that $2k_1 - 1$ and $2k_2 - 1$ are consecutive primes. Suppose (2) holds at $k_1$. Then for $k_1 < k < k_2$, we have

$$P(n(n + 1) \cdots (n + k - 1)) \geq P(n \cdots (n + k_1 - 1)) > 2k_1,$$

implying $P(\Delta(n, k)) \geq 2k_2 - 1 > 2k$. Therefore we may suppose that $k \geq 19$ since $2k - 1$ with $k = 17, 18$ are composites. We assume from now onward in the proof of Theorem 1(a) that $2k - 1$ is prime. We put $x = n + k - 1$. Then $\Delta(n, k) = x(x - 1) \cdots (x - k + 1)$. Let $f_1 < \cdots < f_\mu$ be all the integers in $[0, k)$ such that

$$P((x - f_1) \cdots (x - f_\mu)) \leq k.$$  

We argue as in the proof of [4, Lemma 4] to get

$$k! > x^\mu - \pi(k) \left(1 - \frac{k}{x}\right)^\mu.$$  

We may suppose $\omega(\Delta(n, k)) \leq \pi(2k)$, otherwise (2) follows. Then

$$\mu \geq k - \pi(2k) + \pi(k)$$  

which we use as in [4, Lemma 4] to derive from (9) that

$$x < k^{3/2} \text{ for } k \geq 87; \quad x < k^{7/4} \text{ for } k \geq 40; \quad x < k^2 \text{ for } k \geq 19.$$  

If $x \geq 7k$ and $k > 57$, then as in [4, Lemma 7] we derive from (10) that $x \geq k^{3/2}$. Thus (11) implies that $x < 7k$ for $k \geq 87$. Putting back $n = x - k + 1$, we may assume that $n < 6k + 1$ for $k \geq 87$, $n < k^{7/4} - k + 1$ for $40 \leq k < 87$ and $n < k^2 - k + 1$ for $19 \leq k < 40$. 

Greatest prime divisor of consecutive integers
Let \( k < 87 \). Suppose \( n \geq 2k \). Then \( 2k \leq n < k^{7/4} - k + 1 \) for \( 40 \leq k < 87 \) and \( 2k \leq n < k^2 - k + 1 \) for \( 19 \leq k < 40 \). Thus Lemma 4 with \( N = 87^{7/4} - 87 + 1, k_0 = 35 \) and (6) implies that \( P(\Delta(n, k)) > 2k \) for \( k \geq 35 \). We note here that \( 2k \leq n < N \) for \( 35 \leq k < 40 \). Let \( k < 35 \).

For \( k \geq 35 \), we see from Lemma 4 and (6) that \( P(\Delta(n, k)) > 2k \) for \( k \geq 19 \). Here the case \( k = 20 \) is excluded since \( 2^{20} - 1 \) is composite. Therefore we may assume that \( n < 2k \).

Further we observe that \( \pi(n + k - 1) - \pi(n) > 0 \). This implies that \([2k, n + k - 1]\) contains a prime.

Thus we may assume that \( k \geq 87 \). Then we write \( n = \alpha k + 1 \) with

\[
\begin{align*}
\frac{279}{262} - 1/k &< \alpha \leq 6 & \text{if } k \geq 201, \\
1 + 12/k &< \alpha \leq 6 & \text{if } k < 201.
\end{align*}
\]

Further we consider \( \pi(n + k - 1) - \pi(\max(n - 1, 2k)) \), which is

\[
\begin{align*}
\pi((\alpha + 1)k) - \pi(k) & \text{ for } \alpha \geq 2, \\
\geq \pi \left( \frac{541}{262} \right) - \pi(2k) & \text{ for } \alpha < 2 \text{ and } k \geq 201, \\
\geq \pi(2k + 13) - \pi(2k) & \text{ for } \alpha < 2 \text{ and } k < 201.
\end{align*}
\]

By using exact values of the \( \pi \) function we check that

\[
\begin{align*}
\pi(2k + 13) - \pi(2k) > 0 & \text{ for } k < 201, \\
\pi \left( \frac{541}{262} \right) - \pi(2k) > 0 & \text{ for } 201 \leq k \leq 2616.
\end{align*}
\]

Thus we may suppose that \( k > 2616 \) if \( \alpha < 2 \). Also

\[
\left[ \frac{541}{262} \right] \geq \frac{540}{262} k \text{ for } k > 2616.
\]

Now we apply Lemma 6 with \( X = \alpha k, \theta = 1/\alpha, l = 0 \) if \( \alpha \geq 2 \) and \( X = 2k, \theta = 4/131, l = 1 \) if \( \alpha < 2 \) to get

\[
\pi(n + k - 1) - \pi(\max(n - 1, 2k)) > 0
\]

for \( X > X_0 = 5393/(1 + 1/\alpha) \) if \( \alpha \geq 2 \) and \( X > X_1 = 5393/(1 + 4/131) \) if \( \alpha < 2 \). Further when \( \alpha < 2 \), we observe that \( X = 2k > X_1 \) since \( k > 2616 \). Thus the assertion follows for \( n < 2k \).

It remains to consider the case \( \alpha \geq 2 \) and \( X \leq 5393(1 + 1/\alpha)^{-1} \). Then \( 2k \leq n < n + k - 1 = X(1 + 1/\alpha) \leq 5393 \). Now we apply Lemma 4 with \( N = 5393, k_0 = 35 \) and (6) to conclude that \( P(\Delta(n, k)) > 2k \).}

4. Proof of Theorem 1(b). In view of Lemma 7 and Theorem 1(a), we may assume that \( k \geq 17 \) and \( k < n \leq \frac{279}{262} k \). Let \( X = \frac{279}{262} k, \theta = \frac{245}{279}, l = 0 \).
Then for $k < n \leq X$, we see from Lemma 6 that

$$\pi(2k) - \pi(n - 1) \geq \pi((1 + \theta)X) - \pi(X) > 0$$

for $X > X_0 = 5393(1+\theta)^{-1}$ which is satisfied for $k > 2696$ since $(1 + \theta)X = 2k$. Thus we may suppose that $k \leq 2696$. Now we check with exact values of the $\pi$ function that $\pi(2k) - \pi(\frac{279}{202}k) > 0$. Therefore

$$P(\Delta(n,k)) \geq P((n+1)\cdots2k) \geq p_{\pi(2k)}.$$ 

Further we apply Lemma 6 with $X = 1.97k$, $\theta = 3/197$ and $l = 25$. We calculate that $X_i \leq 284000$. We conclude by Lemma 6 that

$$\pi(2k) - \pi(1.97k) = \pi((1 + \theta)X) - \pi(X) > 0$$

for $k > 145000$. Let $k \leq 145000$. Then we check that $\pi(2k) - \pi(1.97k) > 0$ is valid for $k \geq 680$ by using exact values of the $\pi$ function. Thus

$$p_{\pi(2k)} > 1.97k \quad \text{for} \quad k \geq 680.$$ 

Therefore we may suppose that $k < 680$. Now we observe that for $n > k+13$,

$$\pi(n+k-1) - \pi(1.97k) \geq \pi(2k+13) - \pi(1.97k) > 0;$$

the latter inequality can be checked by using exact values of the $\pi$ function. Hence the assertion follows since $n < 1.97k$. 

5. Proof of Theorem 2. By Theorem 1(b), we may assume that $n \leq k + 13$. Also we may suppose that $k < 680$ by (12). For $k \leq 16$, we calculate $P(\Delta(n,k))$ for all the pairs $(n,k)$ given in the statement of Lemma 7. We find that either $P(\Delta(n,k)) > 1.95k$ or $(n,k)$ is an exception stated in Theorem 1(a). Thus we may suppose that $k \geq 17$. Now we check that $\pi(n+k-1) - \pi(1.95k) > 0$ except when $(n,k) \in [k+1, k, h]$ for $k \in A_h$ with $1 \leq h \leq 11$, and the assertion follows. 

6. Proof of Corollary 1. We calculate $P(\Delta(n,k))$ for all $(n,k)$ with $k \leq 270$ and $k+1 \leq n \leq k + 11$. This contains the set of exceptions given in Theorem 2. We find that $P(\Delta(n,k)) > 1.8k$ unless $(n,k) \in B$. Hence the assertion (5) follows from Theorem 2. 

References

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