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Beilinson–Kato elements in K_2 of modular curves

by

FRANÇOIS BRUNAULT (Lyon)

Introduction. Let X be a smooth projective curve over $\mathbb{Q}$, and let $L(h^1(X), s)$ be the associated L-function. A very special case of Beilinson's conjectures predicts that the special value $L(h^1(X), 2)$ can be expressed in terms of a suitable regulator map on the algebraic K-group $K_2(X)$ (see [8] for a nice overview and a precise statement of this conjecture). Beilinson proved a part of his conjecture in the case where X is a modular curve [18]. Beilinson's work was also partially anticipated by Bloch, who studied the particular case of CM elliptic curves [1].

Despite these profound results, the K-group itself remains very mysterious. There is quite an art to constructing special elements in this group and, as soon as the genus of X is not zero, it is not even known whether $K_2(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a finite-dimensional $\mathbb{Q}$-vector space.

I showed in [4] how Beilinson's theorem can be made explicit in the case of the modular curve $X_1(N)$. This raised the question of determining linear dependence relations in the group $K_2(X_1(N))$ [4, §8].

The main point of this article is to make these relations explicit. Let $Y(N)$ be the open modular curve associated to the congruence subgroup $\Gamma(N)$. By taking cup-products of Siegel units, there is a natural map

$$\varrho : M_2(\mathbb{Z}/N\mathbb{Z}) \to K_2(Y(N)) \otimes \mathbb{Q}. \tag{1}$$

Under the hypothesis that N is not divisible by 3, I show that ϱ satisfies the Manin relations (Theorem 1.4). This was also proved by Goncharov [9] using a different method, and his proof works for all N. Thus ϱ can be seen as a Manin symbol (or modular symbol) with values in $K_2(Y(N)) \otimes \mathbb{Q}$. This result is similar to constructions of Borisov and Gunnells [2, 3] and Paşol [17]

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in the case of modular forms. In these works, the product of two Eisenstein series plays the role of the cup-product.

I then use this result to study the case of the modular curves $X_1(p)$ and $X_0(p)$, where p is prime (Theorems 4.2, 4.4 and 4.8). In particular, the Beilinson conjecture implies that the elements so constructed span the vector space $K_2(X_0(p))_{\mathbb{Z}} \otimes \mathbb{Q}$, and I determine all the relations between them.

Some questions would deserve further study. I do not know (even conjecturally) whether the image of ϱ spans $K_2(Y(N)) \otimes \mathbb{Q}$ (see Remark 1.7). In view of the arithmetic applications of Kato's Euler system [10], it would be also of interest to describe the action of Hecke correspondences on these elements, in the spirit of Merel's result on modular symbols [15].

1. The Beilinson–Kato elements in K_2. Let us first state some standard facts on modular curves (see [20, 13, 7, 11] for more detailed accounts). Let $N \geq 3$ be an integer and $Y(N)$ be the modular curve classifying elliptic curves E with a level N structure, that is, a basis (e_1, e_2) of $E[N]$ over $\mathbb{Z}/N\mathbb{Z}$. The curve $Y(N)$ is a smooth projective curve defined over $\mathbb{Q}$, whose affine ring $\mathcal{O}(Y(N))$ contains the cyclotomic field $\mathbb{Q}(\zeta_N)$ generated by $\zeta_N := e^{2i\pi/N}$. The curve $Y(N)$ is not geometrically connected. Indeed, there is an isomorphism $Y(N)(\mathbb{C}) \cong (\mathbb{Z}/N\mathbb{Z})^* \times (\Gamma(N)\backslash\mathcal{H})$, where $\mathcal{H}$ is the Poincaré upper half-plane and $\Gamma(N) \subset \mathrm{SL}_2(\mathbb{Z})$ is the congruence subgroup of matrices satisfying

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N}.$$

For any $z \in \mathcal{H}$ and $\lambda \in \mathbb{Q}$, let us set $q = e^{2i\pi z}$ and $q^\lambda = e^{2i\pi\lambda z}$.

The curve $Y(N)$ has a smooth compactification $X(N)$ over $\mathbb{Q}$ which is obtained by adding on the cusps. The function field of $X(N)$ will be referred to by $\mathbb{Q}(X(N))$. It is naturally embedded into the function field of the compactification of $\Gamma(N)\backslash\mathcal{H}$. There is also a natural inclusion of $\mathbb{Q}(X(N))$ into the field of formal Laurent series $\mathbb{Q}(\zeta_N)((q^{1/N}))$, by looking at the q-expansion.

1.1. *Siegel units.* Let us give the definition of Siegel units (see [6, 10, 12]). The group of modular units of $X(N)$ will be denoted by $\mathcal{O}^*(Y(N))$. In order to avoid torsion problems, Siegel units will always be considered in the $\mathbb{Q}$-vector space $\mathcal{O}^*(Y(N)) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Let $B_2(X) = X^2 - X + 1/6$ be the second Bernoulli polynomial.

Definition 1.1. For any $(\alpha, \beta) \in (\mathbb{Z}/N\mathbb{Z})^2 - \{(0,0)\}$ let us define

$$g_{\alpha,\beta}(z) = q^{\frac{1}{2}B_2(\widetilde{\alpha}/N)} \prod_{n\geq 0}(1 - q^n q^{\widetilde{\alpha}/N}\zeta_N^{\beta}) \prod_{n\geq 1}(1 - q^n q^{-\widetilde{\alpha}/N}\zeta_N^{-\beta}), \tag{2}$$

where $\widetilde{\alpha} \in \mathbb{Z}$ is the unique representative of α satisfying $0 \leq \widetilde{\alpha} < N$. By convention $g_{0,0} = 1$.

Thus $g_{\alpha,\beta}$ is a holomorphic function on $\mathcal{H}$. It is known that some power of $g_{\alpha,\beta}$ (in fact $g_{\alpha,\beta}^{12N}$) is modular with respect to $\Gamma(N)$, and lies in $\mathcal{O}^*(Y(N))$ [13, Chap. 19, §2]. Therefore $g_{\alpha,\beta}$ is well-defined as an element of $\mathcal{O}^*(Y(N)) \otimes \mathbb{Q}$.

Let G be the group $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$. It acts from the left on $Y(N)$, by the rule

$$(3)\qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (E, e_1, e_2) = (E, ae_1 + be_2, ce_1 + de_2) \qquad \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G\right).$$

This induces on $\mathcal{O}^*(Y(N)) \otimes \mathbb{Q}$ a right action of G. It turns out that G acts on the set of Siegel units. More precisely, we have [10, Lemma 1.7]

$$(4)\qquad g_{\alpha,\beta}|\gamma = g_{(\alpha,\beta)\cdot\gamma} \qquad (\gamma \in G).$$

Since $-1 \in G$ acts trivially on $Y(N)$, we get the relation $g_{-\alpha,-\beta} = g_{\alpha,\beta}$. Kubert and Lang proved that the Siegel units of level N generate $\mathcal{O}^*(Y(N)) \otimes \mathbb{Q}$ [12].

1.2. *The construction of Beilinson and Kato.* Let us consider the Quillen K-group $K_2(Y(N))$, which enjoys a right action of G by functoriality. Beilinson constructed special elements in it using cup-products of modular units. This motivates the following definition.

DEFINITION 1.2. Let ϱ be the map

$$(5)\qquad \varrho : M_2(\mathbb{Z}/N\mathbb{Z}) \to K_2(Y(N)) \otimes_{\mathbb{Z}} \mathbb{Q}, \qquad \begin{pmatrix} s & t \\ u & v \end{pmatrix} \mapsto \{g_{s,t}, g_{u,v}\}.$$

REMARK 1.3. Colmez [6, 1.4.2] constructed an algebraic distribution on $M_2(\mathbb{Q} \otimes \widehat{\mathbb{Z}})$ with values in K_2, which generalizes Definition 1.2. I shall not use this more conceptual point of view in what follows.

Let ε (resp. σ, τ) be the image of $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ (resp. $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$) in G.

THEOREM 1.4. *The elements $\varrho(M)$ satisfy the following relations*:

$$(6)\qquad \varrho(\varepsilon M) = \varrho(M), \qquad \varrho(M) + \varrho(\sigma M) = 0 \qquad (M \in M_2(\mathbb{Z}/N\mathbb{Z})).$$

Suppose further that 3 *does not divide N. Then*

$$(7)\qquad \varrho(M) + \varrho(\tau M) + \varrho(\tau^2 M) = 0 \qquad (M \in M_2(\mathbb{Z}/N\mathbb{Z})).$$

REMARK 1.5. The Manin relations (6) and (7) have also been established by Goncharov [9, Corollary 2.17], without any assumption on the level N, using a different method.

REMARK 1.6. The Manin relations (6) and (7) are consistent with the formula of Kato [10, Thm. 2.6] giving the regulator of $z_N = \varrho(I)$. The element z_N plays a prominent role in the construction of Kato's Euler system [10, §5].

Remark 1.7. It would be interesting to know whether the elements $\varrho(M)$ span the $\mathbb{Q}$-vector space $K_2(Y(N)) \otimes \mathbb{Q}$. A related question is to determine whether $K_2(Y(N))$ is generated by the symbols $\{u,v\}$ with $u, v \in \mathcal{O}^*(Y(N))$. Since $K_2(Y(N)) \otimes \mathbb{Q}$ is in general not known to be finite-dimensional, it is more reasonable to ask whether the Manin relations make up a complete set of relations between the elements $\varrho(M)$. A natural way to tackle this problem would be to compute the Beilinson regulator of $\varrho(M)$. However, the formula of Kato [10, Thm. 2.6] seems to indicate that in general $\varrho(G)$ cannot span $K_2(Y(N)) \otimes \mathbb{Q}$.

Proposition 1.8. *For any $M \in M_2(\mathbb{Z}/N\mathbb{Z})$ the relations* (6) *hold.*

Proof. Let $M = \left(\begin{smallmatrix} s & t \\ u & v \end{smallmatrix}\right)$. We have

$$\varrho(\varepsilon M) = \{g_{-s,-t}, g_{u,v}\} = \{g_{s,t}, g_{u,v}\} = \varrho(M),$$
$$\varrho(\sigma M) = \{g_{-u,-v}, g_{s,t}\} = -\{g_{s,t}, g_{u,v}\} = -\varrho(M),$$

because of the relation $g_{-s,-t} = g_{s,t}$ and the antisymmetry of the Milnor symbol. ■

The relation (7) can be seen as an analogue of the Manin 3-term relation for modular symbols. The proof of this relation lies deeper, and will be given in the next two sections.

2. Weierstrass units. For any $z \in \mathcal{H}$, we let $\wp(z,u)$ be the Weierstrass $\wp$-function associated to the lattice $\Lambda_z = \mathbb{Z}z + \mathbb{Z} \subset \mathbb{C}$. It is defined for $u \in \mathbb{C} - \Lambda_z$.

Definition 2.1. For any $a = (a_1, a_2) \in (\mathbb{Z}/N\mathbb{Z})^2 - \{(0,0)\}$, let us define

$$\wp_a(z) = \wp\left(z, \frac{\widetilde{a}_1 z + \widetilde{a}_2}{N}\right) \qquad (z \in \mathcal{H}), \tag{8}$$

where $\widetilde{a}_1$ and $\widetilde{a}_2$ are any representatives of a_1 and a_2 in $\mathbb{Z}$.

We use these functions to construct the *Weierstrass units*. This classical construction is accomplished in [12, Chap. 2, §6]. We give some details for the sake of completeness.

Theorem ([12]). *Let a, b, c, d be four nonzero elements of $(\mathbb{Z}/N\mathbb{Z})^2$ satisfying $a \neq \pm b$ and $c \neq \pm d$. The function*

$$\frac{\wp_a - \wp_b}{\wp_c - \wp_d} \tag{9}$$

defines an element of $\mathcal{O}^*(Y(N))$.

Proof. The function $\wp_a$ is holomorphic on $\mathcal{H}$ and defines a modular form of weight 2 for the group $\Gamma(N)$. For any $z \in \mathcal{H}$, we have $\wp_a(z) = \wp_b(z)$ if and only if $a = \pm b$. Thus $(\wp_a - \wp_b)/(\wp_c - \wp_d)$ is well-defined and does not vanish on $\mathcal{H}$. The fact that it belongs to $\mathbb{Q}(X(N))$ is a consequence of results of Shimura ([19, §4], [20, Chap. 6]). It essentially amounts to expressing $(\wp_a - \wp_b)/(\wp_c - \wp_d)$ in terms of the x-coordinates of N-torsion points of the universal elliptic curve over $Y(N)$. The fact that (9) is a modular unit is proved in [12, Chap. 2, Thm. 6.1]. ∎

Now we express the Weierstrass units in terms of Siegel units. Once again this is done in [12, Chap. 2, §6].

Proposition 2.2. *Let a, b, c, d be four nonzero elements of $(\mathbb{Z}/N\mathbb{Z})^2$ satisfying $a \neq \pm b$ and $c \neq \pm d$. Then the following identity holds in $\mathcal{O}^*(Y(N)) \otimes \mathbb{Q}$:*

$$\frac{\wp_a - \wp_b}{\wp_c - \wp_d} = \frac{g_{a+b} g_{a-b}}{g_a^2 g_b^2} \cdot \frac{g_c^2 g_d^2}{g_{c+d} g_{c-d}}. \tag{10}$$

Proof. We start with the following classical formula from the theory of elliptic functions [13, Chap. 18, Thm. 2]:

$$\wp(z,u) - \wp(z,v) = -\frac{\sigma(z,u+v)\sigma(z,u-v)}{\sigma^2(z,u)\sigma^2(z,v)} \qquad (z \in \mathcal{H}), \tag{11}$$

where σ is the Weierstrass sigma function. For any $(a_1, a_2) \in \mathbb{Z}^2$, let us define, in the same way as in (8),

$$\sigma_{a_1,a_2}(z) = \sigma\left(z, \frac{a_1 z + a_2}{N}\right) \qquad (z \in \mathcal{H}).$$

We write abusively $a = (a_1, a_2)$ and $b = (b_1, b_2)$ for representatives of a and b in $\mathbb{Z}^2$. The formula (11) can then be rewritten as

$$\wp_a - \wp_b = -\frac{\sigma_{a+b}\sigma_{a-b}}{\sigma_a^2 \sigma_b^2}.$$

Using the expression of σ as an infinite q-product [13, Chap. 18, Thm. 4], we get the following formula (cf. [12, pp. 29 and 51]):

$$\wp_a - \wp_b = (2i\pi)^2 q^{b_1/N} \zeta_N^{b_2} \prod_{n \geq 1} (1 - q^n)^4 \cdot \frac{\gamma(q, a+b)\gamma(q, a-b)}{\gamma^2(q,a)\gamma^2(q,b)},$$

where γ is defined by

$$\gamma(q, a_1, a_2) = \prod_{n \geq 0} (1 - q^n q^{a_1/N} \zeta_N^{a_2}) \cdot \prod_{n \geq 1} (1 - q^n q^{-a_1/N} \zeta_N^{-a_2}).$$

Using the obvious notation for c and d, this gives

$$\frac{\wp_a - \wp_b}{\wp_c - \wp_d} = q^{(b_1 - d_1)/N} \zeta_N^{b_2 - d_2} \frac{\gamma(q, a+b)\gamma(q, a-b)}{\gamma^2(q,a)\gamma^2(q,b)} \cdot \frac{\gamma^2(q,c)\gamma^2(q,d)}{\gamma(q, c+d)\gamma(q, c-d)}.$$

Using the expression (2) for Siegel units, we get the equation

$$\frac{\wp_a-\wp_b}{\wp_c-\wp_d}=\zeta_N^{b_2-d_2}\frac{g_{a+b}g_{a-b}}{g_a^2g_b^2}\cdot\frac{g_c^2g_d^2}{g_{c+d}g_{c-d}}.$$

It is a priori a relation between q-products, but raising it to an appropriate power yields an equality in $\mathbb{Q}(\zeta_N)((q^{1/N}))$ and thus in $\mathcal{O}^*(Y(N))$. Therefore the formula (10) is valid in $\mathcal{O}^*(Y(N))\otimes\mathbb{Q}$. ∎

3. The three-term relation. Weierstrass units (9) satisfy additive relations. These have already been used by Kubert and Lang to get diophantine results on modular curves [12, Chap. 8]. In fact, the whole proof of (7) is based on the following simple identity:

$$\frac{\wp_a-\wp_b}{\wp_a-\wp_c}+\frac{\wp_b-\wp_c}{\wp_a-\wp_c}=1. \tag{12}$$

The relation (12) also has applications to the S-unit equation and is connected to the arithmetic of Fermat curves (see the nice introduction of [12, Chap. 8] for precise statements and references).

Since the canonical bilinear map $\mathcal{O}^*(Y(N))\times\mathcal{O}^*(Y(N))\to K_2(Y(N))$ enjoys Steinberg relations [16, 9.8], the identity (12) implies the following relation in $K_2(Y(N))$:

$$\left\{\frac{\wp_a-\wp_b}{\wp_a-\wp_c},\frac{\wp_b-\wp_c}{\wp_a-\wp_c}\right\}=0. \tag{13}$$

Using the expression of Weierstrass units in terms of Siegel units gives linear dependence relations between the elements $\varrho(M)$ in $K_2(Y(N))\otimes\mathbb{Q}$. The main task will be to show that the 3-term relation is a consequence of these relations.

Let a,b,c be three nonzero elements of $(\mathbb{Z}/N\mathbb{Z})^2$ such that $a\neq\pm b$, $b\neq\pm c$ and $c\neq\pm a$. Using (10) and (13) we have the following identity in $K_2(Y(N))\otimes\mathbb{Q}$:

$$\left\{\frac{g_{a+b}g_{a-b}}{g_a^2g_b^2}\cdot\frac{g_a^2g_c^2}{g_{a+c}g_{a-c}},\frac{g_{b+c}g_{b-c}}{g_b^2g_c^2}\cdot\frac{g_a^2g_c^2}{g_{a+c}g_{a-c}}\right\}=0.$$

Expanding this and using the relation $g_{-a}=g_a$, we get the more symmetric identity

$$\{g_{a+b}g_{a-b}g_c^2,g_{b+c}g_{b-c}g_a^2\}+\{g_{b+c}g_{b-c}g_a^2,g_{c+a}g_{c-a}g_b^2\}+\{g_{c+a}g_{c-a}g_b^2,g_{a+b}g_{a-b}g_c^2\}=0. \tag{14}$$

We remark that when $a=0$ the relation (14) still makes sense and holds. Similarly, it holds in the cases $b=0$, $c=0$, $a=\pm b$, $b=\pm c$ or $c=\pm a$. Thus (14) is true for any values of $a,b,c\in(\mathbb{Z}/N\mathbb{Z})^2$.

We now wish to write (14) as a linear combination of 3-term relations. Let us define $\psi(M)=\varrho(M)+\varrho(\tau M)+\varrho(\tau^2M)$ for any $M\in M_2(\mathbb{Z}/N\mathbb{Z})$.

Let $M = \left(\begin{smallmatrix} s & t \\ u & v \end{smallmatrix}\right)$. An elementary computation yields

$$\psi(M) = \{g_{s,t}, g_{u,v}\} + \{g_{u,v}, g_{s-u,t-v}\} + \{g_{s-u,t-v}, g_{s,t}\}. \tag{15}$$

For any two elements a and b of $(\mathbb{Z}/N\mathbb{Z})^2$, let us write $\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$ for the 2 by 2 matrix with row vectors a and b. Then (15) can be rewritten as

$$\psi\begin{pmatrix} a \\ b \end{pmatrix} = \varrho\begin{pmatrix} a \\ b \end{pmatrix} + \varrho\begin{pmatrix} b \\ a-b \end{pmatrix} + \varrho\begin{pmatrix} a-b \\ a \end{pmatrix}. \tag{16}$$

We also have

$$\psi\begin{pmatrix} a \\ -b \end{pmatrix} = \varrho\begin{pmatrix} a \\ b \end{pmatrix} + \varrho\begin{pmatrix} b \\ a+b \end{pmatrix} + \varrho\begin{pmatrix} a+b \\ a \end{pmatrix}. \tag{17}$$

Lemma 3.1. *For any $a, b, c \in (\mathbb{Z}/N\mathbb{Z})^2$, the left hand side of the relation (14) can be written as*

$$\begin{aligned} &2\psi\begin{pmatrix} a \\ b \end{pmatrix} + 2\psi\begin{pmatrix} a \\ -b \end{pmatrix} + 2\psi\begin{pmatrix} b \\ c \end{pmatrix} + 2\psi\begin{pmatrix} b \\ -c \end{pmatrix} + 2\psi\begin{pmatrix} c \\ a \end{pmatrix} + 2\psi\begin{pmatrix} c \\ -a \end{pmatrix} \\ &\quad + \psi\begin{pmatrix} b+a \\ b+c \end{pmatrix} + \psi\begin{pmatrix} b+a \\ b-c \end{pmatrix} + \psi\begin{pmatrix} b-a \\ b+c \end{pmatrix} + \psi\begin{pmatrix} b-a \\ b-c \end{pmatrix}. \end{aligned} \tag{18}$$

Proof. By expanding (14) completely, we obtain

$$\begin{aligned} &\{g_{a+b}, g_{b+c}\} + \{g_{b+c}, g_{c-a}\} + \{g_{c-a}, g_{a+b}\} \\ &+ \{g_{a+b}, g_{b-c}\} + \{g_{b-c}, g_{c+a}\} + \{g_{c+a}, g_{a+b}\} \\ &+ 2\{g_{a+b}, g_a\} + 4\{g_a, g_b\} + 2\{g_b, g_{a+b}\} \\ &+ \{g_{a-b}, g_{b+c}\} + \{g_{b+c}, g_{c+a}\} + \{g_{c+a}, g_{a-b}\} \\ &+ \{g_{a-b}, g_{b-c}\} + \{g_{b-c}, g_{c-a}\} + \{g_{c-a}, g_{a-b}\} \\ &+ 2\{g_{a-b}, g_a\} + 2\{g_b, g_{a-b}\} \\ &+ 2\{g_c, g_{b+c}\} + 2\{g_{b+c}, g_b\} + 4\{g_b, g_c\} \\ &+ 2\{g_c, g_{b-c}\} + 2\{g_{b-c}, g_b\} \\ &+ 4\{g_c, g_a\} \\ &+ 2\{g_a, g_{c+a}\} + 2\{g_{c+a}, g_c\} \\ &+ 2\{g_a, g_{c-a}\} + 2\{g_{c-a}, g_c\} = 0. \end{aligned} \tag{19}$$

In most lines of (19) we recognize an expression of type (16) or (17), but there are incomplete terms. We can arrange the picture by splitting the terms with a coefficient 4 and moving them to the right places. This gives exactly (18). ∎

We now make use of the relation (14) with a particular choice of a, b and c. Let us assume that $c = a + b$. This gives (for any choice of a and b)

$$(20)\qquad 2\psi\begin{pmatrix} a \\ b \end{pmatrix} + 2\psi\begin{pmatrix} a \\ -b \end{pmatrix} + 2\psi\begin{pmatrix} b \\ a+b \end{pmatrix} + 2\psi\begin{pmatrix} b \\ -a-b \end{pmatrix} + 2\psi\begin{pmatrix} a+b \\ a \end{pmatrix} + 2\psi\begin{pmatrix} a+b \\ -a \end{pmatrix} + \psi\begin{pmatrix} a+b \\ a+2b \end{pmatrix} + \psi\begin{pmatrix} a+b \\ -a \end{pmatrix} + \psi\begin{pmatrix} -a+b \\ a+2b \end{pmatrix} + \psi\begin{pmatrix} -a+b \\ -a \end{pmatrix} = 0.$$

Using the notation $M = \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$ and letting T (resp. T') be the image of $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ (resp. $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$) in G, we can rewrite (20) as

$$2\psi(M) + 2\psi(-\varepsilon M) + 2\psi(-\tau\varepsilon M) + 2\psi(-\tau T^2 M) + 2\psi(\tau^2\varepsilon M) + 3\psi(\tau^2 T'^2 M) + \psi(-\tau^2 T^2 M) + \psi\left(\begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix} M\right) + \psi(\tau^2 M) = 0.$$

Since $\psi(M) = \psi(-M) = \psi(\tau M)$ for any M, this simplifies to

$$(21)\quad 3\psi(M) + 6\psi(\varepsilon M) + 3\psi(T^2 M) + 3\psi(T'^2 M) + \psi\left(\begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix} M\right) = 0.$$

Let us consider the formal linear combination of matrices in $\mathbb{Z}[M_2(\mathbb{Z}/N\mathbb{Z})]$,

$$D(M) = 3[M] + 6[\varepsilon M] + 3[T^2 M] + 3[T'^2 M] + \left[\begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix} M\right].$$

By assumption, we have $\det\left(\begin{smallmatrix} -1 & 1 \\ 1 & 2 \end{smallmatrix}\right) = -3 \in (\mathbb{Z}/N\mathbb{Z})^*$.

Lemma 3.2. *The elements $D(M)$ span $\mathbb{Q}[M_2(\mathbb{Z}/N\mathbb{Z})]$ when M runs through $M_2(\mathbb{Z}/N\mathbb{Z})$.*

Proof. We remark that $D(M)$ is congruent mod 3 to the single matrix $\left(\begin{smallmatrix} -1 & 1 \\ 1 & 2 \end{smallmatrix}\right)M$. Therefore the determinant of the vectors $D(M)$ in the canonical basis of $\mathbb{Z}[M_2(\mathbb{Z}/N\mathbb{Z})]$ is not zero mod 3, and thus a nonzero integer. ∎

Using (21) and Lemma 3.2 gives $\psi(M) = 0$ for any $M \in M_2(\mathbb{Z}/N\mathbb{Z})$, which concludes the proof of Theorem 1.4.

4. Varying the modular curve. In this section I study special elements in the groups $K_2(X_1(N)) \otimes \mathbb{Q}$ and $K_2(X_0(N)) \otimes \mathbb{Q}$, in the case of prime level. In particular, I make explicit the link between the Beilinson–Kato elements and the elements which come up in my PhD thesis [5].

Let us first recall the definition of particular modular units on $X_1(N)$ [4, (95)]. Let $Y_1(N)$ be the modular curve over $\mathbb{Q}$ classifying elliptic curves E

with a point P of order N, and let $X_1(N)$ be the smooth compactification of $Y_1(N)$. The set of cusps of $X_1(N)(\mathbb{C})$ is identified with $\Gamma_1(N)\backslash\mathbb{P}^1(\mathbb{Q})$, and with this convention the cusp $[0]$ is defined over $\mathbb{Q}$. Let $W_N : X_1(N) \to X_1(N)$ be the Atkin–Lehner involution, which is defined over $\mathbb{Q}(\zeta_N)$. For any $\lambda \in (\mathbb{Z}/N\mathbb{Z})^*$, the Diamond operator $\langle\lambda\rangle$ associated to λ is defined by $(E,P) \mapsto (E,\lambda P)$. On the complex points of $X_1(N)$ we have $\langle\lambda\rangle[z] = [m_\lambda z]$, where $m_\lambda \in \mathrm{SL}_2(\mathbb{Z})$ is any matrix congruent to $\left(\begin{smallmatrix}\lambda^{-1} & 0\\ 0 & \lambda\end{smallmatrix}\right)$ mod N.

DEFINITION 4.1. For any $\lambda \in (\mathbb{Z}/N\mathbb{Z})^*$, let $u_\lambda \in \mathcal{O}^*(Y_1(N)) \otimes \mathbb{Q}$ be the unique modular unit satisfying

$$\operatorname{div} u_\lambda = \langle\lambda\rangle[0] - [0] \quad \text{and} \quad u_\lambda \circ W_N \text{ is normalized.} \tag{22}$$

Note that we use the cusp $[0]$ instead of $[\infty]$. It essentially amounts to the same thing, because the two definitions are related by W_N. In [4, Prop. 6.1] I show that the element $\{u_\lambda, u_\mu\}$ belongs to $K_2(X_1(N)) \otimes \mathbb{Q}$ for any choice of $\lambda, \mu \in (\mathbb{Z}/N\mathbb{Z})^*$.

From now on, let us suppose that $N = p$ is an odd prime. In [4, §8] I remark that the Beilinson conjecture should imply some linear dependence relations between the elements $\{u_\lambda, u_\mu\}$. It turns out that these relations can be worked out explicitly and even rigorously proved, as follows.

Let $\overline{B}_2 : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ be the 1-periodic function obtained from B_2 by defining $\overline{B}_2(\bar{t}) = B_2(t)$ for any $0 \le t \le 1$. For any $u, v \in (\mathbb{Z}/p\mathbb{Z})^*$, let us define

$$\gamma(u,v) = \sum_{\lambda,\mu \in (\mathbb{Z}/p\mathbb{Z})^*} \overline{B}_2\left(\frac{\lambda u}{p}\right)\overline{B}_2\left(\frac{\mu v}{p}\right)\{u_\lambda, u_\mu\} \in K_2(X_1(p)) \otimes \mathbb{Q}. \tag{23}$$

By convention, we put $\gamma(u,v) = 0$ when $u = 0$ or $v = 0$.

THEOREM 4.2. *The elements $\gamma(u,v)$ $(u, v \in \mathbb{Z}/p\mathbb{Z})$ satisfy the following relations*:

$$\gamma(u,v) = \gamma(\pm u, v) = \gamma(u, \pm v), \tag{24}$$

$$\gamma(u,v) + \gamma(v,-u) = 0, \tag{25}$$

$$\gamma(u,v) + \gamma(v,-u-v) + \gamma(-u-v,u) = 0. \tag{26}$$

Proof. Since $\overline{B}_2$ is an even function and $u_{-\lambda} = u_\lambda$, we have the relations $\gamma(\pm u, v) = \gamma(u, \pm v) = \gamma(u,v)$. The antisymmetry of the Milnor symbol yields $\gamma(v,u) = -\gamma(u,v)$, which proves (25).

In order to prove the 3-term relation (26), we jump to $X(p)$. We have a finite morphism $\pi : Y(p) \to Y_1(p)$ which is defined over $\mathbb{Q}$, given by $(E, e_1, e_2) \mapsto (E, e_2)$.

Let $\mathcal{M}(p)$ be the field of meromorphic functions on the compactification of $\Gamma(p)\backslash\mathcal{H}$. It is a Galois extension of $\mathbb{C}(j)$ with Galois group $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})/\pm 1$. We say that a function $f \in \mathbb{C}((q^{1/n}))^*$ (for some $n \ge 1$) is *normalized* when the leading coefficient of its q-expansion is 1. This definition extends naturally to $\mathbb{C}((q^{1/n}))^* \otimes \mathbb{Q}$. Two functions $f, g \in \mathcal{M}(p)^*$ coincide if and only if their divisors are equal and f/g is normalized. Since we have an inclusion $\mathcal{O}^*(Y(p)) \subset \mathcal{M}(p)^*$, we will apply this principle to check equality between modular units in $\mathcal{O}^*(Y(p)) \otimes \mathbb{Q}$.

The set of cusps of $\Gamma(p)\backslash\mathcal{H}$ is identified with $\Gamma(p)\backslash\mathbb{P}^1(\mathbb{Q})$, and the restriction of π to the cusps is the natural projection $\Gamma(p)\backslash\mathbb{P}^1(\mathbb{Q}) \to \Gamma_1(p)\backslash\mathbb{P}^1(\mathbb{Q})$. The inverse image of a cusp $[x]$ by π is given by

$$\pi^*[x] = \sum_{k=0}^{p-1} [x+k] \qquad (x \in \mathbb{P}^1(\mathbb{Q})).$$

The set of cusps $\Gamma(p)\backslash\mathbb{P}^1(\mathbb{Q})$ can be identified with the set of nonzero column vectors of $(\mathbb{Z}/p\mathbb{Z})^2$ quotiented by ± 1, the bijection being induced by $[a/c] \in \mathbb{P}^1(\mathbb{Q}) \mapsto [\frac{\overline{a}}{\overline{c}}]$ for any two relatively prime integers a and c. We now consider $\pi^* u_\lambda \in \mathcal{O}^*(Y(p)) \otimes \mathbb{Q} \subset \mathcal{M}(p)^* \otimes \mathbb{Q}$. Its divisor is given by

$$\text{(27)} \qquad \operatorname{div} \pi^* u_\lambda = \pi^* \operatorname{div} u_\lambda = \sum_{k=0}^{p-1} [\langle\lambda\rangle 0 + k] - [k] = \sum_{k \in \mathbb{Z}/p\mathbb{Z}} \begin{bmatrix} k \\ \lambda \end{bmatrix} - \begin{bmatrix} k \\ 1 \end{bmatrix}.$$

On the other hand, the order of the Siegel unit $g_{\alpha,\beta}$ at the cusp $[\infty]$ can be deduced from the q-product (2). Since $q^{1/p}$ is a uniformizing parameter at $[\infty]$, we have

$$\operatorname{ord}_{[\infty]} g_{\alpha,\beta} = \frac{p}{2} \overline{B}_2\left(\frac{\alpha}{p}\right) \qquad ((\alpha,\beta) \neq (0,0)).$$

Using the transformation formula (4), we deduce the order of $g_{\alpha,\beta}$ at any cusp:

$$\operatorname{ord}_{[a/c]} g_{\alpha,\beta} = \frac{p}{2} \overline{B}_2\left(\frac{\alpha\overline{a} + \beta\overline{c}}{p}\right) \qquad ((\alpha,\beta) \neq (0,0)).$$

A straightforward computation gives

$$\text{(28)} \qquad \operatorname{div} g_{0,\beta} = \frac{p}{4} \sum_{\substack{\lambda \in (\mathbb{Z}/p\mathbb{Z})^* \\ k \in \mathbb{Z}/p\mathbb{Z}}} \overline{B}_2\left(\frac{\beta\lambda}{p}\right) \begin{bmatrix} k \\ \lambda \end{bmatrix} + \frac{p}{24} \sum_{k \in (\mathbb{Z}/p\mathbb{Z})^*} \begin{bmatrix} k \\ 0 \end{bmatrix} \qquad (\beta \neq 0).$$

From (27) and (28), it follows that the divisor

$$\operatorname{div} g_{0,\beta} - \frac{p}{4} \sum_{\lambda \in (\mathbb{Z}/p\mathbb{Z})^*} \overline{B}_2\left(\frac{\beta\lambda}{p}\right) \operatorname{div} \pi^* u_\lambda$$

does not depend on $\beta \in (\mathbb{Z}/p\mathbb{Z})^*$. Moreover, we have

$$g_{0,\beta}\left(-\frac{1}{pz}\right) = g_{\beta,0}(pz) \quad \text{ in } \mathbb{C}^* \otimes \mathbb{Q} \quad (z \in \mathcal{H}).$$

and $g_{\beta,0}(pz)$ is a normalized function. Since each $u_\lambda \circ W_p$ is normalized, we can write

$$g_{0,\beta} = h \cdot \prod_{\lambda \in (\mathbb{Z}/p\mathbb{Z})^*} \pi^* u_\lambda \otimes \left(\frac{p}{4}\, \overline{B}_2\left(\frac{\beta\lambda}{p}\right)\right),$$

where $h \in \mathcal{O}^*(Y(p)) \otimes \mathbb{Q}$ is well-defined and independent of β. We then have

$$\left\{\frac{g_{0,u}}{h}, \frac{g_{0,v}}{h}\right\} = \frac{p^2}{16}\, \pi^* \gamma(u,v) \qquad (u, v \in (\mathbb{Z}/p\mathbb{Z})^*). \tag{29}$$

We are now ready to prove (26). Since the map $\pi^* : K_2(Y_1(p)) \otimes \mathbb{Q} \to K_2(Y(p)) \otimes \mathbb{Q}$ is injective, it suffices to work in the latter vector space. The cases $u = 0$, $v = 0$ and $u + v = 0$ are easily treated. In the general case, we write

$$\left\{\frac{g_{0,u}}{h}, \frac{g_{0,v}}{h}\right\} = \{g_{0,u}, g_{0,v}\} + \left\{h, \frac{g_{0,u}}{g_{0,v}}\right\}. \tag{30}$$

Thanks to Theorem 1.4, we already know that $(u, v) \mapsto \{g_{0,u}, g_{0,v}\}$ satisfies the 3-term relation. Since $\{h, g_{0,u}/g_{0,v}\}$ is a "boundary element", we get the desired result. ∎

REMARK 4.3. In general, the relations (24), (25) and (26) between the elements $\gamma(u,v)$ do not make up a complete set of relations. This can be seen by working out the case $p = 5$ explicitly. In that case, $X_1(p)$ is isomorphic to $\mathbb{P}^1$ over $\mathbb{Q}$ and $K_2(X_1(p)) \otimes \mathbb{Q}$ is known to be 0. In the general case however, if we average over the action of Diamond operators (see below), we can produce special elements in $K_2(X_0(p)) \otimes \mathbb{Q}$ together with a full set of relations.

A theorem of Schappacher and Scholl [18, 1.1.2(iii)] implies that $\gamma(u,v)$ belongs to the integral subspace $K_2(X_1(p))_{\mathbb{Z}} \otimes \mathbb{Q}$, and we can ask about the span of the elements $\gamma(u,v)$. Let

$$r_p : K_2(X_1(p))_{\mathbb{Z}} \otimes \mathbb{Q} \to \mathrm{Hom}_{\mathbb{Q}}(\Omega^1(X_1(p)), \mathbb{R}) \tag{31}$$

be the Beilinson regulator map, as defined in [4, §1].

THEOREM 4.4. *The Beilinson conjecture for $L(h^1(X_1(p)), 2)$ implies that $K_2(X_1(p))_{\mathbb{Z}} \otimes \mathbb{Q}$ is generated by the elements $\gamma(u,v)$ with $u, v \in (\mathbb{Z}/p\mathbb{Z})^*$.*

Proof. Beilinson's conjecture predicts that r_p is injective and that its image is a $\mathbb{Q}$-structure of the target vector space. We already know that Beilinson's conjecture implies that $K_2(X_1(p))_{\mathbb{Z}} \otimes \mathbb{Q}$ is generated by the elements $\{u_\lambda, u_\mu\}$ [4, §8]. It is sufficient to show that each $\{u_\lambda, u_\mu\}$ is a $\mathbb{Q}$-linear combination of the elements $\gamma(u,v)$. Let us consider

$$\theta = \sum_{\lambda \in (\mathbb{Z}/p\mathbb{Z})^*} \overline{B}_2\left(\frac{\lambda}{p}\right)[\lambda] \in \mathbb{Q}[(\mathbb{Z}/p\mathbb{Z})^*/\pm 1].$$

For every even Dirichlet character $\chi : (\mathbb{Z}/p\mathbb{Z})^* \to \mathbb{C}^*$, we have

$$\chi(\theta) = \sum_{\lambda \in (\mathbb{Z}/p\mathbb{Z})^*} \overline{B}_2\Big(\frac{\lambda}{p}\Big)\chi(\lambda) = \begin{cases} \dfrac{1-p}{6p} & (\chi = 1), \\ \dfrac{\tau(\chi)}{\pi^2} L(\chi,2) & (\chi \neq 1), \end{cases} \tag{32}$$

where $\tau(\chi) = \sum_{a=1}^{p-1} \chi(a) e^{2ia\pi/p}$ is the Gauß sum of χ. But for any character χ, we have $L(\chi, 2) \neq 0$, so that θ is invertible in the group algebra $\mathbb{Q}[(\mathbb{Z}/p\mathbb{Z})^*/\pm 1]$. ∎

We finally investigate the group $K_2(X_0(p)) \otimes \mathbb{Q}$. The natural morphism $X_1(p) \to X_0(p)$ identifies $K_2(X_0(p)) \otimes \mathbb{Q}$ with the fixed part of $K_2(X_1(p)) \otimes \mathbb{Q}$ under the Diamond operators.

Definition 4.5. For any $x \in (\mathbb{Z}/p\mathbb{Z})^*$, let

$$\gamma_0(x) = \sum_{u \in (\mathbb{Z}/p\mathbb{Z})^*} \gamma(u, ux). \tag{33}$$

Moreover, we define $\gamma_0(0) = \gamma_0(\infty) = 0$.

Lemma 4.6. *For any $x \in (\mathbb{Z}/p\mathbb{Z})^*$, we have*

$$\gamma_0(x) \in K_2(X_0(p)) \otimes \mathbb{Q}.$$

Proof. It suffices to prove that $\pi^*\gamma_0(x)$ is invariant under any matrix $t = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$. Because of (4), we have $g_{0,\beta}|t = g_{0,d\beta}$. Using (29), we remark that

$$\frac{p^2}{16}\pi^*\gamma_0(x) = \sum_{u \in (\mathbb{Z}/p\mathbb{Z})^*} \Big\{\frac{g_{0,u}}{h}, \frac{g_{0,ux}}{h}\Big\} = \sum_{u \in (\mathbb{Z}/p\mathbb{Z})^*} \{g_{0,u}, g_{0,ux}\}$$

which is clearly invariant under t. ∎

Remark 4.7. The element $\gamma_0(x) \in K_2(X_0(p)) \otimes \mathbb{Q}$ is defined only implicitly. By this I mean that the actual definition uses Milnor symbols with functions on $X_1(p)$, and not on $X_0(p)$, which only contains two cusps. We can rewrite $\gamma_0(x)$ as follows:

$$\begin{aligned} \gamma_0(x) &= \sum_{u \in (\mathbb{Z}/p\mathbb{Z})^*} \sum_{\lambda,\mu \in (\mathbb{Z}/p\mathbb{Z})^*} \overline{B}_2\Big(\frac{\lambda u}{p}\Big)\overline{B}_2\Big(\frac{\mu u x}{p}\Big)\{u_\lambda, u_\mu\} \\ &= \sum_{\nu \in (\mathbb{Z}/p\mathbb{Z})^*} \Big(\sum_{u \in (\mathbb{Z}/p\mathbb{Z})^*} \overline{B}_2\Big(\frac{u}{p}\Big)\overline{B}_2\Big(\frac{u\nu x}{p}\Big)\Big)\Big(\sum_{\lambda \in (\mathbb{Z}/p\mathbb{Z})^*} \{u_\lambda, u_{\lambda\nu}\}\Big). \end{aligned} \tag{34}$$

In (34), each sum over λ already lies in $K_2(X_0(p)) \otimes \mathbb{Q}$. Moreover, we recognize the sum over u to be a Dedekind sum.

For any $x \in \mathbb{P}^1(\mathbb{Z}/p\mathbb{Z})$, let $\xi(x) \in H_1(X_0(p)(\mathbb{C}), \text{cusps}, \mathbb{Z})$ be the modular symbol $\{g_x 0, g_x \infty\}$ where $g_x = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ is any matrix satisfying $x = c/d \bmod p$. Let $\xi^{\pm}(x) = \frac{1}{2}(\xi(x) + \xi(-x))$. For any cusp form $f \in S_2(\Gamma_1(p))$, we define $\xi_f(x) = \int_{\xi(x)} \omega_f$ and $\xi_f^{\pm}(x) = \int_{\xi^{\pm}(x)} \omega_f$ where $\omega_f = 2i\pi f(z)dz$.

Theorem 4.8.

(1) *For any newform $f \in S_2(\Gamma_0(p))$, we have*

$$\langle r_p(\gamma_0(x)), f\rangle = \frac{8(p-1)}{p\pi} L(f,2)\xi_f^+(x) \qquad (x \in (\mathbb{Z}/p\mathbb{Z})^*). \tag{35}$$

(2) *For any $x \in \mathbb{P}^1(\mathbb{Z}/p\mathbb{Z})$, the following relations hold*:

$$\begin{gathered} \gamma_0(x) = \gamma_0(-x), \\ \gamma_0(x) + \gamma_0(-1/x) = 0, \\ \gamma_0(x) + \gamma_0\left(-\frac{1}{x-1}\right) + \gamma_0\left(1 - \frac{1}{x}\right) = 0. \end{gathered} \tag{36}$$

(3) *The equations* (36) *make up a complete set of relations for the elements $\gamma_0(x)$.*

Proof. Item (1) will be a consequence of the explicit computation of Beilinson's regulator for the modular curve $X_1(p)$ [4, Thm. 1.1]. Let X be the set of even nontrivial characters of $(\mathbb{Z}/p\mathbb{Z})^*$. For any $\chi \in X$, we define a modular unit $u_\chi \in \mathcal{O}^*(Y_1(p)) \otimes \mathbb{C}$ by

$$u_\chi = \prod_{\lambda \in (\mathbb{Z}/p\mathbb{Z})^*} u_\lambda \otimes \left(-\frac{L(\chi,2)\overline{\chi}(\lambda)}{2\pi^2}\right). \tag{37}$$

Now let us compute the following element in $K_2(X_1(p)) \otimes \mathbb{C}$:

$$\gamma_x = \sum_{\chi \in X} \chi(x)\{u_\chi, u_{\overline{\chi}}\} \qquad (x \in (\mathbb{Z}/p\mathbb{Z})^*).$$

Using (37) gives

$$\gamma_x = \frac{1}{4\pi^4} \sum_{\lambda,\mu \in (\mathbb{Z}/p\mathbb{Z})^*} \left(\sum_{\chi \in X} \chi\left(\frac{x\mu}{\lambda}\right) L(\chi,2)L(\overline{\chi},2)\right)\{u_\lambda, u_\mu\}. \tag{38}$$

The inner sum can be computed using the formula (32), which gives

$$\frac{\pi^4(p-1)}{2p} \sum_{\substack{\alpha,\beta \in (\mathbb{Z}/p\mathbb{Z})^* \\ \alpha x\mu = \pm\beta\lambda}} \overline{B}_2\left(\frac{\alpha}{p}\right)\overline{B}_2\left(\frac{\beta}{p}\right) - \frac{\pi^4}{p} \sum_{\alpha,\beta \in (\mathbb{Z}/p\mathbb{Z})^*} \overline{B}_2\left(\frac{\alpha}{p}\right)\overline{B}_2\left(\frac{\beta}{p}\right). \tag{39}$$

The second term of (39) contributes to zero in (38) by antisymmetry of the Milnor symbol. Finally, we get

$$\gamma_x = \frac{p-1}{4p} \sum_{\substack{\alpha,\beta,\lambda,\mu\in(\mathbb{Z}/p\mathbb{Z})^* \\ \alpha x\mu=\beta\lambda}} \overline{B}_2\left(\frac{\alpha}{p}\right)\overline{B}_2\left(\frac{\beta}{p}\right)\{u_\lambda, u_\mu\} = \frac{p-1}{4p}\,\gamma_0(x).$$

In order to use [4, Thm. 1.1], we have to take care of the Atkin–Lehner involution W_p. Let $w(f)$ be the W_p-eigenvalue of f. We let temporarily $\widetilde{u}_\chi$ (resp. $\widetilde{u}_\lambda$) be the modular unit defined in [4, (5)] (resp. in [4, (95)]). We have $u_\lambda|W_p = \widetilde{u}_{\lambda^{-1}}$ and for any $\chi \in X$,

$$\begin{aligned}\{u_\chi, u_{\overline{\chi}}\}|W_p &= \frac{L(\chi,2)L(\overline{\chi},2)}{4\pi^4} \sum_{\lambda,\mu\in(\mathbb{Z}/p\mathbb{Z})^*} \overline{\chi}(\lambda)\chi(\mu)\{u_\lambda,u_\mu\}|W_p \\ &= \frac{L(\chi,2)L(\overline{\chi},2)}{4\pi^4} \sum_{\lambda,\mu\in(\mathbb{Z}/p\mathbb{Z})^*} \chi(\lambda/\mu)\{\widetilde{u}_\lambda,\widetilde{u}_\mu\} = \{\widetilde{u}_{\overline{\chi}},\widetilde{u}_\chi\}\end{aligned}$$

because of [4, Prop. 5.4]. Let $f \in S_2(\Gamma_0(p))$ be a newform and $w(f)$ be the W_p-eigenvalue of f. Using [4, Thm. 1.1], we have

$$\begin{aligned}\langle r_p(\{u_\chi, u_{\overline{\chi}}\}), f\rangle &= \langle r_p(\{u_\chi, u_{\overline{\chi}}\}|W_p), W_p f\rangle \\ &= w(f)\langle r_p(\{\widetilde{u}_\chi, \widetilde{u}_{\overline{\chi}}\}), f\rangle \\ &= \frac{2(p-1)w(f)}{p\pi\tau(\chi)}\,L(f,2)L(f,\chi,1).\end{aligned}$$

A classical computation [14] yields

$$L(f,\chi,1) = -\frac{w(f)\tau(\chi)}{p} \sum_{a\in(\mathbb{Z}/p\mathbb{Z})^*} \chi(a)\xi_f^+(a) \qquad (\chi \in X).$$

By taking the sum over characters χ, we obtain

$$\langle r_p(\gamma_x), f\rangle = \frac{2(p-1)^2}{p^2\pi}\,L(f,2)\xi_f^+(x).$$

This proves (35).

The relations (36) are easy consequences of Theorem 4.2 and the definition (33) of $\gamma_0(x)$. Note that they are consistent with the regulator formula (35).

Finally, for item (3), let $\widetilde{\gamma}_0$ be the map

$$\widetilde{\gamma}_0 : \mathbb{Q}[(\mathbb{Z}/p\mathbb{Z})^*] \to K_2(X_0(p)) \otimes \mathbb{Q}, \qquad [x] \mapsto \gamma_0(x).$$

Let R be the kernel of $\widetilde{\gamma}_0$. We wish to show that R is generated by the relations (36). For this we use the theory of Manin symbols. For any $x \in (\mathbb{Z}/p\mathbb{Z})^*$, the cycle $\xi(x)$ has trivial boundary. Thus we have a map

$$\xi^+ : \mathbb{Q}[(\mathbb{Z}/p\mathbb{Z})^*] \to H_1^+(X_0(p)(\mathbb{C}), \mathbb{Q}).$$

Manin's theorem implies that the kernel of ξ^+ is generated by the relations (36), so that $\ker \xi^+ \subset R$. In order to prove the reverse inclusion, it suffices to consider the dimensions. Let $g(X_0(p))$ be the genus of $X_0(p)$. From (35) we know that the image of $\widetilde{\gamma}_0$ has dimension at least $g(X_0(p))$. Manin's theorem implies that the dimension of the image of ξ^+ is precisely $g(X_0(p))$ (the element $\xi(0) = \{0, \infty\} = -\xi(\infty)$ has nontrivial boundary). We conclude that $\dim R \leq \dim \ker \xi^+$, so that R is generated by (36). ∎

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Université de Lyon
E.N.S. Lyon, U.M.P.A.
46 allée d'Italie
F-69007 Lyon, France
E-mail: brunault@umpa.ens-lyon.fr
http://www.umpa.ens-lyon.fr/~brunault/

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