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On the behavior close to the unit circle of the power series whose coefficients are squared Möbius function values

by

Oleg Petrushov (Moscow)

1. Notation and introduction. In this paper we study the series

$$\mathfrak{M}_0(z) = \sum_{n=1}^{\infty} \mu^2(n) z^n.$$

We set

$$e^{2\pi i\theta} = e(\theta), \quad S_N(\beta) = \sum_{n<N} \mu^2(n) e(n\beta), \quad \tau(\chi, l) = \sum_{k=1}^{q} \chi(k) e(lk/q)$$

for a character χ modulo q, and $\bar{\chi}$ is a character conjugate to χ.

Let $g(x) \geq 0$. The equality $f(x) = \Omega(g(x))$ when $x \to a$ means that there is an infinite sequence $t_k \to a$ such that $|f(t_k)| > \delta g(t_k)$ for some $\delta > 0$. Let $f(x)$ be real. The equality $f(x) = \Omega_{\pm}(g(x))$ when $x \to a$ means that there are infinite sequences $t_k \to a$, $u_k \to a$ such that $f(t_k) > \delta g(t_k)$, $f(u_k) < -\delta g(u_k)$ for some $\delta > 0$. The notations $A \ll B$ or $B \gg A$ mean $|A| = O(|B|)$.

In 1991 R. S. Baker and G. Harman [BH] obtained results for $\mu(n)$ from which it follows that if for each Dirichlet character χ the function $L(s, \chi)$ has no zeros in the half-plane $\{\Re s > a\}$ then for any $\beta \in \mathbb{R}$,

$$\sum_{n=1}^{\infty} \mu(n) e(n\beta) r^n = O((1-r)^{-b-\varepsilon}), \quad r \to 1-, \tag{1.1}$$

where

$$b = \begin{cases} a + 1/4 & \text{if } 1/2 \leq a < 11/20, \\ 4/5 & \text{if } 11/20 \leq a < 3/5, \\ (1/2)(a+1) & \text{if } 3/5 \leq a < 1. \end{cases}$$

This result is conditional and depends on the bound on L-function zeros.

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In this paper we study the series $\mathfrak{M}_0(z) = \sum_{n=1}^{\infty} \mu^2(n) z^n$ for which we give an unconditional nontrivial estimate. The sum $M_0(x) = \sum_{n<x} \mu^2(n)$ is equal to the number of square-free numbers less than x. From the classical result [HW, p. 269]

$$M_0(x) = \frac{6}{\pi^2} x + O(\sqrt{x})$$

it easily follows that

$$\mathfrak{M}_0(r) = \frac{6}{\pi^2}(1-r)^{-1} + O((1-r)^{-1/2}), \quad r \to 1-. \tag{1.2}$$

In 1967 I. Katai [Ka] proved the following result on the oscillation of the remainder term in (1.2):

$$\mathfrak{M}_0(r) = \frac{6}{\pi^2}(1-r)^{-1} + \Omega_{\pm}((1-r)^{-0.25}), \quad r \to 1-.$$

In our paper we study the series $\mathfrak{M}_0(z)$, where z tends to the unit circle along the radius $z = e(\beta)r$ where $\beta \in \mathbb{R}$.

We will use the following notation: $f(q)$ is a function of an integer argument defined by

$$f(q) = \begin{cases} 0 & \text{if } p^3 \mid q \text{ for some prime } p, \\ \displaystyle\prod_{p \mid q} \Big(-\frac{1}{p^2-1}\Big) & \text{otherwise,} \end{cases}$$

where the product is taken over prime divisors of q.

In Sections 2–4 we prove some useful estimates.

In Section 5 we prove the following theorem on the behavior of $\mathfrak{M}_0(e(\beta)r)$ where $\beta \in \mathbb{Q}$.

THEOREM 1.1. *For coprime integers $l, q > 0$ and real $\varepsilon > 0$,*

$$\mathfrak{M}_0(e(l/q)r) = \frac{f(q)}{\zeta(2)}(1-r)^{-1} + O\big((1-r)^{-1/2-\varepsilon} q^{3/4+\varepsilon}\big), \quad r \to 1-. \tag{1.3}$$

The O constant depends only on ε.

Theorem 1.1 is applied to study the behavior of $\mathfrak{M}_0(e(\beta)r)$ where β is irrational.

In Section 6 we obtain some results on Diophantine approximation.

DEFINITION. The *irrationality exponent* of a real number β is the least upper bound of real numbers a such that

$$0 < |\beta - p/q| < 1/q^a$$

is satisfied by an infinite number of integer pairs (p, q) with $q > 0$.

In Section 7 using the results of [S] we prove the following theorem.

Main Theorem 1.2. *If the irrationality exponent of* β *is* 2 *then for any* $\varepsilon > 0$,

$$\mathfrak{M}_0(e(\beta)r) = O((1-r)^{-1/2-\varepsilon}), \qquad r \to 1-. \tag{1.4}$$

The O *constant does not depend on* r.

In Section 8 we prove that the asymptotic equality (1.4) cannot be extended to all irrational numbers: for each δ we construct an irrational number such that $\mathfrak{M}_0(e(\beta)r) = \Omega((1-r)^{-1+\delta})$. We first prove

Theorem 1.3. *Let* β *be an irrational number,* $\delta > 0$ *and* $\gamma - 2 > \max\{11/2, 2/\delta\}$. *If there is a sequence of rational numbers* l_n/q_n *with square-free denominators such that*

$$|\beta - l_n/q_n| \le c/q_n^\gamma,$$

then

$$\mathfrak{M}_0(e(\beta)r) = \Omega((1-r)^{-1+\delta}), \qquad r \to 1-.$$

From this theorem we deduce

Main Theorem 1.4. *For any* $\delta > 0$ *there exist irrational numbers* β *such that*

$$\mathfrak{M}_0(e(\beta)r) = \Omega((1-r)^{-1+\delta}), \qquad r \to 1-. \tag{1.5}$$

Corollary 1.5. *For any* $\delta > 0$ *there exist irrational numbers* β *such that*

$$|S_N(\beta)| = \Omega(N^{1-\delta}), \qquad N \to \infty. \tag{1.6}$$

2. Preliminary results. Let $\alpha(n)$ be a function of a natural variable. Let

$$\mathfrak{A}(z) = \sum_{n=1}^{\infty} \alpha(n) z^n, \qquad F(s) = \sum_{n=1}^{\infty} \alpha(n) n^{-s}.$$

For the Dirichlet series $F(s)$, a Dirichlet character χ, and $\beta \in \mathbb{R}$ we define

$$F(s,\chi) = \sum_{n=1}^{\infty} \alpha(n)\chi(n) n^{-s}, \qquad F[\beta](s) = \sum_{n=1}^{\infty} \frac{\alpha(n)}{n^s} e(\beta n).$$

We will write $A \ll_{a_0,\ldots,a_k} B$ if $A = O(B)$ and the O constant depends only on $a_0,\ldots,a_k$.

Let q be a positive integer and $q = \prod_{i=1}^k p_i^{l_i}$ be its canonical representation. Let $K(q) = \{n \in \mathbb{N} \mid n = \prod_{i=1}^k p_i^{m_i}\}$ where the m_i are arbitrary nonnegative integers. From the fundamental theorem of arithmetic it easily follows that each $n \in \mathbb{N}$ has a unique representation

$$n = km, \tag{2.1}$$

where $k \in K(q)$ and $(m,q) = 1$.

Lemma 2.1. *Let $\alpha(n)$ be an arbitrary sequence of complex numbers, and $l \in \mathbb{Z}$. Suppose that the Dirichlet series $F[l/q](s) = \sum_{n=1}^{\infty} \alpha(n)e(ln/q)n^{-s}$ is convergent for $\sigma = \Re s > \sigma_0 > 0$. Then*

$$\Gamma(s)F[l/q](s) = \int_0^{\infty} t^{s-1}\mathfrak{A}(e(l/q)e^{-t})\,dt.$$

Proof. This follows from the results of [H]. ∎

Let $\beta \in \mathbb{Q}$, $\beta = l/q$, $q > 0$, where l and q are coprime. For the Dirichlet series $F(s)$ we define

$$C_\chi(s) = \sum_{k \in K(q)} \tau(\bar{\chi}, lk)\alpha(k)k^{-s}.$$

Lemma 2.2. *Let $\alpha(n)$ be an arbitrary sequence of complex numbers, and $q > 1$. Suppose the Dirichlet series $F(s) = \sum_{n=1}^{\infty} \alpha(n)n^{-s}$ is absolutely convergent for $\Re s > \sigma_0$. Then for any $l \in \mathbb{Z}$ and s with $\Re s > \sigma_0$,*

$$\sum_{(n,q)=1} \frac{\alpha(n)e(ln/q)}{n^s} = \frac{1}{\phi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \tau(\bar{\chi}, l)F(s,\chi).$$

Proof. We have

$$\sum_{(n,q)=1} \frac{\alpha(n)e(ln/q)}{n^s} = \sum_{n=1}^{\infty} \alpha(n)\frac{u(n)}{n^s}, \tag{2.2}$$

where $u(n) = e(ln/q)$ if $(n,q) = 1$, and $u(n) = 0$ if $(n,q) \neq 1$. Since $u(n) = \frac{1}{\phi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \tau(\bar{\chi}, l)\chi(n)$ we obtain

$$\sum_{(n,q)=1} \frac{\alpha(n)e(ln/q)}{n^s} = \frac{1}{\phi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \tau(\bar{\chi}, l)F(s,\chi). \ ∎$$

Lemma 2.3. *Let $\alpha(n)$ be a multiplicative function and suppose the Dirichlet series $F(s)$ is absolutely convergent in $\{\Re s > \sigma_1\}$. Then*

$$F[\beta](s) = \frac{1}{\phi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} C_\chi(s)F(s,\chi). \tag{2.3}$$

Proof. Let A_k be the set of natural numbers that have k in the representation (2.1). Then

$$\mathbb{N} = \bigsqcup_{k \in K} A_k.$$

Let $S_k = \sum_{n \in A_k} \frac{\alpha(n)}{n^s} e(\beta n)$. Then

$$F[\beta](s) = \sum_{k \in K} S_k, \tag{2.4}$$

where

$$S_k = \alpha(k)k^{-s}\sum_{(n,q)=1} e(lkn/q)\frac{\alpha(n)}{n^s}.$$

By Lemma 2.2,

$$S_k = \frac{\alpha(k)}{k^s}\frac{1}{\phi(q)}\sum_{\chi\,(\mathrm{mod}\,q)} \tau(\bar{\chi}, lk)F(s,\chi). \tag{2.5}$$

From (2.4), (2.5) and the definition of $C_\chi(s)$ we obtain (2.3). ∎

The following lemma reduces the calculation of $C_{\chi_0}(s)$ to the calculation of $C_{\chi_i}(s)$ where χ_i are principal characters modulo $p_i^{m_i}$.

Lemma 2.4. *Let χ_0 be a principal character modulo $q = p_1^{m_1}\ldots p_r^{m_r}$. Let χ_i be principal characters modulo $p_i^{m_i}$. Then*

$$C_{\chi_0}(s) = C_{\chi_1}(s)\ldots C_{\chi_r}(s).$$

Proof. From the properties of the Ramanujan sum [MV, p. 110] it easily follows that for any l satisfying $(l,q)=1$ we have $\tau(\chi_0, lk) = \tau(\chi_1, lk)\ldots \tau(\chi_r, lk) = \tau(\chi_1, p_1^{m_1})\ldots\tau(\chi_r, p_r^{m_r})$. Using the multiplicativity of $\alpha(n)$ we obtain

$$\begin{aligned} C_{\chi_0}(s) &= \sum_{i_1=0,\ldots,i_r=0}^{\infty} \frac{\alpha(p_1^{i_1})\ldots\alpha(p_r^{i_r})}{p_1^{i_1 s}\ldots p_r^{i_r s}}\tau(\chi_1, p_1^{i_1})\ldots\tau(\chi_r, p_r^{i_r}) \\ &= \sum_{i_1=0}^{\infty}\alpha(p_1^{i_1})\tau(\chi_1, p_1^{i_1})p_1^{-i_1 s}\ldots\sum_{i_r=0}^{\infty}\alpha(p_r^{i_r})\tau(\chi_r, p_r^{i_r})p_r^{-i_r s} \\ &= C_{\chi_1}(s)\ldots C_{\chi_r}(s). \ \blacksquare \end{aligned}$$

Let $\omega(n)$ be the number of prime divisors of n. We will use the simple estimate

$$\omega(n) = O\biggl(\frac{\ln n}{\ln\ln n}\biggr). \tag{2.6}$$

The proof can be found in [MV, p. 55]. From (2.6) we obtain, for all A, ε,

$$A^{\omega(n)} \ll_{A,\varepsilon} n^{\varepsilon}. \tag{2.7}$$

Hence we easily derive an estimate for the function $f(q)$ of Section 1. Since $f(p) = \frac{1}{p^2-1} = \frac{p^{-2}}{1-p^{-2}} \le 2p^{-1}$, $f(p^2) = \frac{1}{p^2-1} \le 2p^{-2}$, and $f(p^3) = 0 \le 2p^{-3}$, we have

$$f(q) \le q^{-1}2^{\omega(q)} \ll_\varepsilon q^{-1+\varepsilon}. \tag{2.8}$$

3. Residues and asymptotic formulas. In this section, as usual, $\sigma = \Re s$, $t = \Im s$, and χ_0 is a principal character modulo q. Denote by $F_1(s)$

the function $\zeta(s)/\zeta(2s)$. It can be represented by the Dirichlet series

$$F_1(s) = \sum_{n=1}^{\infty} \frac{\mu^2(n)}{n^s}. \tag{3.1}$$

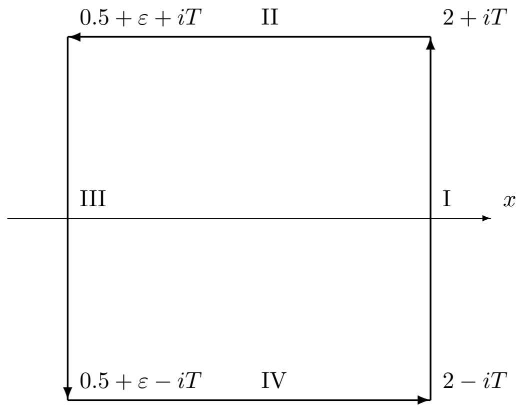
For the Dirichlet series $F_1(s,\chi)$ we have the representation

$$F_1(s,\chi) = \frac{L(s,\chi)}{L(2s,\chi^2)}.$$

Using the Mellin inversion formula [FGD, p. 4], Lemma 2.1 and representation (3.1) we deduce

$$\mathfrak{M}_0(e(l/q)e^{-x}) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} x^{-s}\Gamma(s)F_1[l/q](s)\,ds. \tag{3.2}$$

Let Π be the rectangle with vertices $2-iT$, $2+iT$, $0.5+\varepsilon+iT$, $0.5+\varepsilon-iT$. Let $I=[2-iT,2+iT]$, $II=[2+iT,0.5+\varepsilon+iT]$, $III=[0.5+\varepsilon+iT,0.5+\varepsilon-iT]$, $IV=[0.5+\varepsilon-iT,2-iT]$.



The contour Π

Let $E_1(q)$, $E_2(q)$ be constants depending only on q and ε. Using, for each character χ modulo q, the simple estimate

$$\left|\frac{1}{L(s,\chi)}\right| \le \zeta(\sigma), \qquad \sigma > 1,$$

and the estimate

$$|L(s,\chi)| \le E_1(q)|t|, \qquad |t| \to \infty,$$

uniformly with respect to $\sigma \in [0.5+\varepsilon, 2]$, we obtain

$$\left|\frac{L(s,\chi)}{L(2s,\chi^2)}\right| \le E_2(q)|t|, \qquad |t| \to \infty, \tag{3.3}$$

uniformly with respect to $\sigma \in [0.5+\varepsilon, 2]$. Since $C_\chi(s) = \sum_{k\in K(q)} \frac{\mu^2(k)}{k^s}\tau(\bar{\chi}, lk)$ and $\mu^2(n) = 0$ if $p^2 \mid n$ for some prime p, the sum in $C_\chi(s)$ is finite. Hence the functions $C_\chi(s)$ are entire and

$$|C_\chi(s)| \le D(q), \qquad \Re s \in [0.5, 2], \tag{3.4}$$

where $D(q)$ is a constant depending only on q. Since $\Gamma(s) \ll |t|^C e^{-\frac{\pi}{2}|t|}$ when $|t| \to \infty$ and $\sigma \in [0.5+\varepsilon, 2]$, from (3.3) and (3.4) we deduce

$$\Gamma(\sigma+iT)F_1[l/q](\sigma+iT) \ll e^{-\alpha|T|}$$

for some $\alpha > 0$. Hence

$$\int_{II} x^{-s}\Gamma(s)F_1[l/q](s)\,ds \to 0, \qquad |t| \to \infty, \tag{3.5}$$

$$\int_{IV} x^{-s}\Gamma(s)F_1[l/q](s)\,ds \to 0, \qquad |t| \to \infty. \tag{3.6}$$

Since for each nonprincipal Dirichlet character χ the function $L(s,\chi)$ is holomorphic in Π and $L(s,\chi_0)$ has a simple pole at $s = 1$ in Π, by the Cauchy theorem on residues we have

$$\int_{\Pi} x^{-s}\Gamma(s)F_1[l/q](s)\,ds = \operatorname{res}_{s=1} x^{-s}\Gamma(s)F_1[l/q](s).$$

Using Lemma 2.3, relations (3.5)–(3.6) and equation (3.2) we obtain

$$\begin{aligned}\mathfrak{M}_0(e(l/q)e^{-x}) &= \operatorname{res}_{s=1} x^{-s}\Gamma(s)F_1[l/q](s)\\ &\quad+\frac{1}{2\pi i}\int_{1/2+\varepsilon-i\infty}^{1/2+\varepsilon+i\infty} x^{-s}\Gamma(s)F_1[l/q](s)\,ds\\ &= \operatorname{res}_{s=1} x^{-s}\Gamma(s)C_{\chi_0}(s)F_1(s,\chi_0)\\ &\quad+\frac{1}{2\pi i}\int_{1/2+\varepsilon-i\infty}^{1/2+\varepsilon+i\infty} x^{-s}\Gamma(s)\frac{1}{\phi(q)}\sum_{\chi\,(\mathrm{mod}\,q)} C_\chi(s)F_1(s,\chi)\,ds.\end{aligned} \tag{3.7}$$

Let us calculate the residue. As usual $q = p_1^{m_1}\dots p_r^{m_r}$. Applying Lemma 2.4 to $\mu^2(n)$ we obtain

$$C_{\chi_0}(1) = C_{\chi_1}(1)\dots C_{\chi_r}(1), \quad \text{where} \quad C_{\chi_j}(s) = \sum_{k=0}^{\infty}\frac{\tau(\chi_j, p_j^k)}{p_j^{ks}}\mu^2(p_j^k)$$

and χ_j are principal characters modulo $p_j^{m_j}$. Note that $C_{\chi_j}(s) = 0$ if $m_j > 2$. If $m_j = 1$, then $C_{\chi_j}(s) = -1 + (p_j-1)/p_j^s$ and $C_{\chi_j}(1) = -1 + (p_j-1)/p_j = -1/p_j$. If $m_j = 2$, then $C_{\chi_j}(s) = -p_j/p_j^s = -1$. Hence

$$\operatorname{res}_{s=1} x^{-s}\Gamma(s)F_1[l/q](s) = \operatorname{res}_{s=1} x^{-s}\Gamma(s)\frac{C_{\chi_0}(s)}{\phi(q)}F_1(s,\chi_0)$$
$$= \frac{x^{-1}}{\zeta(2)}\frac{C_{\chi_0}(1)}{\phi(q)}\prod_{p|q}\Big(1+\frac{1}{p}\Big)^{-1} = \frac{f(q)}{\zeta(2)}x^{-1},$$

where $f(q)$ is the multiplicative function defined in Section 1. Consequently,

$$\mathfrak{M}_0(e(l/q)e^{-x}) = \frac{f(q)}{\zeta(2)}x^{-1} + \int_{1/2+\varepsilon-i\infty}^{1/2+\varepsilon+i\infty} x^{-s}\Gamma(s)F_1[l/q](s)\,ds. \tag{3.8}$$

4. Inequalities for L-functions. The following lemma enables us to give a uniform (with respect to q) estimate of $L(s,\chi)$.

LEMMA 4.1. *For each nonprincipal character χ modulo q, and for $0<\sigma<1$,*

$$\Big|\sum_{n<x}\chi(n)n^{-\sigma}\Big| \ll_\sigma q^{\frac{1}{2}(1-\sigma)}\log q.$$

Proof. Let $x>\sqrt{q}$. Then using the Abel transform [K, p. 43] we obtain

$$\sum_{n<x}\chi(n)n^{-\sigma} = X(x)x^{-\sigma} + \sigma\int_1^x X(t)t^{-\sigma-1}\,dt,$$

where $X(t)=\sum_{n\le t}\chi(n)$. Hence

$$\sum_{n<x}\chi(n)n^{-\sigma} = X(x)x^{-\sigma} + \sigma\int_1^{\sqrt{q}} X(t)t^{-\sigma-1}\,dt + \sigma\int_{\sqrt{q}}^{x} X(t)t^{-\sigma-1}\,dt.$$

Using Pólya's estimate $X(x)\ll\sqrt{q}\log q$ (see [MV, p. 307]) we obtain $|X(x)x^{-\sigma}|\ll q^{1/2}(\log q)x^{-\sigma}\le q^{(1-\sigma)/2}\log q$. Further, using the trivial inequality $|X(t)|\le t$ we deduce

$$\Big|\sigma\int_1^{\sqrt{q}} X(t)t^{-\sigma-1}\,dt\Big| \le \sigma\int_1^{\sqrt{q}} t^{-\sigma}\,dt \ll_\sigma q^{\frac{1}{2}(1-\sigma)}.$$

Applying Pólya's estimate again we obtain

$$\Big|\int_{\sqrt{q}}^{x} X(t)t^{-\sigma-1}\,dt\Big| \le \sqrt{q}\log q\int_{\sqrt{q}}^{x} t^{-\sigma-1}\,dt$$
$$\le \sqrt{q}\log q\int_{\sqrt{q}}^{\infty} t^{-\sigma-1}\,dt \ll_\sigma \sqrt{q}\,q^{-\sigma/2}\log q = q^{\frac{1}{2}(1-\sigma)}\log q,$$

so the lemma is proved in the case $x>\sqrt{q}$.

If $x \le \sqrt{q}$, then

$$\Big|\sum_{n<x} \chi(n) n^{-\sigma}\Big| \le \sum_{n<\sqrt{q}} n^{-\sigma} \ll_\sigma (\sqrt{q})^{1-\sigma} = q^{\frac12(1-\sigma)},$$

and the lemma is also proved for $x \le \sqrt{q}$. ■

The following result will be applied in the proof of Theorem 1.1.

Theorem 4.2. *Let χ be a character modulo q, and $\sigma = \Re s$, $0 < \sigma < 1$. Then*

$$|L(s,\chi)| \ll_{\sigma,\varepsilon} |s| q^{(1-\sigma)/2+\varepsilon}.$$

Proof. Consider first the case $\chi \ne \chi_0$ modulo q. Using the Abel transform we get

$$L(\sigma - \varepsilon + s, \chi) = s \int_1^\infty u^{-s-1} \sum_{n<u} \chi(n) n^{\varepsilon-\sigma}\, du.$$

Hence

$$L(s,\chi) = |L(\sigma-\varepsilon+\varepsilon+it,\chi)| \le |it+\varepsilon| \int_1^\infty u^{-\varepsilon-1} \Big|\sum_{n<u} \chi(n) n^{\varepsilon-\sigma}\Big|\, du.$$

Thus by Lemma 4.1,

$$|L(s,\chi)| \ll |s| q^{(1-\sigma+\varepsilon)/2} (\log q) \int_1^\infty u^{-\varepsilon-1}\, du \ll_\varepsilon |s| q^{(1-\sigma)/2+\varepsilon}.$$

If $\chi = \chi_0$ modulo q then by (2.7),

$$\begin{aligned} |L(s,\chi_0)| &= \Big|\prod_{p|q} (1-p^{-s}) \zeta(s)\Big| \ll |s| \prod_{p|q} (1+p^{-\sigma}) \le |s|(1+2^{-\sigma})^{\omega(q)} \\ &\ll_{\sigma,\varepsilon} |s| q^{\varepsilon}. \ ■ \end{aligned}$$

5. Proof of Theorem 1.1

Lemma 5.1. *For each $m \in \mathbb{N}$,*

$$\sum_{\chi \,(\mathrm{mod}\, q)} |\tau(\chi,m)|^2 = \phi^2(q).$$

Proof. We have

$$\begin{aligned} \sum_\chi |\tau(\chi,m)|^2 &= \sum_\chi \Big(\sum_{x=0}^{q-1} \chi(x) e(mx/q) \sum_{y=0}^{q-1} \bar\chi(y) e(-my/q) \Big) \\ &= \sum_\chi \sum_{x,y}{}' \chi(xy^{-1}) e(m(x-y)/q) \end{aligned}$$

$$= {\sum_{x,y}}' \Big(\sum_{\chi} \chi(xy^{-1})\Big) e(m(x-y)/q)$$
$$= {\sum_{x=y}}' \sum_{\chi} \chi(1) = \phi(q)^2,$$

where ${\sum}'_{x,y}$ is the sum over invertible elements x, y of $\mathbb{Z}_q$. ∎

Proof of Theorem 1.1. Since $\left|\frac{L(s,\chi)}{L(2s,\chi^2)}\right| \le |L(s,\chi)|\zeta(2\sigma)$ for $\Re s \ge 1/2+\varepsilon$ we have

$$\left|\frac{L(s,\chi)}{L(2s,\chi^2)}\right| \ll_\varepsilon |L(s,\chi)|. \tag{5.1}$$

Using the definition of $C_\chi(s)$ and Lemma 2.3, and changing the order of summation, we obtain

$$F_1[l/q](s) = \frac{1}{\phi(q)} \sum_{k\in K(q)} \frac{\mu^2(k)}{k^s} \sum_{\chi \,(\mathrm{mod}\, q)} \tau(\bar\chi, lk) \frac{L(s,\chi)}{L(2s,\chi^2)}. \tag{5.2}$$

Using (5.1), (5.2) and Hölder's inequality we deduce

$$\begin{aligned} |F_1[l/q](s)| &= \left|\sum_{n=1}^{\infty} \frac{\mu^2(n)e(nl/q)}{n^s}\right| \\ &\ll \frac{1}{\phi(q)} \sum_{k\in K(q)} \frac{\mu^2(k)}{k^\sigma} \sum_{\chi \,(\mathrm{mod}\, q)} |\tau(\bar\chi, lk)| \left|\frac{L(s,\chi)}{L(2s,\chi^2)}\right| \\ &\ll_\varepsilon \frac{1}{\phi(q)} \sum_{k\in K(q)} \frac{\mu^2(k)}{k^\sigma} \sum_{\chi \,(\mathrm{mod}\, q)} |\tau(\bar\chi, lk)|\, |L(s,\chi)| \\ &\ll \frac{1}{\phi(q)} \sum_{k\in K(q)} \frac{\mu^2(k)}{k^\sigma} \sqrt{\sum_{\chi \,(\mathrm{mod}\, q)} |\tau(\chi, lk)|^2} \sqrt{\sum_{\chi \,(\mathrm{mod}\, q)} |L(s,\chi)|^2}. \end{aligned}$$

By Lemma 5.1 and Theorem 4.2 for $\Re s = 1/2+\varepsilon$ we obtain

$$\begin{aligned} |F_1[l/q](s)| &\ll_\varepsilon \frac{1}{\phi(q)} \prod_{p|q} (1+p^{-1/2-\varepsilon}) \phi(q) \sqrt{\phi(q)|s|^2 q^{1/2+\varepsilon}} \\ &= |s| \prod_{p|q} (1+p^{-1/2-\varepsilon}) \sqrt{\phi(q) q^{1/2+\varepsilon}} \le |s| \prod_{p|q} (1+p^{-1/2-\varepsilon}) q^{3/4+\varepsilon/2}. \end{aligned}$$

Since for fixed $\sigma > 0$ by (2.7) we have $\prod_{p|q}(1+p^{-\sigma}) \le 2^{\omega(q)} \ll_\varepsilon q^{\varepsilon/2}$, for $\Re s = 1/2+\varepsilon$ we obtain

$$|F_1[l/q](s)| \ll |s| q^{3/4+\varepsilon}.$$

Further, from (3.8) and rapid decrease of $|\Gamma(s)|$ when $|\Im s| \to \infty$ we deduce

$$\mathfrak{M}_0(e(l/q)e^x) = \frac{f(q)}{\zeta(2)}x^{-1} + O\Big(\int_{1/2+\varepsilon-i\infty}^{1/2+\varepsilon+i\infty} x^{-1/2-\varepsilon}|\Gamma(1/2+\varepsilon+it)|\,|1/2+\varepsilon+it|q^{3/4+\varepsilon}\,dt\Big) = \frac{f(q)}{\zeta(2)}x^{-1} + O(x^{-1/2-\varepsilon}q^{3/4+\varepsilon}). \ \blacksquare$$

6. The behavior of convergents of a continued fraction

Lemma 6.1. *Let β be a real number with irrationality exponent 2. Let p_n/q_n be the convergents of its continued fraction. Then for each $\varepsilon > 0$,*

$$q_{n+1} \le q_n^{1+\varepsilon}, \tag{6.1}$$

where $n > N(\varepsilon, \beta)$.

Proof. Assume that there exist a subsequence $n_k \to \infty$ and $\varepsilon > 0$ such that

$$q_{n_k+1} \ge q_{n_k}^{1+\varepsilon}.$$

Then using [Kh, p. 16, Theorem 9], we obtain

$$\left|\beta - \frac{p_{n_k}}{q_{n_k}}\right| \le \frac{1}{q_{n_k}q_{n_k+1}} \le \frac{1}{q_{n_k}^{2+\varepsilon}}, \tag{6.2}$$

contrary to the assumption that the irrationality exponent of β equals 2. Hence there exists an $N(\varepsilon, \beta)$ such that (6.1) is true for all $n > N(\varepsilon, \beta)$. ■

7. Proof of Main Theorem 1.2

Lemma 7.1. *Let β be a real number with irrationality exponent 2. Let q_n be the denominators of convergents of its continued fraction. Then for any c, d satisfying $0 < c < d$ there exists an x_0 such that every $x > x_0$ can be represented in the form $x = q_n^A$, where $n \in \mathbb{N}$ and $A \in [c, d]$.*

Proof. Note that q_n^c is an increasing sequence tending to ∞. Take an ε such that $c(1+\varepsilon) \le d$. Let $x_0 = q_{N(\epsilon,\beta)}^c$, where $N(\varepsilon, \beta)$ is defined in Lemma 6.1. Then if $x > x_0$, we have $x \in [q_n^c, q_{n+1}^c]$, where $n > N(\varepsilon, \beta)$, and by Lemma 6.1, $\log_{q_n} x \le \log_{q_n} q_{n+1}^c \le \log_{q_n} q_n^{c(1+\varepsilon)} = c(1+\varepsilon) \le d$. Hence $x = q_n^A$, where $A \in [c, d]$. ■

Lemma 7.2 (see [S]). *If q and a are integers satisfying $|\beta q - a| \le q^{-1}$, then*

$$|S_N(\beta)| \ll_{\varepsilon,\beta} N^{1+\varepsilon} + N^{\varepsilon}q.$$

Lemma 7.3. *We have*

$$S_N(\beta) \ll_{\beta,\varepsilon} N^{1/2+\varepsilon}.$$

Proof. Let l_n/q_n be the convergents of the continued fraction of β. Then $|\alpha - l_n/q_n| < 1/q_n^2$, so by Lemma 7.2,

$$S_N(\beta) \ll_{\beta,\varepsilon} N^{1+\varepsilon}q^{-1} + N^{\varepsilon}q.$$

Since $q_{n+1} \le q_n^{1+\varepsilon}$ if $n > N_0$ and $q_{n+1} > q_n$, each sufficiently large N satisfies the inequality

$$q_n^2 \le N \le q_n^{2(1+\varepsilon)}.$$

Hence $q_n^{-1} \le N^{-\frac{1}{2(1+\varepsilon)}}$ and $q_n \le \sqrt{N}$. Thus $S_N(\beta) \ll N^{1/2+\varepsilon_1}$ for each $\varepsilon_1 > 0$. ■

Using the Abel transform, from Lemma 7.3 we obtain (1.4). Thus Theorem 1.2 is proved.

Since the irrationality exponent of every algebraic number equals 2, by Theorem 1.2 we obtain

COROLLARY 7.4. *Let β be an algebraic number. Then*

$$\mathfrak{M}_0(e(\beta)r) = O((1-r)^{-1/2-\varepsilon}), \qquad r \to 1-.$$

8. Proof of Theorems 1.3 and 1.4. In this section we consider the case of numbers β that are well approximated by rational numbers with square-free denominators.

LEMMA 8.1. *Let β be an irrational number, $\gamma > 0$ and l_m/q_m be a sequence of rational numbers such that*

$$|\beta - l_m/q_m| \ll_\beta 1/q_m^\gamma.$$

Then

$$|\mathfrak{M}_0(e(\beta)e^{-x}) - \mathfrak{M}_0(e(l_m/q_m)e^{-x})| \ll_\beta q_m^{-\gamma}x^{-2}.$$

Proof. Since $|\beta - l_m/q_m| \ll_\beta 1/q_m^\gamma$ we have

$$\sum_{n=1}^{\infty}\mu^2(n)e(n\beta)e^{-nx} - \sum_{n=1}^{\infty}\mu^2(n)e(nl_m/q_m)e^{-nx} \ll \sum_{n=1}^{\infty}\mu^2(n)|e(n(\beta - l_m/q_m)) - 1|e^{-nx}.$$

Using the estimate $|e(x) - 1| \ll |x|$ we obtain

$$\Big|\sum_{n=1}^{\infty}\mu^2(n)e(n\beta)e^{-nx} - \sum_{n=1}^{\infty}\mu^2(n)e(nl_m/q_m)e^{-nx}\Big| \ll \sum_{n=1}^{\infty}\mu^2(n)|n(\beta - l_m/q_m)|e^{-nx} \ll_\beta \sum_{n=1}^{\infty}\frac{n}{q_m^\gamma}e^{-nx} \ll x^{-2}q_m^{-\gamma}. \ ■$$

Proof of Theorem 1.3. Let $x = q_n^{-A}$, where $A = \frac{1}{2}(\max\{11/2, 2/\delta\} + \gamma - 2)$. Note that $A > \max\{11/2, 2/\delta\}$ and $A < \gamma - 2$. By Theorem 1.1 and

Lemma 8.1, for any $\varepsilon > 0$,

$$\begin{aligned}\mathfrak{M}_0(e(\beta)e^{-x}) &= \frac{f(q)}{\zeta(2)} q_n^A + O(q_n^{A(1/2+\varepsilon)} q_n^{3/4+\varepsilon} + q_n^{-\gamma} q_n^{2A}) \\ &= C(q_n) q_n^{A-2} + O(q_n^{A(1/2+\varepsilon)+3/4+\varepsilon} + q_n^{2A-\gamma}),\end{aligned}$$

where $C(q_n) \gg 1$.

Since $A < \gamma - 2$ we have $A - 2 > 2A - \gamma$. The inequality $A > 11/2$ yields $A(1/2+\varepsilon) + 3/4 + \varepsilon < A - 2$ for some $\varepsilon > 0$. Hence

$$\mathfrak{M}_0(e(\beta)e^{-x}) = C(q_n) q_n^{A-2} + o(q_n^{A-2}), \qquad n \to \infty.$$

Using the inequality $A > 2/\delta$ we obtain $\log_{x^{-1}} q_n^{A-2} = \frac{\log q_n^{A-2}}{\log q_n^A} = 1 - 2/A > 1 - \delta$. Since $x^{-1} > 1$ we have $q_n^{A-2} > x^{-1+\delta}$. Hence

$$\begin{aligned}|\mathfrak{M}_0(e(\beta)e^{-x})| &= |q_n^{A-2}(C(q_n) + o(1))| = q_n^{A-2}(C(q_n) + o(1)) \\ &\gg q_n^{A-2} > x^{-1+\delta}. \ \blacksquare\end{aligned}$$

Proof of Main Theorem 1.4. Such numbers can be constructed by means of the method of inserted segments. Let $\gamma > 2 + \max\{11/2, 2/\delta\}$. Let us find a rational number l_1/q_1 where q_1 is a square-free integer and find a real θ_1 with $|\theta_1| \le 1/q_1^\gamma$. Let us find a square-free number q_2 and an integer l_2 such that $l_2/q_2 \ne l_1/q_1$ and

$$\left[\frac{l_2 - 1}{q_2}, \frac{l_2 + 1}{q_2}\right] \subset R_1 = \left[\frac{l_1}{q_1} - \theta_1, \frac{l_1}{q_1} + \theta_1\right],$$

and a θ_2 such that $|\theta_2| \le 1/q_2^\gamma$. Let us find a square-free number q_3 and an integer l_3 such that $l_3/q_3 \ne l_2/q_2$ and

$$\left[\frac{l_3 - 1}{q_3}, \frac{l_3 + 1}{q_3}\right] \subset R_2 = \left[\frac{l_2}{q_2} - \theta_2, \frac{l_2}{q_2} + \theta_2\right],$$

and a θ_3 such that $|\theta_3| \le 1/q_3^\gamma$, etc. Thus we construct a sequence of segments R_i with length tending to zero and with $R_{i+1} \subseteq R_i$. Let $\alpha = \bigcap_i R_i$. Since $\alpha \in R_i$ for each i we have $|\alpha - l_i/q_i| \le 2\theta_i = 2/q_i^\gamma$. By Theorem 1.3 we obtain inequality (1.5). ■

Using the Abel transform we obtain Corollary 1.5.

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Oleg Petrushov
Moscow State University
Vorobyovy Gory
Moscow, Russia
E-mail: olegAP86@yandex.ru

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