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Representations of squares by certain quinary quadratic forms

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1. Introduction. More than 100 years ago, Hurwitz determined the number of ways a square can be expressed as a sum of five squares:

THEOREM 1.1 (Hurwitz). *Let n be a positive integer with prime factorization $n = \prod_p p^{\lambda_p}$. Then the number of solutions in integers of*

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = n^2$$

is given by

$$10\left(\frac{2^{3\lambda_2+3}-1}{2^3-1}\right)\prod_{p\geq 3}\left(\frac{p^{3\lambda_p+3}-1}{p^3-1} - p\,\frac{p^{3\lambda_p}-1}{p^3-1}\right).$$

See [6], [11, p. 311] or [12] for more details. The goal of this work is to state and prove the following three analogues of Hurwitz's theorem.

THEOREM 1.2. *Let n be a positive integer with prime factorization $n = \prod_p p^{\lambda_p}$. Then the number of solutions in integers of*

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + 2x_5^2 = n^2$$

is given by

$$c(\lambda_2)\prod_{p\geq 3}\left(\frac{p^{3\lambda_p+3}-1}{p^3-1} - p\left(\frac{2}{p}\right)\frac{p^{3\lambda_p}-1}{p^3-1}\right)$$

where

$$c(\lambda_2) = \begin{cases} 8 & \text{if } n \text{ is odd,}\\ 24\left(\dfrac{2^{3\lambda_2+1}+5}{2^3-1}\right) & \text{if } n \text{ is even,}\end{cases}$$

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and the values of the Legendre symbol $\left(\frac{2}{p}\right)$ *for primes* $p \geq 3$ *are given by*

$$\left(\frac{2}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \text{ or } 7 \pmod 8, \\ -1 & \text{if } p \equiv 3 \text{ or } 5 \pmod 8. \end{cases}$$

THEOREM 1.3. *Let* n *be a positive integer with prime factorization* $n = \prod_p p^{\lambda_p}$. *Then the number of solutions in integers of*

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + 3x_5^2 = n^2$$

is given by

$$8\left(\frac{2^{3\lambda_2+2}+3}{2^3-1}\right)\left(\frac{36\cdot 3^{3\lambda_3}-10}{3^3-1}\right)\prod_{p\geq 5}\left(\frac{p^{3\lambda_p+3}-1}{p^3-1} - p\left(\frac{3}{p}\right)\frac{p^{3\lambda_p}-1}{p^3-1}\right)$$

where the values of the Legendre symbol $\left(\frac{3}{p}\right)$ *for primes* $p \geq 5$ *are given by*

$$\left(\frac{3}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \text{ or } 11 \pmod{12}, \\ -1 & \text{if } p \equiv 5 \text{ or } 7 \pmod{12}. \end{cases}$$

THEOREM 1.4. *Let* n *be a positive integer with prime factorization* $n = \prod_p p^{\lambda_p}$. *Then the number of solutions in integers of*

$$x_1^2 + x_2^2 + x_3^2 + 2x_4^2 + 3x_5^2 = n^2$$

is given by

$$12\left|\frac{2^{3\lambda_2+3}-15}{2^3-1}\right|\left(\frac{3^{3\lambda_3+2}+4}{3^3-1}\right)\prod_{p\geq 5}\left(\frac{p^{3\lambda_p+3}-1}{p^3-1} - p\left(\frac{6}{p}\right)\frac{p^{3\lambda_p}-1}{p^3-1}\right)$$

where the values of the Legendre symbol $\left(\frac{6}{p}\right)$ *for primes* $p \geq 5$ *are given by*

$$\left(\frac{6}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1,\ 5,\ 19 \text{ or } 23 \pmod{24}, \\ -1 & \text{if } p \equiv 7,\ 11,\ 13 \text{ or } 17 \pmod{24}. \end{cases}$$

Theorems 1.2 and 1.3 are analogues of results for $n^2 = x_1^2 + x_2^2 + 2x_3^2$ and $n^2 = x_1^2 + x_2^2 + 3x_3^2$ that have been studied recently in [10]. Theorems 1.2, 1.3 and 1.4 were discovered by computer investigation, and an experimental search for other examples was performed. This yielded 15 additional results.

This work is organized as follows. In Section 2 we define some notation and list some preliminary results for theta functions. A proof of Theorem 1.3 is given in Sections 3 and 4: the primes $p = 2$ and $p = 3$ are handled in Section 3 and the remaining primes $p \geq 5$ are treated in Section 4. The proof of Theorem 1.2 is given in Section 5. It is similar to, but simpler than, the proof of Theorem 1.3. The proof of Theorem 1.4 is given in Section 6. It is also similar to the proof of Theorem 1.3. The 15 additional results discovered by computer search are presented in Section 7. All can be deduced fairly simply from one of Theorems 1.1, 1.2 or 1.3.

2. Notation and background results. The theta functions $\varphi(q)$, $\psi(q)$ and $X(q)$ are defined by

$$\varphi(q)=\sum_{j=-\infty}^{\infty} q^{j^2},\quad \psi(q)=\sum_{j=0}^{\infty} q^{j(j+1)/2}\quad\text{and}\quad X(q)=\sum_{j=-\infty}^{\infty} q^{3j^2+2j},$$

and for any positive integer k we define φ_k, ψ_k and X_k by

$$\varphi_k=\varphi(q^k),\quad \psi_k=\psi(q^k)\quad\text{and}\quad X_k=X(q^k). \tag{2.1}$$

For positive integers a, b, c, d and e and for any nonnegative integer n let $r_{(a,b,c,d,e)}(n)$ denote the number of solutions in integers of

$$ax_1^2+bx_2^2+cx_3^2+dx_4^2+ex_5^2=n.$$

Clearly,

$$\sum_{m=0}^{\infty} r_{(a,b,c,d,e)}(m)q^m=\varphi(q^a)\varphi(q^b)\varphi(q^c)\varphi(q^d)\varphi(q^e)=\varphi_a\varphi_b\varphi_c\varphi_d\varphi_e.$$

We will need

LEMMA 2.1. *Let φ_k, ψ_k and X_k be the theta functions defined by* (2.1). *Then*

$$\varphi_1\psi_2=\psi_1^2.$$

The following dissections into even and odd parts hold:

$$\varphi_1=\varphi_4+2q\psi_8,\qquad \psi_1^2=\varphi_4\psi_2+2q\psi_2\psi_8,$$
$$\varphi_1^2=\varphi_2^2+4q\psi_4^2,\qquad \psi_1\psi_3=\varphi_6\psi_4+q\varphi_2\psi_{12}.$$

Moreover,

$$\varphi_1=\varphi_9+2qX_3 \tag{2.2}$$

and

$$8q\varphi_3X_1^3=\varphi_1^4-\varphi_3^4. \tag{2.3}$$

Proof. The first result is given in [5, p. 40]. The dissections into even and odd parts are (i), (ii), (xiii) and (xxxiii), respectively, in [9]. The last two identities are (v) and (vi) in [9]. ∎

In addition to the Legendre symbols $\left(\frac{2}{p}\right)$, $\left(\frac{3}{p}\right)$ and $\left(\frac{6}{p}\right)$ given in Theorems 1.2–1.4, we shall also need values of Jacobi symbols, defined for any positive integer m by

$$\left(\frac{-2}{m}\right)=\begin{cases}1 & \text{if } m\equiv 1 \text{ or } 3 \pmod 8,\\ -1 & \text{if } m\equiv 5 \text{ or } 7 \pmod 8,\\ 0 & \text{otherwise,}\end{cases}$$

$$\left(\frac{-3}{m}\right)=\begin{cases}1 & \text{if } m\equiv 1 \pmod 3,\\ -1 & \text{if } m\equiv 2 \pmod 3,\\ 0 & \text{otherwise,}\end{cases}$$

$$\left(\frac{-4}{m}\right)=\begin{cases}1 & \text{if } m\equiv 1 \pmod 4,\\ -1 & \text{if } m\equiv 3 \pmod 4,\\ 0 & \text{otherwise,}\end{cases}$$

$$\left(\frac{6}{m}\right)=\left(\frac{-2}{m}\right)\left(\frac{-3}{m}\right)=\begin{cases}1 & \text{if } p\equiv 1,\ 5,\ 19 \text{ or } 23 \pmod{24},\\ -1 & \text{if } p\equiv 7,\ 11,\ 13 \text{ or } 17 \pmod{24},\\ 0 & \text{otherwise,}\end{cases}$$

$$\left(\frac{12}{m}\right)=\left(\frac{-3}{m}\right)\left(\frac{-4}{m}\right)=\begin{cases}1 & \text{if } m\equiv 1 \text{ or } 11 \pmod{12},\\ -1 & \text{if } m\equiv 5 \text{ or } 7 \pmod{12},\\ 0 & \text{otherwise.}\end{cases}$$

3. Proof of Theorem 1.3: Part 1. In this section we will establish the parts of the formula in Theorem 1.3 that involve the primes 2 and 3. We begin with the prime 2.

LEMMA 3.1. *Fix an odd integer j. For any nonnegative integer k let*

$$f(k)=r_{(1,1,1,1,3)}(2^{2k}j^2).$$

Then

$$f(k+2)=9f(k+1)-8f(k), \tag{3.1}$$

$$f(1)=5f(0). \tag{3.2}$$

Hence,

$$f(k)=\left(\frac{2^{3k+2}+3}{2^3-1}\right)f(0). \tag{3.3}$$

Proof. By Lemma 2.1 we have

$$\varphi_1^4\varphi_3=(\varphi_4+2q\psi_8)^4(\varphi_{12}+2q^3\psi_{24}).$$

Expanding, extracting the terms of the form q^{4n}, and then replacing q^4 with q we deduce

$$\sum_{n=0}^{\infty} r_{(1,1,1,1,3)}(4n)q^n=\varphi_1^4\varphi_3+16q\varphi_1^3\psi_2\psi_6+16q\psi_2^4\varphi_3, \tag{3.4}$$

while a similar process applied to the terms of the form q^{4n+1} leads to

$$\sum_{n=0}^{\infty} r_{(1,1,1,1,3)}(4n+1)q^n = 8(\varphi_1^3\psi_2\varphi_3 + 6q\varphi_1^2\psi_2^2\psi_6). \tag{3.5}$$

Next, applying Lemma 2.1 to (3.4) gives

$$\begin{aligned}\sum_{n=0}^{\infty} r_{(1,1,1,1,3)}(4n)q^n = {} & (\varphi_4 + 2q\psi_8)^4(\varphi_{12} + 2q^3\psi_{24}) \\ & + 16q(\varphi_4 + 2q\psi_8)^3(\varphi_{12}\psi_8 + q^2\varphi_4\psi_{24}) \\ & + 16q(\varphi_8\psi_4 + 2q^2\psi_4\psi_{16})^2(\varphi_{12} + 2q^3\psi_{24}).\end{aligned} \tag{3.6}$$

On extracting the terms of the form q^{4n} and then replacing q^4 with q we find that

$$\sum_{n=0}^{\infty} r_{(1,1,1,1,3)}(16n)q^n = \varphi_1^4\varphi_3 + 112q\varphi_1^3\psi_2\psi_6 + 144q\psi_2^4\varphi_3 + 32q\psi_1^2\psi_6(\varphi_2^2 + 4q\psi_4^2).$$

Similarly, extracting the terms of the form q^{4n+1} in (3.6) and then replacing q^4 with q leads to

$$\sum_{n=0}^{\infty} r_{(1,1,1,1,3)}(16n+4)q^n = 24\varphi_1^3\psi_2\varphi_3 + 240q\varphi_1^2\psi_2^2\psi_6 + 16\psi_1^2\varphi_3(\varphi_2^2 + 4q\psi_4^2).$$

On applying Lemma 2.1, the two equations above simplify to

$$\sum_{n=0}^{\infty} r_{(1,1,1,1,3)}(16n)q^n = \varphi_1^4\varphi_3 + 144q\varphi_1^3\psi_2\psi_6 + 144q\psi_2^4\varphi_3 \tag{3.7}$$

and

$$\sum_{n=0}^{\infty} r_{(1,1,1,1,3)}(16n+4)q^n = 40(\varphi_1^3\psi_2\varphi_3 + 6q\varphi_1^2\psi_2^2\psi_6), \tag{3.8}$$

respectively. From (3.4) and (3.7) we deduce that

$$\sum_{n=0}^{\infty} r_{(1,1,1,1,3)}(16n)q^n = 9\sum_{n=0}^{\infty} r_{(1,1,1,1,3)}(4n)q^n - 8\varphi_1^4\varphi_3$$

and this implies (3.1). From (3.5) and (3.8) we deduce that

$$r_{(1,1,1,1,3)}(16n+4) = 5r_{(1,1,1,1,3)}(4n+1)$$

and this implies (3.2). Finally, (3.3) follows from (3.1) and (3.2). ∎

In preparation for the result involving the prime 3 we will need:

LEMMA 3.2. *The following identities hold:*

$$\sum_{n=0}^{\infty} r_{(1,1,1,1,3)}(3n)q^n = 4\varphi_1^5 - 3\varphi_1\varphi_3^4, \tag{3.9}$$

$$\sum_{n=0}^{\infty} r_{(1,1,1,1,3)}(9n)q^n = 37\varphi_1^4\varphi_3 - 36\varphi_3^5, \tag{3.10}$$

$$\sum_{n=0}^{\infty} r_{(1,1,1,1,3)}(27n)q^n = 112\varphi_1^5 - 111\varphi_1\varphi_3^4, \tag{3.11}$$

$$\sum_{n=0}^{\infty} r_{(1,1,1,1,3)}(81n)q^n = 1009\varphi_1^4\varphi_3 - 1008\varphi_3^5. \tag{3.12}$$

Proof. We utilize the last two parts of Lemma 2.1. By (2.2) we have

$$\sum_{n=0}^{\infty} r_{(1,1,1,1,3)}(n)q^n = \varphi_1^4\varphi_3 = (\varphi_9 + 2qX_3)^4\varphi_3.$$

On expanding, extracting the terms of the form q^{3n} and then replacing q^3 with q we get

$$\sum_{n=0}^{\infty} r_{(1,1,1,1,3)}(3n)q^n = \varphi_3^4\varphi_1 + 32q\varphi_3X_1^3\varphi_1 = 4\varphi_1^5 - 3\varphi_1\varphi_3^4,$$

where (2.3) was used for the last step. This proves the first result. The others can be obtained in succession by applying the same procedure; we omit the details. ∎

LEMMA 3.3. *Fix an integer j that is not divisible by* 3. *For any nonnegative integer k let*

$$g(k) = r_{(1,1,1,1,3)}(3^{2k}j^2).$$

Then

$$g(k+2) = 28g(k+1) - 27g(k), \tag{3.13}$$

$$g(1) = 37g(0). \tag{3.14}$$

Hence,

$$g(k) = \left(\frac{36\cdot 3^{3k} - 10}{3^3 - 1}\right)g(0). \tag{3.15}$$

Proof. From (3.10) and (3.12) we have

$$\sum_{n=0}^{\infty} r_{(1,1,1,1,3)}(81n)q^n = 28\sum_{n=0}^{\infty} r_{(1,1,1,1,3)}(9n)q^n - 27\varphi_1^4\varphi_3$$

and this implies (3.13). Next, by (3.10) we have

$$\sum_{n=0}^{\infty} r_{(1,1,1,1,3)}(9n)q^n = 37\sum_{n=0}^{\infty} r_{(1,1,1,1,3)}(n)q^n - 36\varphi_3^5.$$

Equating the coefficients of q^{3n+1} on both sides we deduce that

$$r_{(1,1,1,1,3)}(27n+9) = 37r_{(1,1,1,1,3)}(3n+1)$$

and this implies (3.14). Finally, (3.15) follows from (3.13) and (3.14). ■

From Lemmas 3.1 and 3.3 we immediately deduce:

PROPOSITION 3.4. *Let n be a positive integer with prime factorization*

$$n = \prod_p p^{\lambda_p} = 2^{\lambda_2}3^{\lambda_3}m \quad \text{where} \quad m = \prod_{p\geq 5} p^{\lambda_p}.$$

Then the number of solutions in integers of

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + 3x_5^2 = n^2$$

is given by

$$r_{(1,1,1,1,3)}(n^2) = \left(\frac{2^{3\lambda_2+2}+3}{2^3-1}\right)\left(\frac{36\cdot 3^{3\lambda_3}-10}{3^3-1}\right)r_{(1,1,1,1,3)}(m^2).$$

It remains to determine the value of $r_{(1,1,1,1,3)}(m^2)$ in the case that $\gcd(m,6)=1$. This will be done in the next section.

4. Proof of Theorem 1.3: Part 2. In this section we will prove:

PROPOSITION 4.1. *Let m be a positive integer that is relatively prime to* 6 *and has prime factorization*

$$m = \prod_{p\geq 5} p^{\lambda_p}.$$

Then

$$r_{(1,1,1,1,3)}(m^2) = 8\prod_{p\geq 5}\left(\frac{p^{3\lambda_p+3}-1}{p^3-1} - p\left(\frac{3}{p}\right)\frac{p^{3\lambda_p}-1}{p^3-1}\right).$$

Note that Propositions 3.4 and 4.1 immediately imply Theorem 1.3.

In order to prove Proposition 4.1 we will need some background information and several lemmas. Let f_1, f_2, f_3, f_4 be the infinite products defined by

$$f_1(q) = q\prod_{j=1}^{\infty}\frac{(1-q^{2j})^2(1-q^{3j})^2(1-q^{4j})(1-q^{12j})}{(1-q^j)^2},$$

$$f_2(q) = q\prod_{j=1}^{\infty}\frac{(1-q^{j})^2(1-q^{4j})(1-q^{6j})^2(1-q^{12j})}{(1-q^{3j})^2},$$

$$f_3(q) = q\prod_{j=1}^{\infty}\frac{(1-q^{j})(1-q^{2j})^2(1-q^{3j})(1-q^{12j})^2}{(1-q^{4j})^2},$$

$$f_4(q) = \prod_{j=1}^{\infty}\frac{(1-q^{j})(1-q^{3j})(1-q^{4j})^2(1-q^{6j})^2}{(1-q^{12j})^2}.$$

Let their series expansions be given by

$$f_1(q)=\sum_{n=0}^{\infty}a_1(n)q^n,\qquad f_2(q)=\sum_{n=0}^{\infty}a_2(n)q^n,\qquad f_3(q)=\sum_{n=0}^{\infty}a_3(n)q^n \tag{4.1}$$

and

$$f_4(q) = -\sum_{n=0}^{\infty}a_4(n)q^n, \tag{4.2}$$

where

$$a_1(0)=a_2(0)=a_3(0)=0 \quad\text{and}\quad a_4(0)=-1.$$

The reason for the negative sign in the definition of $a_4(n)$ in (4.2) is that Lemma 4.6 below will hold for this sequence; see also (4.6).

For $j\in\{1,2,3,4\}$, define $A_j(n)$ by

$$\sum_{n=0}^{\infty}A_j(n)q^n = \left(\sum_{n=0}^{\infty}a_j(n)q^n\right)^2.$$

The next three lemmas involve the functions f_1, f_2, f_3, f_4 and their series expansions.

LEMMA 4.2 ([8, Cor. 4.6]). *The following identity holds:*

$$\varphi^3(q)\varphi(q^3) = 6f_1(q)-2f_2(q)+3f_3(q)+f_4(q).$$

LEMMA 4.3 ([1, Theorem 3.1] or [8, Theorem 4.1 and Remark 4.2]). *Let n be any positive integer with prime factorization*

$$n = 2^{\lambda_2}3^{\lambda_3}m \quad \textit{where} \quad m=\prod_{p\ge 5}p^{\lambda_p}.$$

Then

$$a_1(n) = 2^{\lambda_2}3^{\lambda_3}a_1(m), \tag{4.3}$$

$$a_2(n) = 2^{\lambda_2}(-1)^{\lambda_2+\lambda_3}\left(\frac{-3}{m}\right)a_1(m), \tag{4.4}$$

$$a_3(n) = 3^{\lambda_3}(-1)^{\lambda_2+\lambda_3}\left(\frac{-4}{m}\right)a_1(m), \tag{4.5}$$

$$a_4(n) = \left(\frac{12}{m}\right)a_1(m), \tag{4.6}$$

where

$$a_1(m) = \sum_{d\mid m}\frac{m}{d}\left(\frac{12}{d}\right) = \prod_{p\ge 5}\frac{p^{\lambda_p+1} - \left(\frac{12}{p}\right)^{\lambda_p+1}}{p - \left(\frac{12}{p}\right)}. \tag{4.7}$$

Lemma 4.4. *For any positive integer k let*

$$Q_k = 1 + 240\sum_{j=1}^{\infty}\frac{j^3q^{jk}}{1-q^{jk}}.$$

Then

$$6f_1^2 - 2f_2^2 + 3f_3^2 - f_4^2 = \tfrac{1}{300}(Q_1 + Q_2 + 9Q_3 - 32Q_4 + 9Q_6 - 288Q_{12}) + \tfrac{6}{5}e(q) \tag{4.8}$$

where

$$e(q) = q\prod_{j=1}^{\infty}(1-q^j)^2(1-q^{2j})^2(1-q^{3j})^2(1-q^{6j})^2 + 2q^2\prod_{j=1}^{\infty}(1-q^{2j})^2(1-q^{4j})^2(1-q^{6j})^2(1-q^{12j})^2$$

and $e(q)$ is an odd function.

Proof. The identity (4.8) follows from the two-variable parameterizations in [2, Sect. 3]. Since

$$\prod_{j=1}^{\infty}(1-(-q)^j) = \prod_{j=1}^{\infty}\frac{(1-q^{2j})^3}{(1-q^j)(1-q^{4j})},$$

the identity $e(q) + e(-q) = 0$ to be shown is

$$q\prod_{j=1}^{\infty}(1-q^j)^2(1-q^{2j})^2(1-q^{3j})^2(1-q^{6j})^2$$

$$- q\prod_{j=1}^{\infty}\frac{(1-q^{2j})^8(1-q^{6j})^8}{(1-q^{j})^2(1-q^{3j})^2(1-q^{4j})^2(1-q^{12j})^2}$$
$$+ 4q^2\prod_{j=1}^{\infty}(1-q^{2j})^2(1-q^{4j})^2(1-q^{6j})^2(1-q^{12j})^2 = 0.$$

This follows from the two-variable parameterizations in [2, Sect. 3]. ∎

LEMMA 4.5. *Let m be a positive integer relatively prime to* 6 *with prime factorization*

$$m = \prod_{p\geq 5} p^{\lambda_p}.$$

Let $c(m)$ be the coefficient of q^{2m} in

$$6f_1^2 - 2f_2^2 + 3f_3^2 - f_4^2.$$

Then

$$c(m) = 8\sum_{d|m} d^3 = 8\prod_{p\geq 5}\frac{p^{3\lambda_p+3}-1}{p^3-1}.$$

Proof. This follows immediately from Lemma 4.4, for the only term on the right hand side of (4.8) that contains terms of the form q^{12j+2} or q^{12j+10} is $(Q_1+Q_2)/300$. ∎

LEMMA 4.6. *Let $j \in \{1,2,3,4\}$ and let $a_j(n)$ be defined by* (4.1) *and* (4.2), *or equivalently, by* (4.3)–(4.7). *For any nonnegative integer n and any prime p we have*

$$a_j(pn) = a_j(p)a_j(n) - \chi(p)a_j(n/p) \tag{4.9}$$

where χ is the completely multiplicative function defined on the positive integers by

$$\chi(r) = r\left(\frac{12}{r}\right) \tag{4.10}$$

and $a_j(x)$ is defined to be 0 *if x is not an integer.*

Proof. The result follows from (4.3)–(4.7) on checking each of the four functions separately, and considering the cases $p=2$, $p=3$ and $p\geq 5$ one at a time. The details are straightforward and we omit them. ∎

The next result is due to Hurwitz.

LEMMA 4.7 ([13, Sect. 2]). *Suppose that $a(n)$ is a function, defined for all nonnegative integers n, that has the property*

$$a(pn) = a(p)a(n) - \chi(p)a\left(\frac{n}{p}\right) \tag{4.11}$$

for all primes p, where χ is a completely multiplicative function. Then the coefficient of q^{n^2} in

$$\Big(\sum_{j=-\infty}^{\infty} q^{j^2}\Big)\Big(\sum_{k=0}^{\infty} a(k)q^k\Big)$$

is equal to

$$\sum_{r=1}^{\infty} A\Big(\frac{2n}{r}\Big)\chi(r)\mu(r)$$

where μ is the Möbius function, $A(n)$ is defined by

$$\sum_{n=0}^{\infty} A(n)q^n = \Big(\sum_{k=0}^{\infty} a(k)q^k\Big)^2$$

and $A(x)$ is defined to be 0 if x is not a nonnegative integer.

Proof of Proposition 4.1. Let $[q^k]f(q)$ denote the coefficient of q^k in the Taylor expansion of $f(q)$. In this notation,

$$r_{(1,1,1,1,3)}(m^2) = [q^{m^2}](\varphi_1^4\varphi_3).$$

By Lemma 4.2 and (4.1) and (4.2), this is

$$\begin{aligned} r_{(1,1,1,1,3)}(m^2) &= [q^{m^2}]\big(\varphi(q)(6f_1(q) - 2f_2(q) + 3f_3(q) + f_4(q))\big) \\ &= 6[q^{m^2}]\Big(\varphi(q)\sum_{j=0}^{\infty} a_1(j)q^j\Big) - 2[q^{m^2}]\Big(\varphi(q)\sum_{j=0}^{\infty} a_2(j)q^j\Big) \\ &\quad + 3[q^{m^2}]\Big(\varphi(q)\sum_{j=0}^{\infty} a_3(j)q^j\Big) - [q^{m^2}]\Big(\varphi(q)\sum_{j=0}^{\infty} a_4(j)q^j\Big). \end{aligned}$$

By Lemmas 4.6 and 4.7 this is equivalent to

$$\begin{aligned} r_{(1,1,1,1,3)}(m^2) &= 6\sum_{r=1}^{\infty} A_1\Big(\frac{2m}{r}\Big)\chi(r)\mu(r) - 2\sum_{r=1}^{\infty} A_2\Big(\frac{2m}{r}\Big)\chi(r)\mu(r) \\ &\quad + 3\sum_{r=1}^{\infty} A_3\Big(\frac{2m}{r}\Big)\chi(r)\mu(r) - \sum_{r=1}^{\infty} A_4\Big(\frac{2m}{r}\Big)\chi(r)\mu(r) \\ &= \sum_{r=1}^{\infty} [q^{2m/r}](6f_1^2 - 2f_2^2 + 3f_3^2 - f_4^2)\chi(r)\mu(r), \end{aligned}$$

where $\chi(r)$ is the completely multiplicative function defined by (4.10). Since $\chi(r) = 0$ if r is even, the last sum is over odd r only. Moreover, since m is relatively prime to 6, we may apply Lemma 4.5 to deduce that

$$\begin{aligned} r_{(1,1,1,1,3)}(m^2) &= \sum_{r=1}^{\infty} c(m/r)\chi(r)\mu(r) = c(m)\sum_{r\mid m}\frac{c(m/r)}{c(m)}\chi(r)\mu(r) \\ &= c(m)\prod_{p\geq 5}\left(1-\chi(p)\frac{c(m/p)}{c(m)}\right) \\ &= \left(8\prod_{p\geq 5}\frac{p^{3\lambda_p+3}-1}{p^3-1}\right)\left(\prod_{p\geq 5}\left(1-p\left(\frac{12}{p}\right)\frac{p^{3\lambda_p}-1}{p^{3\lambda_p+3}-1}\right)\right) \\ &= 8\prod_{p\geq 5}\left(\frac{p^{3\lambda_p+3}-1}{p^3-1}-p\left(\frac{3}{p}\right)\frac{p^{3\lambda_p}-1}{p^3-1}\right). \ \blacksquare \end{aligned}$$

5. Proof of Theorem 1.2. In this section we will outline the proof of Theorem 1.2. We require four lemmas:

LEMMA 5.1 ([7, Theorem 3.2]). *The following identity holds:*

$$\varphi^3(q)\varphi(q^2) = \varphi(-q)\varphi(-q^2)\varphi^2(-q^4) + 8q\psi^2(q)\psi(q^2)\psi(q^4).$$

LEMMA 5.2. *The following series expansion holds:*

$$q\psi^2(q)\psi(q^2)\psi(q^4) = \sum_{j=1}^{\infty} a(j)q^j$$

where

$$a(j) = 2^{\lambda_2}\prod_{p\geq 3}\frac{p^{\lambda_p+1}-\left(\frac{2}{p}\right)^{\lambda_p+1}}{p-\left(\frac{2}{p}\right)} \quad \textit{and} \quad j=\prod_p p^{\lambda_p}.$$

Proof. This follows from [7, Lemma 4.1]. ■

LEMMA 5.3 ([5, pp. 36, 40]). *The following theta function identities hold:*

$$\varphi^2(q)-\varphi^2(-q) = 16q\psi^2(q^4),$$

$$\varphi(-q) = \prod_{j=1}^{\infty}\frac{(1-q^j)^2}{1-q^{2j}}, \qquad \psi(q) = \prod_{j=1}^{\infty}\frac{(1-q^{2j})^2}{1-q^j}.$$

LEMMA 5.4. *Let j be an odd positive integer with prime factorization*

$$j = \prod_{p\geq 3} p^{\lambda_p}.$$

Denote the coefficient of q^{2j} in $q\psi^8(q)$ by $c(j)$. Then

$$c(j) = 8\sum_{d\mid j} d^3 = 8\prod_{p\geq 3}\frac{p^{3\lambda_p+3}-1}{p^3-1}.$$

Proof. This follows from the sum of 8 triangular numbers formula; see e.g. [5, p. 139]. ■

We are now ready for

Proof of Theorem 1.2. Let m be an odd integer. For any nonnegative integer k let

$$f(k) = r_{(1,1,1,1,2)}(2^{2k}m^2).$$

By the methods in Section 3 we may deduce that

$$f(k+2) = 9f(k+1) - 8f(k) \quad \text{for } k \geq 1,$$
$$f(1) = 9f(0), \quad f(2) = 57f(0),$$

and it follows that

$$f(k) = 3\left(\frac{2^{3k+1}+5}{2^3-1}\right)f(0) \quad \text{for } k \geq 1. \tag{5.1}$$

It remains to determine $f(0)$, that is, $r_{(1,1,1,1,2)}(m^2)$. By Lemma 5.1 we have

$$\begin{aligned} r_{(1,1,1,1,2)}&(m^2) \\ &= [q^{m^2}](\varphi^4(q)\varphi(q^2)) \\ &= [q^{m^2}]\big(\varphi(q)\varphi(-q)\varphi(-q^2)\varphi^2(-q^4)\big) + 8[q^{m^2}]\big(q\varphi(q)\psi^2(q)\psi(q^2)\psi(q^4)\big). \end{aligned}$$

The first term on the right hand side is zero because m is odd and because $\varphi(q)\varphi(-q)\varphi(-q^2)\varphi^2(-q^4)$ is an even function of q. Therefore,

$$r_{(1,1,1,1,2)}(m^2) = 8[q^{m^2}]\Big(\varphi(q)\sum_{j=1}^{\infty}a(j)q^j\Big)$$

where the value of $a(j)$ is given by Lemma 5.2. By Lemma 4.7 with $\chi(p) = p\big(\frac{2}{p}\big)$ we deduce that

$$r_{(1,1,1,1,2)}(m^2) = 8\sum_{r=1}^{\infty}[q^{2m/r}](q^2\psi^4(q)\psi^2(q^2)\psi^2(q^4))\chi(r)\mu(r). \tag{5.2}$$

Since m is odd and $\chi(r) = 0$ for r even, the sum in (5.2) is over odd r only. Hence, we are only concerned with even powers of q. By Lemma 5.3,

$$\begin{aligned} 8q^2\psi^4(q)\psi^2(q^2)\psi^2(q^4) &= 8q\psi^4(q)\psi^2(q^2)(\varphi^2(q) - \varphi^2(-q)) \\ &= q\psi^8(q) - q\prod_{j=1}^{\infty}(1-q^j)^4(1-q^{2j})^4. \end{aligned}$$

Therefore, from (5.2) we deduce that

$$r_{(1,1,1,1,2)}(m^2) = \sum_{r=1}^{\infty}[q^{2m/r}](q\psi^8(q))\chi(r)\mu(r) = \sum_{r|m}c(m/r)\chi(r)\mu(r)$$

where the value of $c(j)$ is given by Lemma 5.4. The remainder of the proof follows the final steps of the proof of Theorem 1.3 at the end of Section 4, but with $\left(\frac{2}{p}\right)$ in place of $\left(\frac{12}{p}\right)$, and the final result is

$$r_{(1,1,1,1,2)}(m^2) = \prod_{p\ge 3}\left(\frac{p^{3\lambda_p+3}-1}{p^3-1} - p\left(\frac{2}{p}\right)\frac{p^{3\lambda_p}-1}{p^3-1}\right).$$

On combining this with (5.1) we complete the proof of Theorem 1.2. ∎

6. Proof of Theorem 1.4. In this section we will outline the proof of Theorem 1.4. The proof is similar to the proof of Theorem 1.3. The main difference is that whereas the functions f_1, f_2, f_3, f_4 used in Section 4 are single infinite products, the corresponding functions we shall encounter in this section are sums of four infinite products. We rely on some recent results of [4].

The two lemmas below may be deduced by the methods of Section 3.

LEMMA 6.1. *Fix an odd integer j. For any nonnegative integer k let*

$$f(k) = r_{(1,1,1,2,3)}(2^{2k}j^2).$$

Then

$$f(k+3) = 9f(k+2) - 8f(k+1), \quad f(1) = 7f(0), \quad f(2) = 71f(0).$$

Hence,

$$f(k) = \left|\frac{2^{3k+3}-15}{2^3-1}\right| f(0). \tag{6.1}$$

LEMMA 6.2. *Fix an integer j that is not divisible by 3. For any nonnegative integer k let*

$$g(k) = r_{(1,1,1,2,3)}(3^{2k}j^2).$$

Then

$$g(k+2) = 28g(k+1) - 27g(k) \quad \text{and} \quad g(1) = 19g(0).$$

Hence,

$$g(k) = 2\left(\frac{3^{3k+2}+4}{3^3-1}\right) g(0). \tag{6.2}$$

From Lemmas 6.1 and 6.2 we immediately deduce:

PROPOSITION 6.3. *Let n be a positive integer with prime factorization*

$$n = \prod_p p^{\lambda_p} = 2^{\lambda_2}3^{\lambda_3}m \quad \text{where} \quad m = \prod_{p\ge 5} p^{\lambda_p}.$$

Then the number of solutions in integers of

$$x_1^2 + x_2^2 + x_3^2 + 2x_4^2 + 3x_5^2 = n^2$$

is given by

$$r_{(1,1,1,2,3)}(n^2) = 2\left|\frac{2^{3\lambda_2+3}-15}{2^3-1}\right|\left(\frac{3^{3\lambda_3+2}+4}{3^3-1}\right)r_{(1,1,1,2,3)}(m^2).$$

It remains to determine $r_{(1,1,1,2,3)}(m^2)$ in the case that $\gcd(m,6)=1$.

PROPOSITION 6.4. *Let m be a positive integer relatively prime to* 6 *with prime factorization*

$$m = \prod_{p\geq 5} p^{\lambda_p}.$$

Then

$$r_{(1,1,1,2,3)}(m^2) = 6\prod_{p\geq 5}\left(\frac{p^{3\lambda_p+3}-1}{p^3-1} - p\left(\frac{6}{p}\right)\frac{p^{3\lambda_p}-1}{p^3-1}\right).$$

Note that Propositions 6.3 and 6.4 immediately imply Theorem 1.4.

To prove Proposition 6.4, we will need some relevant lemmas. Let

$$\begin{aligned}
f_1(q) &= \tfrac{1}{4}\varphi_1^2\varphi_2\varphi_3 + \tfrac{1}{4}\varphi_1\varphi_2^2\varphi_6 - \tfrac{1}{4}\varphi_1\varphi_3^2\varphi_6 - \tfrac{1}{4}\varphi_2\varphi_3\varphi_6^2,\\
f_2(q) &= -\tfrac{1}{2}\varphi_1^2\varphi_2\varphi_3 + \varphi_1\varphi_2^2\varphi_6 + \tfrac{1}{2}\varphi_1\varphi_3^2\varphi_6 - \varphi_2\varphi_3\varphi_6^2,\\
f_3(q) &= -\tfrac{1}{4}\varphi_1^2\varphi_2\varphi_3 + \tfrac{1}{4}\varphi_1\varphi_2^2\varphi_6 + \tfrac{3}{4}\varphi_1\varphi_3^2\varphi_6 - \tfrac{3}{4}\varphi_2\varphi_3\varphi_6^2,\\
f_4(q) &= \tfrac{1}{2}\varphi_1^2\varphi_2\varphi_3 + \varphi_1\varphi_2^2\varphi_6 - \tfrac{3}{2}\varphi_1\varphi_3^2\varphi_6 - 3\varphi_2\varphi_3\varphi_6^2,
\end{aligned}$$

where φ_k is given by (2.1). Let their series expansions be given by

$$f_1(q) = \sum_{n=0}^{\infty} a_1(n)q^n, \quad f_2(q) = \sum_{n=0}^{\infty} a_2(n)q^n, \tag{6.3}$$

$$f_3(q) = \sum_{n=0}^{\infty} a_3(n)q^n \quad f_4(q) = \sum_{n=0}^{\infty} a_4(n)q^n, \tag{6.4}$$

where

$$a_1(0) = a_2(0) = a_3(0) = 0 \quad \text{and} \quad a_4(0) = -3.$$

For $j \in \{1,2,3,4\}$, define $A_j(n)$ by

$$\sum_{n=0}^{\infty} A_j(n)q^n = \Big(\sum_{n=0}^{\infty} a_j(n)q^n\Big)^2.$$

LEMMA 6.5 ([4, Th. 4.1 and Section 4]). *The following identity holds:*

$$\varphi^2(q)\varphi(q^2)\varphi(q^3) = 4f_1(q) - f_2(q) + \tfrac{4}{3}f_3(q) - \tfrac{1}{3}f_4(q).$$

LEMMA 6.6 ([4, Section 4]). *Let n be any positive integer with prime factorization*

$$n = 2^{\lambda_2}3^{\lambda_3}m \quad \textit{where} \quad m = \prod_{p\geq 5} p^{\lambda_p}.$$

Then

$$a_1(n) = 2^{\lambda_2}3^{\lambda_3}a_1(m), \tag{6.5}$$

$$a_2(n) = (-1)^{\lambda_2}3^{\lambda_3}\left(\frac{-2}{m}\right)a_1(m), \tag{6.6}$$

$$a_3(n) = (-1)^{\lambda_2}2^{\lambda_2}\left(\frac{-3}{m}\right)a_1(m), \tag{6.7}$$

$$a_4(n) = \left(\frac{6}{m}\right)a_1(m), \tag{6.8}$$

where

$$a_1(m) = \sum_{d|m}\frac{m}{d}\left(\frac{6}{d}\right) = \prod_{p\geq 5}\frac{p^{\lambda_p+1} - \left(\frac{6}{p}\right)^{\lambda_p+1}}{p - \left(\frac{6}{p}\right)}. \tag{6.9}$$

LEMMA 6.7. *Define*

$$\begin{aligned} e(q) = {} & (4f_1^2 - f_2^2 + \tfrac{4}{3}f_3^2 - \tfrac{1}{3}f_4^2) \\ & - \left(\tfrac{1}{40}Q_2 - \tfrac{1}{20}Q_4 - \tfrac{9}{40}Q_6 + \tfrac{2}{5}Q_8 + \tfrac{9}{20}Q_{12} - \tfrac{18}{5}Q_{24}\right), \end{aligned}$$

where Q_k is defined in Lemma 4.4. Then $e(q)$ is an odd function.

Proof. The equality $e(q) + e(-q) = 0$ follows from the two-variable parameterizations of $\varphi(q)$, $\varphi(q^2)$, $\varphi(q^3)$, $\varphi(q^6)$, $\varphi(-q)$, $\varphi(-q^3)$ and Q_k for $k \in \{2, 4, 6, 8, 12, 24\}$ in [3, Lemma 3.1] and [1, Theorems 2.4 and 2.5]. ∎

LEMMA 6.8. *Let m be a positive integer relatively prime to 6, with prime factorization*

$$m = \prod_{p\geq 5} p^{\lambda_p}.$$

Let $c(m)$ be the coefficient of q^{2m} in

$$4f_1^2 - f_2^2 + \tfrac{4}{3}f_3^2 - \tfrac{1}{3}f_4^2.$$

Then

$$c(m) = 6\sum_{d|m} d^3 = 6\prod_{p\geq 5}\frac{p^{3\lambda_p+3} - 1}{p^3 - 1}.$$

Proof. This follows immediately from Lemma 6.7, for the only term in $\frac{1}{40}Q_2 - \frac{1}{20}Q_4 - \frac{9}{40}Q_6 + \frac{2}{5}Q_8 + \frac{9}{20}Q_{12} - \frac{18}{5}Q_{24}$ that contains terms of the form q^{12j+2} or q^{12j+10} is $Q_2/40$. ∎

LEMMA 6.9. *Let $j \in \{1, 2, 3, 4\}$ and let $a_j(n)$ be defined by* (6.3) *and* (6.4), *or equivalently, by* (6.5)–(6.9). *For any nonnegative integer n and any prime p we have*

$$a_j(pn) = a_j(p)a_j(n) - \chi(p)a_j(n/p) \tag{6.10}$$

where χ is the completely multiplicative function defined on the positive integers by

$$\chi(r) = r\left(\frac{6}{r}\right) \tag{6.11}$$

and $a_j(x)$ is defined to be 0 *if x is not an integer.*

Proof. The result follows from (6.5)–(6.9) on checking each of the four functions separately, and considering the cases $p = 2$, $p = 3$ and $p \geq 5$ one at a time. The details are straightforward and we omit them. ∎

Proof of Proposition 6.4. Let $[q^k]f(q)$ denote the coefficient of q^k in the Taylor expansion of $f(q)$. In this notation,

$$r_{(1,1,1,2,3)}(m^2) = [q^{m^2}](\varphi_1^3\varphi_2\varphi_3).$$

By Lemma 6.5 and (6.3) and (6.4), we have

$$\begin{aligned} r_{(1,1,1,2,3)}(m^2) &= [q^{m^2}](\varphi(q)(4f_1(q) - f_2(q) + \tfrac{4}{3}f_3(q) - \tfrac{1}{3}f_4(q))) \\ &= 4[q^{m^2}]\Big(\varphi(q)\sum_{j=0}^{\infty} a_1(j)q^j\Big) - [q^{m^2}]\Big(\varphi(q)\sum_{j=0}^{\infty} a_2(j)q^j\Big) \\ &\quad + \tfrac{4}{3}[q^{m^2}]\Big(\varphi(q)\sum_{j=0}^{\infty} a_3(j)q^j\Big) - \tfrac{1}{3}[q^{m^2}]\Big(\varphi(q)\sum_{j=0}^{\infty} a_4(j)q^j\Big). \end{aligned}$$

By Lemmas 4.7 and 6.9 this is equivalent to

$$\begin{aligned} r_{(1,1,1,2,3)}(m^2) &= 4\sum_{r=1}^{\infty} A_1\left(\frac{2m}{r}\right)\chi(r)\mu(r) - \sum_{r=1}^{\infty} A_2\left(\frac{2m}{r}\right)\chi(r)\mu(r) \\ &\quad + \frac{4}{3}\sum_{r=1}^{\infty} A_3\left(\frac{2m}{r}\right)\chi(r)\mu(r) - \frac{1}{3}\sum_{r=1}^{\infty} A_4\left(\frac{2m}{r}\right)\chi(r)\mu(r) \\ &= \sum_{r=1}^{\infty}[q^{2m/r}](4f_1^2 - f_2^2 + \tfrac{4}{3}f_3^2 - \tfrac{1}{3}f_4^2)\chi(r)\mu(r), \end{aligned}$$

where $\chi(r)$ is the completely multiplicative function defined by (6.11). Since $\chi(r) = 0$ if r is even, the last sum is over odd r only. Moreover, since m is relatively prime to 6, we may apply Lemma 6.8 to deduce that

$$\begin{aligned} r_{(1,1,1,2,3)}(m^2) &= \sum_{r=1}^{\infty} c(m/r)\chi(r)\mu(r) = c(m)\sum_{r\mid m}\frac{c(m/r)}{c(m)}\chi(r)\mu(r) \\ &= c(m)\prod_{p\geq 5}\left(1 - \chi(p)\frac{c(m/p)}{c(m)}\right) \end{aligned}$$

$$= \left(6\prod_{p\ge 5}\frac{p^{3\lambda_p+3}-1}{p^3-1}\right)\left(\prod_{p\ge 5}\left(1-p\left(\frac{6}{p}\right)\frac{p^{3\lambda_p}-1}{p^{3\lambda_p+3}-1}\right)\right)$$
$$= 6\prod_{p\ge 5}\left(\frac{p^{3\lambda_p+3}-1}{p^3-1}-p\left(\frac{6}{p}\right)\frac{p^{3\lambda_p}-1}{p^3-1}\right). \blacksquare$$

7. Further results. In this section we state 15 further results and prove one as an illustration. Proofs for the others are similar.

THEOREM 7.1. *Let* $\mathbf{a} = (a_1, a_2, a_3, a_4, a_5)$. *Let* n *be a positive integer with prime factorization*

$$n = 2^{\lambda_2} m \quad \textit{where} \quad m = \prod_{p\ge 3} p^{\lambda_p}.$$

Then

$$r_{\mathbf{a}}(n^2) = c_{\mathbf{a}}(\lambda_2) r_{(1,1,1,1,1)}(m^2)$$

for the values of $\mathbf{a}$ *and* $c_{\mathbf{a}}(\lambda_2)$ *given in Table* 1. *The value of* $r_{(1,1,1,1,1)}(m^2)$ *is given by Hurwitz's Theorem* 1.1.

Table 1

$\mathbf{a}$	$c_{\mathbf{a}}$
$(1,1,1,2,2)$	$\frac{1}{5}\left(\frac{26\cdot 2^{3\lambda_2}-5}{2^3-1}\right)$
$(1,1,1,1,4)$	$\begin{cases} \frac{4}{5} & \text{if } n \text{ is odd} \\ \frac{1}{5}\left(\frac{3\cdot 2^{3\lambda_2+2}-5}{2^3-1}\right) & \text{if } n \text{ is even} \end{cases}$
$(1,1,1,4,4)$	$\begin{cases} \frac{3}{5} & \text{if } n \text{ is odd} \\ \frac{2^{3\lambda_2}-1}{2^3-1} & \text{if } n \text{ is even} \end{cases}$
$(1,1,2,2,4)$	$\begin{cases} \frac{2}{5} & \text{if } n \text{ is odd} \\ \frac{1}{5}\left(\frac{3\cdot 2^{3\lambda_2+2}-5}{2^3-1}\right) & \text{if } n \text{ is even} \end{cases}$
$(1,1,4,4,4)$	$\begin{cases} \frac{2}{5} & \text{if } n \text{ is odd} \\ \frac{2^{3\lambda_2}-1}{2^3-1} & \text{if } n \text{ is even} \end{cases}$
$(1,2,2,2,2)$	$\frac{1}{5}\left(\frac{3\cdot 2^{3\lambda_2+2}-5}{2^3-1}\right)$
$(1,2,2,4,4)$	$\begin{cases} \frac{1}{5} & \text{if } n \text{ is odd} \\ \frac{2^{3\lambda_2}-1}{2^3-1} & \text{if } n \text{ is even} \end{cases}$
$(1,4,4,4,4)$	$\begin{cases} \frac{1}{5} & \text{if } n \text{ is odd} \\ \frac{2^{3\lambda_2}-1}{2^3-1} & \text{if } n \text{ is even} \end{cases}$

THEOREM 7.2. *Let* $\mathbf{a} = (a_1, a_2, a_3, a_4, a_5)$. *Let* n *be a positive integer with prime factorization*

$$n = 2^{\lambda_2} 3^{\lambda_3} \ell \quad \textit{where} \quad \ell = \prod_{p\ge 5} p^{\lambda_p}.$$

Then

$$r_{\mathbf{a}}(n^2) = c_{\mathbf{a}}(\lambda_2)d_{\mathbf{a}}(\lambda_3)\, r_{(1,1,1,1,1)}(\ell^2)$$

for the values of $\mathbf{a}$, $c_{\mathbf{a}}(\lambda_2)$ *and* $d_{\mathbf{a}}(\lambda_3)$ *given in Table* 2*, while the value of* $r_{(1,1,1,1,1)}(\ell^2)$ *is given by Hurwitz's Theorem* 1.1*.*

Table 2

$\mathbf{a}$	$c_{\mathbf{a}}$	$d_{\mathbf{a}}$
$(1,2,2,3,3)$	$\frac{2}{5}(\frac{6\cdot 2^{3\lambda_2}+1}{2^3-1})$	$\frac{2\cdot 3^{3\lambda_3+2}-5}{3^3-1}$
$(1,3,3,3,3)$	$\frac{2}{5}(\frac{2^{3\lambda_2+3}-1}{2^3-1})$	$\frac{8\cdot 3^{3\lambda_3}+5}{3^3-1}$

THEOREM 7.3. *Let* $\mathbf{a} = (a_1, a_2, a_3, a_4, a_5)$*. Let* n *be a positive integer with prime factorization*

$$n = 2^{\lambda_2} m \quad \textit{where} \quad m = \prod_{p\geq 3} p^{\lambda_p}.$$

Then

$$r_{\mathbf{a}}(n^2) = c_{\mathbf{a}}(\lambda_2) r_{(1,1,1,1,2)}(m^2)$$

for the values of $\mathbf{a}$ *and* $c_{\mathbf{a}}(\lambda_2)$ *given in Table* 3*. The value of* $r_{(1,1,1,1,2)}(m^2)$ *is given by Theorem* 1.2*.*

Table 3

$\mathbf{a}$	$c_{\mathbf{a}}$
$(1,1,2,2,2)$	$\begin{cases} \frac{1}{2} & \text{if } n \text{ is odd} \\ \frac{5}{2}(\frac{2^{3\lambda_2}+6}{2^3-1}) & \text{if } n \text{ is even} \end{cases}$
$(1,2,2,2,4)$	$\begin{cases} \frac{1}{4} & \text{if } n \text{ is odd} \\ 2 & \text{if } \lambda_2 = 1 \\ \frac{13\cdot 2^{3\lambda_2-3}+15}{2^3-1} & \text{if } \lambda_2 \geq 2 \end{cases}$

THEOREM 7.4. *Let* $\mathbf{a} = (a_1, a_2, a_3, a_4, a_5)$*. Let* n *be a positive integer with prime factorization*

$$n = 2^{\lambda_2} 3^{\lambda_3} \ell \quad \textit{where} \quad \ell = \prod_{p\geq 5} p^{\lambda_p}.$$

Then

$$r_{\mathbf{a}}(n^2) = c_{\mathbf{a}}(\lambda_2)d_{\mathbf{a}}(\lambda_3)\, r_{(1,1,1,1,2)}(\ell^2)$$

for the values of $\mathbf{a}$, $c_{\mathbf{a}}(\lambda_2)$ *and* $d_{\mathbf{a}}(\lambda_3)$ *given in Table* 4*, while the value of* $r_{(1,1,1,1,2)}(\ell^2)$ *is given by Theorem* 1.2*.*

Table 4

$\mathbf{a}$	$c_{\mathbf{a}}$	$d_{\mathbf{a}}$
$(1,1,2,3,3)$	$\left\vert\frac{2^{3\lambda_2+3}-15}{2^3-1}\right\vert$	$\frac{3^{3\lambda_3+2}+4}{3^3-1}$
$(1,3,3,3,6)$	$\frac{1}{2}\left\vert\frac{2^{3\lambda_2+3}-15}{2^3-1}\right\vert$	$\frac{5\cdot 3^{3\lambda_3}+8}{3^3-1}$

THEOREM 7.5. *Let* $\mathbf{a} = (a_1, a_2, a_3, a_4, a_5)$. *Let* n *be a positive integer with prime factorization*

$$n = 2^{\lambda_2} 3^{\lambda_3} \ell \quad \text{where} \quad \ell = \prod_{p \ge 5} p^{\lambda_p}.$$

Then

$$r_{\mathbf{a}}(n^2) = c_{\mathbf{a}}(\lambda_2) d_{\mathbf{a}}(\lambda_3)\, r_{(1,1,1,1,3)}(\ell^2)$$

for the values of $\mathbf{a}$, $c_{\mathbf{a}}(\lambda_2)$ *and* $d_{\mathbf{a}}(\lambda_2)$ *given in Table* 5, *while the value of* $r_{(1,1,1,1,3)}(\ell^2)$ *is given by Theorem* 1.3.

Table 5

$\mathbf{a}$	$c_{\mathbf{a}}$	$d_{\mathbf{a}}$
$(1,2,2,2,6)$	$\frac{1}{2}\left\vert\frac{2^{3\lambda_2+3}-15}{2^3-1}\right\vert$	$\frac{3^{3\lambda_3+2}+4}{3^3-1}$

We shall prove the case $\mathbf{a} = (1,1,1,2,2)$ of Theorem 7.1 to illustrate the technique. The proofs in the other cases are similar.

Proof of Theorem 7.1 for $\mathbf{a} = (1,1,1,2,2)$. Let n be any positive integer and write its prime factorization as

$$n = 2^{\lambda_2} m \quad \text{where} \quad m \text{ is odd and } m = \prod_{p \ge 3} p^{\lambda_p}.$$

By the techniques of Section 3 we may deduce that

$$\sum_{j=0}^{\infty} r_{(1,1,1,1,1)}(4j+1) q^j = 10\varphi_1^4\psi_2 + 32q\psi_2^5, \tag{7.1}$$

$$\sum_{j=0}^{\infty} r_{(1,1,1,2,2)}(4j+1) q^j = 6\varphi_1^4\psi_2 + 32q\psi_2^5, \tag{7.2}$$

$$\sum_{j=0}^{\infty} r_{(1,1,1,2,2)}(16j+4) q^j = 58\varphi_1^4\psi_2 + 32q\psi_2^5, \tag{7.3}$$

$$\sum_{j=0}^{\infty} r_{(1,1,1,2,2)}(64j+16) q^j = 474\varphi_1^4\psi_2 + 32q\psi_2^5. \tag{7.4}$$

Moreover,

$$\sum_{j=0}^{\infty} r_{(1,1,1,2,2)}(4j)q^j = \varphi_1^5 + 48q\varphi_1\psi_2^4, \tag{7.5}$$

$$\sum_{j=0}^{\infty} r_{(1,1,1,2,2)}(16j)q^j = \varphi_1^5 + 464q\varphi_1\psi_2^4, \tag{7.6}$$

$$\sum_{j=0}^{\infty} r_{(1,1,1,2,2)}(64j)q^j = \varphi_1^5 + 3792q\varphi_1\psi_2^4. \tag{7.7}$$

For a fixed odd m, let

$$f(k) = r_{(1,1,1,2,2)}(2^{2k}m^2).$$

On comparing the coefficients of q^{2j} in (7.1) and (7.2) we get

$$\tfrac{1}{6} r_{(1,1,1,2,2)}(8j+1) = [q^{2j}]\varphi_1^4\psi_2 = \tfrac{1}{10} r_{(1,1,1,1,1)}(8j+1).$$

It follows that

$$f(0) = r_{(1,1,1,2,2)}(m^2) = \tfrac{3}{5} r_{(1,1,1,1,1)}(m^2)$$

and therefore by Hurwitz's Theorem 1.1 we deduce that

$$f(0) = r_{(1,1,1,2,2)}(m^2) = 6\prod_{p\geq 3}\left(\frac{p^{3\lambda_p+3}-1}{p^3-1} - p\,\frac{p^{3\lambda_p}-1}{p^3-1}\right). \tag{7.8}$$

Next, on comparing the coefficients of q^{2j} in (7.2)–(7.4) we get

$$\tfrac{1}{6} r_{(1,1,1,2,2)}(8j+1) = \tfrac{1}{58} r_{(1,1,1,2,2)}(32j+4) = \tfrac{1}{474} r_{(1,1,1,2,2)}(128j+16)$$

and it follows that

$$f(1) = \tfrac{29}{3} f(0) \quad \text{and} \quad f(2) = 79 f(0). \tag{7.9}$$

Finally, from (7.5)–(7.7) we deduce

$$\sum_{j=0}^{\infty} r_{(1,1,1,2,2)}(64j)q^j = 9\sum_{j=0}^{\infty} r_{(1,1,1,2,2)}(16j)q^j - 8\sum_{j=0}^{\infty} r_{(1,1,1,2,2)}(4j)q^j$$

and it follows that

$$f(k+2) = 9f(k+1) - 8f(k) \quad \text{for } k \geq 1. \tag{7.10}$$

The solution of the recurrence relation (7.10) that satisfies the initial condition (7.9) is given by

$$f(k) = \frac{1}{3}\left(\frac{26\cdot 2^{3k}-5}{2^3-1}\right) f(0). \tag{7.11}$$

On combining (7.8) and (7.11) we deduce that

$$r_{(1,1,1,2,2)}(n^2) = 2\left(\frac{26\cdot 2^{3k}-5}{2^3-1}\right)\prod_{p\geq 3}\left(\frac{p^{3\lambda_p+3}-1}{p^3-1} - p\,\frac{p^{3\lambda_p}-1}{p^3-1}\right). \blacksquare$$

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