Tetragonal modular curves

by

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0. Introduction. A smooth projective curve $X$ defined over an algebraically closed field $k$ is called $d$-gonal if it admits a map $\phi : X \to \mathbb{P}^1$ over $k$ of degree $d$. If the genus $g \geq 2$ and $d = 2$ then $X$ is called hyperelliptic. We will say that $X$ is trigonal, tetragonal and pentagonal for $d = 3, d = 4$ and $d = 5$ respectively.

Let $N$ be a positive integer, and let

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \mid c \equiv 0 \text{ mod } N \right\}.$$ 

Let $X_0(N)$ denote the modular curve corresponding to $\Gamma_0(N)$. Then Zograf [Z] gave a linear bound on the level $N$ of $d$-gonal modular curves $X_0(N)$. Also Nguyen and Saito [N-Sa] proved an analogue of the strong Uniform Boundedness Conjecture for elliptic curves defined over function fields of dimension one by using the connection with giving a bound on the level $N$ of $d$-gonal modular curves $X_0(N)$.

Recently, Hasegawa and Shimura [H-S] gave a highly sharpened upper bound for $3 \leq d \leq 5$ by trying to determine $d$-gonal modular curves $X_0(N)$ for such $d$. For $d = 2$ it was done by Ogg [O]. Actually Hasegawa and Shimura succeeded in determining all trigonal modular curves $X_0(N)$ but failed for tetragonal and pentagonal $X_0(N)$.

The following lists $N$ for which they did not know whether $X_0(N)$ was tetragonal or not:


In this work we prove that $X_0(N)$ is tetragonal only for $N = 88, 99, 109$ among the above 18 numbers. Combining with the result of [H-S] we get the following:

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Theorem 0.1. The modular curve $X_0(N)$ has $\text{Gon}(X_0(N)) = 4$ if and only if

$g = 5$: $N = 42, 51, 52, 55, 56, 57, 63, 65, 67, 72, 73, 75,$

$g = 6$: $N = 58, 79, 121,$

$g = 7$: $N = 60, 62, 68, 69, 77, 80, 83, 85, 89, 91, 98, 100,$

$g = 8$: $N = 74, 101, 103, 109, 125,$

$g = 9$: $N = 66, 70, 87, 88, 95, 96, 99, 107,$

$g = 10$: $N = 92,$

$g = 11$: $N = 78, 94, 104, 111, 119, 131,$

$g = 13$: $N = 143,$

$g = 14$: $N = 167,$

$g = 16$: $N = 191,$

$g = 17$: $N = 142.$

Remark 0.2. Note that rational, elliptic and hyperelliptic curves always admit tetragonal maps. One can find a list of those curves $X_0(N)$ in [O]. For $d \geq \frac{1}{2}g + 1$, any curve of genus $g$ has a $d$-gonal map [K-L]. Since all the modular curves $X_0(N)$ with $\text{Gon}(X_0(N)) = 3$ are of genus $g = 3, 4$ (see [H-S]), they are tetragonal. Combining with Theorem 0.1 one can get a complete list of $X_0(N)$ which are tetragonal.

1. Definitions and general results. Let $X$ be a smooth projective curve of genus $g \geq 4$.

1.1. Gonality. For a line bundle $L \in \text{Pic} X$, a subspace $V \subset H^0(X, L)$ is said to be a $g^r_d$ if $\text{deg}(L) = d$ and $\dim(V) = r + 1$. The gonality of $X$ is

$$\text{Gon}(X) := \min\{d \mid X \text{ has a } g^1_d \}.$$ 

In this paper, when we say that $X$ is $d$-gonal it does not mean $\text{Gon}(X) = d$ but just that $X$ admits a $g^1_d$. Thus $d$-gonal curves $X$ may have $\text{Gon}(X) < d$. For example all hyperelliptic curves are automatically tetragonal.

1.2. Clifford index. For a line bundle $L \in \text{Pic} X$, the Clifford index of $L$ is the integer

$$\text{Cliff}(L) := \text{deg}(L) - 2(h^0(X, L) - 1)$$

and the Clifford index of $X$ itself is defined as

$$\text{Cliff}(X) := \min\{\text{Cliff}(L) \mid h^0(X, L) \geq 2, h^1(X, L) \geq 2\}.$$ 

It is well known that $\text{Cliff}(X) + 2 \leq \text{Gon}(X) \leq \text{Cliff}(X) + 3$ (see [C-M]).

1.3. Clifford dimension. We will say that $L \in \text{Pic} X$ contributes to the Clifford index of $X$ if both $h^0(X, L) \geq 2$, and $h^1(X, L) \geq 2$, and that $L$
computes the Clifford index of $X$ if in addition $\text{Cliff}(X) = \text{Cliff}(L)$. The Clifford dimension of $X$ is defined as

$$\min\{h^0(X, L) - 1 \mid L \text{ computes the Clifford index of } X\}.$$ 

If a line bundle $L$ achieves the minimum and computes the Clifford index, then we will say that $L$ computes the Clifford dimension. In most cases, the Clifford dimension $\ell$ is equal to 1 and the curves with $\ell \geq 2$ are rather rare. It is a classical result that $\ell = 2$ if and only if $X$ is a smooth plane curve of degree $\geq 5$. The case $\ell = 3$ was settled by Martens [M]. He proved that $\ell = 3$ if and only if $X$ is a complete intersection of two irreducible cubic surfaces in $\mathbb{P}^3$, and hence its genus $g$ is 10.

1.4. Property $N_p$. If $X$ is a non-hyperelliptic curve, then the canonical line bundle $\omega_X$ defines an embedding $X \hookrightarrow \mathbb{P}^{g-1}$. Consider the minimal free resolution

$$0 \rightarrow F_{g-2} \rightarrow \cdots \rightarrow F_2 \rightarrow F_1 \rightarrow S \rightarrow S_X \rightarrow 0$$

of the homogeneous coordinate ring $S_X = S/I_X$ as an $S$-module where $S = \mathbb{C}[X_0, X_1, \ldots, X_{g-1}]$ and $F_i = \bigoplus_{j \in \mathbb{Z}} S(-i - j)^{\beta_{i,j}}$. We call $\beta_{i,j}$ the graded Betti numbers. Due to Green and Lazarsfeld [G-L], $X \hookrightarrow \mathbb{P}^{g-1}$ is said to have property $N_p$ if the resolution is of the form

$$\cdots \rightarrow S^{\beta_{p,1}}(-p - 1) \rightarrow \cdots \rightarrow S^{\beta_{2,1}}(-3) \rightarrow S^{\beta_{1,1}}(-2) \rightarrow S \rightarrow S_X \rightarrow 0.$$ 

Therefore property $N_1$ holds if and only if the homogeneous ideal is generated by quadrics, and property $N_p$ holds for $p \geq 2$ if and only if it has property $N_1$ and the $k$th syzygies among the quadrics are generated by linear syzygies for all $1 \leq k \leq p - 1$. Now we recall the following:

Theorem 1.1 (M. Green and R. Lazarsfeld, Appendix in [G]). Let $X$ be a smooth non-hyperelliptic curve of genus $g \geq 3$. Then the canonical embedding $X \hookrightarrow \mathbb{P}^{g-1}$ fails to have property $N_p$ for $p \geq \text{Cliff}(X)$.

Thus if the canonical embedding $X \hookrightarrow \mathbb{P}^{g-1}$ has property $N_p$, then $\text{Cliff}(X) \geq p + 1$ and $\text{Gon}(X) \geq p + 3$.

1.5. Betti numbers. When $\text{Cliff}(X) \leq 2$, Schreyer clarified the relation between the minimal free resolution of the canonical embedding $X \hookrightarrow \mathbb{P}^{g-1}$ and the existence of special linear series of divisors on $X$. For $g \leq 7$, see Table 1 in [Sch1]. For $g \geq 8$, the graded Betti number $\beta_{2,2}$ has one of the values given in the following table [Sch2]:

<table>
<thead>
<tr>
<th>$\beta_{2,2}$</th>
<th>$(g-4)(g-2)$</th>
<th>$(g-2)^2 - 1$</th>
<th>$g-4$</th>
<th>$0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear series</td>
<td>$\exists g_1^1$</td>
<td>$\exists g_2^2$ or $g_8^3$</td>
<td>$\exists$ a single $g_1^1$</td>
<td>$\text{Cliff}(C) \geq 3$</td>
</tr>
</tbody>
</table>
2. Proof. We proceed to prove that $X_0(N)$ is tetragonal only for $N = 88, 99, 109$ among the 18 numbers $N$ in §0. For this purpose we compute the graded Betti numbers of the canonical embedding of $X_0(N)$. We use the computer programs “Maple” and “Singular”. First we calculate the homogeneous ideal of the canonical embedding of $X_0(N)$ by using Maple.

Note that, for such $N$, $X_0(N)$ is not hyperelliptic [O]. Thus $X_0(N)$ can be identified with the canonical curve which is the image of the canonical embedding

$$X_0(N) \ni P \mapsto (f_1(P) : \cdots : f_g(P)) \in \mathbb{P}^{g-1}$$

where $\{f_1, \ldots, f_g\}$ is a basis of the space of cusp forms of weight 2 on $X_0(N)$. One can get such a basis and the corresponding Fourier coefficients from [St].

Then to obtain the minimal generating system of the homogeneous ideal $I(X_0(N))$, we only have to compute the relations of the $f_if_j$ ($1 \leq i, j \leq g$) by Petri’s theorem. Since there are $(g-2)(g-3)/2$ linear relations among the $f_if_j$, we get quadric generators $Q_k(x_1, \ldots, x_g)$ with $1 \leq k \leq (g-2)(g-3)/2$ by assigning $x_i$ to $f_i$ (for details see [H-S]).

Now we compute the Betti numbers by using Singular. In fact when the genus of $X_0(N)$ is big then Singular does not work efficiently. Note that since the canonical embedding is always projectively Cohen–Macaulay, the Betti numbers of the canonical curve are equal to those of the hyperplane section, which allows us to get Betti numbers easier.

We exhibit the so-called Betti table of the canonical embedding for our cases in Table 1.

**Case 1.** The canonical embeddings of $X_0(N)$ for

$N = 76, 82, 84, 90, 93, 97, 106, 108, 113, 115, 128, 133, 137, 157, 169$

have property $N_p$ for $p \geq 2$. Thus $\operatorname{Gon}(X_0(N)) \geq 5$ by §1.4.

**Case 2.** The curve $X_0(88)$ is of genus 9 and $\beta_{2,2} = 9 - 4 = 5$. Thus it is tetragonal by §1.5.

**Case 3.** For $N = 99$ and 109, $\operatorname{Cliff}(X_0(N)) = 2$. Indeed, by §1.4, $\operatorname{Cliff}(X_0(N)) \geq 2$ because the canonical embeddings of these curves have property $N_1$. Also $\operatorname{Cliff}(X_0(N)) \leq 2$ since they have a $g_6^2$ or a $g_8^3$ by §1.4. Therefore $\operatorname{Cliff}(X_0(N))$ is computed by a $g_{2+2\ell}^2$ where $\ell$ denotes the Clifford dimension. It remains to show that $\ell = 1$. We already know that $\ell \leq 3$ from the existence of a $g_6^2$ or a $g_8^3$. Since a smooth plane curve of degree $d$ is of genus $(d-1)(d-2)/2$, the curves $X_0(99)$ and $X_0(109)$ cannot be plane curves. Thus $\ell \neq 2$ by §1.3. Also $\ell \neq 3$ because their genera are less than 10. Therefore $\operatorname{Cliff}(X_0(N))$ is computed by a $g_4^1$ and hence $X_0(99)$ and $X_0(109)$ are tetragonal.
Table 1. The graded Betti numbers for the canonical embedding

<table>
<thead>
<tr>
<th>Genus</th>
<th>$X_0(N)$</th>
<th>$\beta_{1,2}$</th>
<th>$\beta_{2,2}$</th>
<th>$\beta_{3,2}$</th>
<th>Genus</th>
<th>$X_0(N)$</th>
<th>$\beta_{1,2}$</th>
<th>$\beta_{2,2}$</th>
<th>$\beta_{3,2}$</th>
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<tbody>
<tr>
<td>7</td>
<td>$X_0(97)$</td>
<td>0</td>
<td>0</td>
<td>16</td>
<td>9</td>
<td>$X_0(128)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>10</td>
<td>16</td>
<td>0</td>
<td></td>
<td></td>
<td>21</td>
<td>64</td>
<td>70</td>
</tr>
<tr>
<td>8</td>
<td>$X_0(76)$</td>
<td>0</td>
<td>0</td>
<td>21</td>
<td>10</td>
<td>$X_0(108)$</td>
<td>0</td>
<td>0</td>
<td>20</td>
</tr>
<tr>
<td></td>
<td></td>
<td>15</td>
<td>35</td>
<td>21</td>
<td></td>
<td></td>
<td>28</td>
<td>105</td>
<td>162</td>
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<tr>
<td></td>
<td>$X_0(109)$</td>
<td>0</td>
<td>14</td>
<td>35</td>
<td>11</td>
<td>$X_0(84)$</td>
<td>0</td>
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<td>0</td>
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<td></td>
<td>15</td>
<td>35</td>
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<td>36</td>
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<td></td>
<td>$X_0(169)$</td>
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<td>$X_0(90)$</td>
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<td></td>
<td>36</td>
<td>160</td>
<td>315</td>
</tr>
<tr>
<td>9</td>
<td>$X_0(82)$</td>
<td>0</td>
<td>0</td>
<td>8</td>
<td></td>
<td>$X_0(115)$</td>
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<td>0</td>
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<td>21</td>
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<td></td>
<td>36</td>
<td>160</td>
<td>315</td>
</tr>
<tr>
<td></td>
<td>$X_0(88)$</td>
<td>0</td>
<td>5</td>
<td>24</td>
<td></td>
<td>$X_0(133)$</td>
<td>0</td>
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<tr>
<td></td>
<td>$X_0(93)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td>$X_0(137)$</td>
<td>0</td>
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<td>315</td>
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<td></td>
<td>$X_0(99)$</td>
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<td>20</td>
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<td>0</td>
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<td></td>
<td></td>
<td>21</td>
<td>64</td>
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<td></td>
<td>45</td>
<td>231</td>
<td>550</td>
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<td></td>
<td>$X_0(113)$</td>
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<td>$X_0(157)$</td>
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References


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