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A note on the paper by K. Feng "Non-congruent numbers, odd graphs and the Birch–Swinnerton-Dyer conjecture"

(Acta Arith. 75 (1996), 71–83)

by

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There is a mistake in Theorem 2.4(1) of [2], which says that for the imaginary quadratic field $K = \mathbb{Q}(\sqrt{D})$,

$$2^{t-1} \parallel h_K \Leftrightarrow \text{the directed graph } FG(D) \text{ is odd},$$

where D is the discriminant of K and t is the number of distinct prime factors of D. The correct statement is:

$$2^{t-1} \parallel h_K \Leftrightarrow \text{the directed graph } RG(D) \text{ is odd}.$$

(For the definition of the graph $FG(D)$ and the odd graph, see [2]. Notice that our notation $FG(D)$ is just the notation $G(-D)$ in [2]. The definition of the Rédei graph $RG(D)$ is given in Definition 0.3 below.) In the following, we will give the proof of the correction and a counterexample (Example 0.5).

LEMMA 0.1 ([3, Proposition 2.2]). *D can be uniquely decomposed as $D = D_1 \ldots D_t$, where D_i is the discriminant of $\mathbb{Q}(\sqrt{D_i})$ and a prime power (up to sign). Explicitly, $D_1 = -4$, 8, or -8 if $2 \mid D_1$ (and then put $p_1 = 2$), otherwise $D_i = (-1)^{(p_i-1)/2} p_i$ with p_i an odd prime, for $1 \le i \le t$.*

DEFINITION 0.2. The *Rédei matrix* $RM(D) = (r_{ij})$ is the $t \times t$ matrix over $\mathbb{F}_2$ such that $\left(\frac{D_j}{p_i}\right) = (-1)^{r_{ij}}$, and $r_{ii} = \sum_{j \ne i} r_{ij}$, where $1 \le i, j \le t$, $i \ne j$ and $\left(\frac{D_j}{p_i}\right)$ is the Kronecker symbol.

DEFINITION 0.3. Following the notation of [2], the *Rédei graph* $RG(D)$ for $\mathbb{Q}(\sqrt{D})$ is defined as the simple directed graph with vertices $\{D_1, \ldots, D_t\}$ such that there is an arc $\overrightarrow{D_i D_j}$ if and only if $\left(\frac{D_j}{p_i}\right) = -1$ (i.e. $r_{ij} = 1$ in the Rédei matrix $RM(D) = (r_{ij})$).

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The following theorem is well known. For a proof, see Rédei and Reichardt [5], [6] or Morton [4].

THEOREM 0.4 (Rédei and Reichardt). *Let* r_4 *be the* 4*-rank of* Cl_K*. Then* $r_4 = t - 1 - \operatorname{rank} RM(D)$.

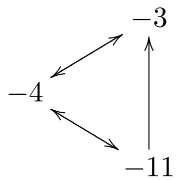
Now we can prove the correction of Theorem 2.4(1) of [2].

Proof. Gauss's genus theory tells us that the 2-rank of Cl_K equals $t-1$. So $2^{t-1} \parallel h_K$ is equivalent to $r_4 = 0$. By Rédei and Reichardt's theorem, this is equivalent to $\operatorname{rank} RM(D) = t-1$. By Lemma 2.2 of [2], it is also equivalent to $RG(D)$ being odd. ∎

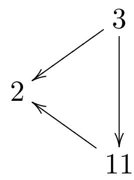
EXAMPLE 0.5. Consider the imaginary quadratic field $\mathbb{Q}(\sqrt{-33})$. The table III in the appendix of Cohn [1] shows that

$$\mathrm{Cl}_{\mathbb{Q}(\sqrt{-33})} \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

The discriminant of $\mathbb{Q}(\sqrt{-33})$ is equal to -132. The graphs $RG(-132)$ and $FG(-132)$ are as follows:



$RG(-132)$



$FG(-132)$

It is easily seen that $RG(-132)$ is odd but $FG(-132)$ is not.

REMARK 0.6. Suppose $D = -8p_2 \dots p_t$, $p_2 \equiv \pm 3 \bmod 8$ and $p_i \equiv 1 \bmod 8$ for $i \geq 3$. Then it can be deduced that $RG(D)$ is odd if and only if $FG(D)$ is odd. So Theorem 2.4(2) of [2] is correct. Since the remaining part of [2] only uses Theorem 2.4(2), the mistake does not affect the main result of [2].

References

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