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Convergence of series of dilated functions and spectral norms of GCD matrices

by

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1. Introduction. Carleson's theorem [11] states that the series

$$\sum_{k=1}^{\infty} c_k \sin 2\pi kx \quad \text{and} \quad \sum_{k=1}^{\infty} c_k \cos 2\pi kx \tag{1.1}$$

are convergent for almost every x in $[0,1]$ provided that the sequence $(c_k)_{k\geq 1}$ of coefficients (assumed to be real) satisfies

$$\sum_{k=1}^{\infty} c_k^2 < \infty. \tag{1.2}$$

By orthogonality, condition (1.2) is also necessary and sufficient for the L^2 norm convergence of the two series in (1.1). A much studied problem is what happens with the convergence in either sense if the functions $\sin 2\pi x$ and $\cos 2\pi x$ are replaced by more general periodic functions. More precisely, the question is what we can say about the convergence of the series

$$\sum_{k=1}^{\infty} c_k f(kx) \tag{1.3}$$

when $f : \mathbb{R} \to \mathbb{R}$ is a measurable function satisfying

$$f(x+1) = f(x), \qquad \int_0^1 f(x)\,dx = 0, \qquad \int_0^1 f^2(x)\,dx < \infty. \tag{1.4}$$

In general, (1.2) will not be a sufficient condition either for convergence in L^2 or for almost everywhere convergence of (1.3), and the problem is to find alternative conditions on the coefficients $(c_k)_{k\geq 1}$ when f belongs to a

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prescribed class of functions. For a survey of existing results in this direction and recent results we refer to [3, 6]. For a recent survey on Carleson's theorem, see [23].

In this paper, we will be interested in the case when f belongs to the class C_α for $\alpha > 1/2$, i.e. when the Fourier series of f is of the form

$$\sum_{j=1}^{\infty}(a_j \sin 2\pi jx + b_j \cos 2\pi jx)$$

with

$$a_j = \mathcal{O}(j^{-\alpha}), \quad b_j = \mathcal{O}(j^{-\alpha}) \quad \text{as } j \to \infty.$$

The important limiting case $\alpha = 1$ is essentially covered by the results of [3] (see Section 3 below for details). We will extend the methods of [3] to cover also the range $1/2 < \alpha < 1$, and will give sharp conditions for the L^2 convergence and the almost everywhere convergence of (1.3) as well as of the related series

$$\sum_{k=1}^{\infty} c_k f(n_k x), \tag{1.5}$$

where $(n_k)_{k\geq 1}$ is a sequence of distinct positive integers.

Problems concerning the convergence of (1.3) or (1.5) can be traced back to Riemann's *Habilitationsschrift* (1852). They exhibit profound interrelations between various parts of analysis and number theory, as illustrated by the following list of important contributions: classical formulas of Franel and Landau connecting the convergence theory of (1.3) and (1.5) with sums of greatest common divisors (*GCD sums*); their generalization to the Hurwitz zeta function due to Mikolás; the work of Koksma, Erdős, Gál, LeVeque, and others in Diophantine approximation and uniform distribution theory; the results of Dyer and Harman in the context of the Duffin–Schaeffer conjecture in metric Diophantine approximation; upper and lower bounds for GCD sums obtained by the authors of the present paper; and problems concerning the magnitude of the largest eigenvalue of GCD matrices, which were studied by Wintner, by Lindqvist and Seip (in the context of questions about Riesz bases), and by Hilberdink (in the context of the Riemann zeta function). Basic work on the convergence and divergence of dilated series and their relation to lacunary series was done by Gaposhkin, Nikishin, Philipp, and Kaufman, just to mention a few.

In view of this multitude of connections, we have found it appropriate to give a fairly detailed presentation of those ideas and lines of research that are most relevant to our particular problem. To this end, following the statement of our three main theorems in the next section, Section 3 gives an extensive survey of the relevant background material. Section 4 contains auxiliary results, and the proofs are given in Section 5.

2. Results. Throughout this paper we write $K, \hat{K}, K_1, K_2, \ldots$ for appropriate positive constants, not always the same, which only depend (at most) on α and f. We will use the Vinogradov symbols "$\ll$" and "$\gg$" in the same sense. Throughout this paper, we assume that $(c_k)_{1\le k\le N}$ and $(c_k)_{k\ge 1}$ denote sequences of real numbers and that $(n_k)_{1\le k\le N}$ and $(n_k)_{k\ge 1}$ denote sequences of distinct positive integers. For notational convenience, throughout this paper we will read $\log x$ as $\max\{1, \log x\}$; in particular, this implies that iterated logarithms are defined and positive.

THEOREM 1. *Assume that $f \in C_\alpha$ for some $\alpha \in (1/2, 1)$. Then the series* (1.3) *is convergent in L^2 norm and almost everywhere provided*

$$\sum_{k=1}^{\infty} c_k^2 \exp\left(\frac{K(\log k)^{1-\alpha}}{\log\log k}\right) < \infty, \tag{2.1}$$

where

$$K = 3/(1-\alpha) + 4/\sqrt{2\alpha - 1}.$$

Conversely, for every $\alpha \in (1/2, 1)$ there exist a function $f \in C_\alpha$ and a sequence $(c_k)_{k\ge1}$ such that (2.1) *holds with K replaced by $(1-\varepsilon)/(1-\alpha)$ for any $0 < \varepsilon < 1$, but the series* (1.3) *is not convergent in L^2.*

THEOREM 2. *Assume that $f \in C_\alpha$ for some $\alpha \in (1/2, 1)$. Then the series* (1.5) *is convergent in L^2 norm and almost everywhere if*

$$\sum_{k=1}^{\infty} c_k^2 \exp\left(\frac{K(\log k)^{1-\alpha}}{(\log\log k)^{\alpha}}\right) < \infty, \tag{2.2}$$

where

$$K = 6/(1-\alpha) + 7(|\log(2\alpha - 1)|^{1/2} + 1).$$

Conversely, for every $\alpha \in (1/2, 1)$ there exist a function $f \in C_\alpha$, a sequence $(c_k)_{k\ge1}$, a sequence $(n_k)_{k\ge1}$, and a constant $\hat{K} = \hat{K}(\alpha)$ such that (2.2) *holds with K replaced by $\hat{K}$, but the series* (1.5) *is not convergent in L^2 norm and is divergent almost everywhere.*

Theorem 1 improves results of Brémont [10], who proved that (1.3) is convergent in L^2 norm and almost everywhere provided

$$\sum_{k=1}^{\infty} c_k^2 \exp\left(\frac{(1+\varepsilon)(\log k)^{2-2\alpha}}{2(1-\alpha)\log\log k}\right) < \infty \quad \text{for some } \varepsilon > 0.$$

Brémont also proved that there exists a sequence $(c_k)_{k\ge1}$ satisfying (1.2) such that the series (1.3) does not converge in L^2 norm and is almost everywhere divergent.

As the second part of Theorem 2 shows, condition (2.2) is optimal both for convergence in L^2 and almost everywhere convergence, except for the precise value of the constant, thus providing a nearly complete solution of the problem of norm convergence and almost everywhere convergence of series of the form (1.5). In Theorem 1, we claim the optimality of condition (2.1) only for the norm convergence of (1.3); we do not know whether (2.1) is also optimal for almost everywhere convergence. However, we do know that, in general, condition (1.2) is not sufficient for the almost everywhere convergence of (1.3). This follows from our proof of the optimality of the convergence condition in Theorem 2 for almost everywhere convergence of (1.5). In fact, for the proof of the optimality of Theorem 2 for given $\alpha \in (1/2, 1)$ and an appropriate function $f \in C_\alpha$, we construct sequences $(c_k)_{k\geq 1}$ and $(n_k)_{k\geq 1}$ such that (2.2) holds for a certain value of K, but the series (1.5) is almost everywhere divergent. The proof reveals that n_k is of asymptotic order at most $R^{k \log k}$ for some constant $R = R(\alpha)$. Consequently, setting $d_{n_k} = c_k$ when $n = n_k$ and $d_n = 0$ otherwise, we see that $\sum_{n=1}^{\infty} d_n f(nx)$ is divergent almost everywhere, but

$$\sum_{n=1}^{\infty} d_n \exp\left(\frac{\hat{K}(\log\log n)^{1-\alpha}}{(\log\log\log n)^{\alpha}}\right) = \sum_{k=1}^{\infty} d_{n_k} \exp\left(\frac{\hat{K}(\log\log n_k)^{1-\alpha}}{(\log\log\log n_k)^{\alpha}}\right)$$
$$\leq \sum_{k=1}^{\infty} c_k \exp\left(\frac{K(\log k)^{1-\alpha}}{(\log\log k)^{\alpha}}\right) < \infty$$

for some (sufficiently small) positive constant $\hat{K}$. Hence, in the condition for almost everywhere convergence in Theorem 1, a Weyl factor of order at least

$$\exp\left(\frac{\hat{K}(\log\log k)^{1-\alpha}}{(\log\log\log k)^{\alpha}}\right)$$

is necessary. This leaves a rather large gap in comparison to the Weyl factor in (2.1).

As already noted, Theorem 1 gives an optimal condition for the problem of L^2 convergence of series of the form (1.3). More precisely, this statement is true as long as one requires the Weyl multiplier to be a "simple", slowly varying function. On the other hand, the situation is totally different if one allows the Weyl mutiplier $\psi(k)$ to depend on number-theoretic properties of k and to be strongly fluctuating as k increases. In this sense, Theorem 1 may be said to conceal the arithmetical nature of our problem.

To state the next result, we introduce the divisor function

$$\sigma_s(k) = \sum_{d|k} d^s.$$

THEOREM 3. *Assume that $f \in C_\alpha$ for some $\alpha \in (1/2, 1)$. Assume also that*

$$\sum_{k=1}^{\infty} c_k^2 \sigma_{1-2\alpha+\varepsilon}(k) < \infty \tag{2.3}$$

for some $\varepsilon > 0$. *Then* (1.3) *is convergent in* L^2. *On the other hand, for every* $\alpha \in (1/2, 1)$ *and every* $0 < \beta < 1$ *there exist a function* $f \in C_\alpha$ *and a real sequence* $(c_k)_{k\geq 1}$ *such that*

$$\sum_{k=1}^{\infty} c_k^2 \sigma_{-\alpha}(k)^{\beta} < \infty, \tag{2.4}$$

but (1.3) *is not convergent in* L^2.

In Berkes and Weber [5] it is proved that

$$\sum_{k=1}^{\infty} c_k^2 \sigma_{1-2\alpha}(k)(\log k)^2 < \infty \tag{2.5}$$

implies the convergence of (1.3) in L^2 norm and almost everywhere. Despite the similarity of (2.3) and (2.5), there is a crucial difference between the corresponding convergence statements. Clearly, for every $s > 0$,

$$\sum_{k=1}^{n} \sigma_{-s}(k) = \sum_{k=1}^{n} \sum_{d|k} d^{-s} = \sum_{d=1}^{\infty} \left\lfloor \frac{n}{d} \right\rfloor d^{-s} \sim n \sum_{d=1}^{\infty} d^{-1-s} \qquad \text{as } n \to \infty,$$

showing that the average value of $\sigma_{-s}(k)$ is $\sum_{d=1}^{\infty} d^{-1-s} < \infty$. This implies that given any positive function $\omega(k) \to \infty$, the asymptotic density of the set $\{k : \sigma_{-s}(k) \leq \omega(k)\}$ is 1, and thus for $\alpha > 1/2$ and sufficiently small $\varepsilon > 0$, the Weyl factor $\sigma_{1-2\alpha+\varepsilon}(k)$ in (2.3) is of order $\mathcal{O}(\omega(k))$ for "most" k. Thus, despite the optimality of the condition

$$\sum_{k=1}^{\infty} c_k^2 \exp\left(\frac{K(\log k)^{1-\alpha}}{\log\log k}\right) < \infty$$

in Theorem 1, for most k the much smaller Weyl factor $\omega(k)$ suffices for the norm convergence of $\sum_{k=1}^{\infty} c_k f(kx)$. This effect will be apparent from the proofs of the divergence results in Theorems 1–3.

The construction of $(c_k)_{k\geq 1}$ and $(n_k)_{k\geq 1}$ in the examples of divergence uses, roughly speaking, the eigenvectors of suitable GCD matrices belonging to the maximal eigenvalue, which, as is seen from [3] and [15], are concentrated on indices k with many small prime factors. These are also the indices k where the divisor functions $\sigma_{-s}(k)$ are large: as Gronwall [14] showed,

$$\sigma_{-s}(k) \leq \exp\left(\frac{1+o(1)}{1-s}\,\frac{(\log k)^{1-s}}{\log\log k}\right) \tag{2.6}$$

and $\sigma_{-s}(k)$ reaches the order of magnitude on the right-hand side along the sequence $k_r = p_1 \cdots p_r$, $r = 1, 2, \ldots$, where $(p_r)_{r\geq 1}$ is the sequence of primes. There is a gap between (2.3) and (2.4), and the problem of finding the optimal arithmetic function required for the L^2 norm convergence of (1.3) remains open.

As mentioned in the Introduction, the case $\alpha = 1$ is essentially covered by the results of [3]. We refer here to [3, Theorem 3], concerning the almost everywhere convergence of (1.5) for functions f of bounded variation. The only property used in the proof of that result is that a function of bounded variation belongs to C_1. It therefore follows from [3, Theorem 3] that (1.5) is almost everywhere convergent when $f \in C_1$ provided

$$\sum_{k=1}^{\infty} c_k^2 (\log\log k)^{\gamma} < \infty \tag{2.7}$$

for some $\gamma > 4$ (under the additional assumption that $(n_k)_{k\geq 1}$ is strictly increasing). Moreover, it was proved in [3, Theorem 7] that this statement becomes false for $\gamma < 2$. Naturally, under (2.7) with $\gamma > 4$ the series (1.3) is also a.e. convergent, but the divergence statement for $\gamma < 2$ remains open. Concerning L^2 convergence, the situation is different. Using [3, Lemma 4] it can be shown that the series (1.5) is convergent in L^2 norm for all $f \in C_1$ provided (2.7) holds for some $\gamma > 4$, and by the results in [13] this statement becomes false for $\gamma < 2$. On the other hand, using the results from [15] it is possible to show that (1.3) is convergent in L^2 norm for all $f \in C_1$ provided (2.7) holds for some $\gamma > 2$, and this statement becomes false for $\gamma < 2$. Thus the problem of L^2 and almost everywhere convergence of the series (1.3) and (1.5) is solved, up to powers of $\log\log k$ in the extra convergence conditions ([1]).

The problem of norm and almost everywhere convergence of (1.3) when (1.4) is our only assumption on f is considerably harder. The reason is that while for $f \in C_\alpha$ we have

$$\Big|\int_0^1 f(kx)f(\ell x)\,dx\Big| \leq K\frac{(\gcd(k,\ell))^{2\alpha}}{(k\ell)^{\alpha}}, \qquad k,\ell \geq 1, \tag{2.8}$$

for some constant $K > 0$, for general f satisfying (1.4) the integral in (2.8) depends on k, ℓ and the Fourier coefficients of f in a rather complicated way, and the arithmetic machinery involving GCD sums and eigenvalues of GCD matrices used in the proofs of our theorems breaks down. Assuming that the complex Fourier coefficients a_j of f satisfy $|a_j| \leq \phi(j)$, where the positive function ϕ has the homogeneity property $|\phi(jk)| \ll k^{-\gamma}\phi(j)$ for some $\gamma > 0$, much of what is developed in the present paper will carry over to this situation. Estimates as those found in [8] could then, for instance, be used to obtain fairly sharp analogues of Theorems 1 and 2 for the function classes considered.

([1]) **Added in proof.** In a recent paper [24], Lewko and Radziwiłł showed that for $f \in C_1$, the series (1.5) converges a.e. provided (2.7) holds for some $\gamma > 2$, closing the gap between the direct and converse part of the results above. They also showed that $(\log\log N)^4$ in (3.16) can be improved to $(\log\log N)^2$, reaching another optimal result.

In the case of arithmetic criteria like the one in Theorem 3, Berkes and Weber [7] proved that if f satisfies (1.4) with complex Fourier coefficients a_j, then the series (1.3) converges almost everywhere provided

$$\sum_{k=1}^{\infty} c_k^2 \psi(k)(\log k)^2 < \infty, \tag{2.9}$$

where the arithmetic function ψ is defined by

$$\psi(k) = \sum_{d|k} (dg(d) + G(d)) \tag{2.10}$$

with

$$g(r) = \sum_{j=1}^{\infty} |a_{jr}|^2 \quad \text{and} \quad G(r) = \sum_{j \le 2r} g(j).$$

For example, if $|a_j| \le Kj^{-1/2}(\log j)^{-\gamma}$, $\gamma > 1/2$, then $\psi(k)$ reduces to

$$\psi(k) = \sum_{d|k} (\log d)^{-(2\gamma-1)}. \tag{2.11}$$

Note that the arithmetic function ψ in (2.11) is larger than the one in (2.3), which is of course to be expected. Note also that if $j^{-\gamma}|a_j|$ is non-increasing for some $\gamma > 0$, then in (2.9) we can choose

$$\psi(k) = d(k) = \sum_{d|k} 1.$$

The same criterion holds if the function f satisfies a Hölder continuity condition (see [5, 31]). These remarks show again the strong arithmetic character of our convergence problem. In [7] it is also shown that except for the factor $(\log k)^2$, condition (2.9) is optimal. However, just as in Theorem 3, the arithmetic criterion (2.9) is not as sharp as those in Theorems 1 and 2.

Note that if (1.3) converges almost everywhere for $c_k = 1/k$, then by the Kronecker lemma we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(kx) = 0 \quad \text{a.e.}, \tag{2.12}$$

and thus the almost everywhere convergence problem of (1.3) under (1.4) is closely connected with the classical problem of the convergence of averages in (2.12). Khinchin [18] conjectured that under (1.4) (even without the third condition) the convergence relation (2.12) holds. This conjecture was disproved nearly 50 years later by a famous counterexample of Marstrand [26]. In the positive direction, Koksma [21] proved that (2.12) holds provided the complex Fourier coefficients a_j of f satisfy

$$\sum_{j=1}^{\infty} |a_j|^2 \sigma_{-1}(j) < \infty.$$

Bourgain [9] gave a new, much simplified counterexample to Khinchin's conjecture, and claimed, without proof, that Koksma's criterion is essentially optimal. This claim was proved recently by Berkes and Weber [7]. Thus while the almost everywhere convergence problem for (1.3) under (1.4) remains open, the closely related problem of almost everywhere convergence of the averages (2.12) is essentially settled.

3. The role of GCD matrices and certain extremal functions in C_α. We will now review the key ideas used in both [3] and the present paper. We begin by introducing the special functions $f_\alpha(x)$ and $\bar{f}_\alpha(x)$ in C_α defined by

$$f_\alpha(x) = \sum_{j=1}^{\infty} \frac{\sin 2\pi jx}{j^\alpha} \quad \text{and} \quad \bar{f}_\alpha(x) = \sum_{j=1}^{\infty} \frac{\cos 2\pi jx}{j^\alpha}. \tag{3.1}$$

Informally speaking, these functions are extremal in C_α in the sense that their Fourier coefficients are of maximal size. Furthermore, all Fourier coefficients are positive, which makes it relatively easy to obtain lower bounds for L^2 norms of sums of dilated functions.

When $\alpha = 1$, the first series in (3.1) is the Fourier series of the function

$$f_1(x) = \pi(1/2 - \{x\}),$$

where $\{\cdot\}$ denotes fractional part. This means that, up to multiplication by a constant, f_1 is the first Bernoulli polynomial on $[0, 1]$, extended with period one. Convergence problems for (1.3) and (1.5) have been investigated extensively for $f = f_1$, starting probably with Riemann's *Habilitationsschrift* of 1852. Such series have been called *Davenport series* in honor of Harold Davenport, who was the first to study them in this general form [12]. See [16] for a survey on the history of the subject and several results on the convergence problem for series involving this function. Convergence problems for Davenport series have an interesting connection with fractal geometry: see for example [17].

The convergence problem for series involving the function f_1 is connected with sums involving greatest common divisors through the formula

$$\int_0^1 (\{kx\} - 1/2)(\{\ell x\} - 1/2)\, dx = \frac{1}{12} \frac{(\gcd(k, \ell))^2}{k\ell} \tag{3.2}$$

for positive integers k, ℓ, which was first stated by Franel and formally proved by Landau in 1924. Consequently,

$$\int_0^1 \Big(\sum_{k=1}^N c_k f_1(n_k x)\Big)^2\,dx = \frac{\pi^2}{12}\sum_{k,\ell=1}^N c_k c_\ell \frac{(\gcd(n_k,n_\ell))^2}{n_k n_\ell}. \tag{3.3}$$

But much more is true since the Fourier coefficients of f_1 are positive and maximal: By an observation of Koksma [20] we have

$$\int_0^1 \Big(\sum_{k=1}^N c_k f(n_k x)\Big)^2\,dx \ll \sum_{k,\ell=1}^N |c_k c_\ell| \frac{(\gcd(n_k,n_\ell))^2}{n_k n_\ell} \tag{3.4}$$

for every function f in C_1.

The relation between L^2 norms of sums of dilated functions and sums involving greatest common divisors extends to the classes C_α for $1/2 < \alpha < 1$. This was first observed by Mikolás [27], who proved that for the Hurwitz zeta function $\zeta(1-\alpha,\cdot)$,

$$\int_0^1 \zeta(1-\alpha,\{kx\})\zeta(1-\alpha,\{\ell x\})\,dx = 2\Gamma(\alpha)^2\frac{\zeta(2\alpha)}{(2\pi)^{2\alpha}}\,\frac{(\gcd(k,\ell))^{2\alpha}}{(k\ell)^\alpha} \tag{3.5}$$

for positive integers k,ℓ and for $\alpha > 1/2$. Hurwitz's formula states that for $\alpha > 1$ and $x \in [0,1]$,

$$\zeta(1-\alpha,x) = \frac{\Gamma(\alpha)}{(2\pi)^\alpha}\Big(e^{-\pi i\alpha/2}\sum_{j=1}^\infty \frac{e^{2\pi ijx}}{j^\alpha} + e^{\pi i\alpha/2}\sum_{j=1}^\infty \frac{e^{-2\pi ijx}}{j^\alpha}\Big)$$

(see for example [19] for a simple proof), which implies that

$$\zeta(1-\alpha,x) = \frac{2\Gamma(\alpha)}{(2\pi)^\alpha}\Big(\cos\frac{\pi\alpha}{2}\sum_{j=1}^\infty \frac{\cos 2\pi jx}{j^\alpha} + \sin\frac{\pi\alpha}{2}\sum_{j=1}^\infty \frac{\sin 2\pi jx}{j^\alpha}\Big). \tag{3.6}$$

Thus $\zeta(1-\alpha,x)$ has Fourier coefficients of asymptotic order precisely $j^{-\alpha}$, and in particular $\zeta(1-\alpha,x) \in C_\alpha$. As Mikolás showed, the formula (3.6) continues to hold for $\alpha > 1/2$ and $0 < x < 1$, which leads to (3.5) by the orthogonality of the trigonometric system. By the same argument as for $\alpha = 1$, we find that

$$\int_0^1 \Big(\sum_{k=1}^N c_k f(n_k x)\Big)^2\,dx \ll \sum_{k,\ell=1}^N |c_k c_\ell| \frac{(\gcd(n_k,n_\ell))^{2\alpha}}{(n_k n_\ell)^\alpha} \tag{3.7}$$

for every f in C_α (see Lemma 1 below). For the special function $f_\alpha(x)$ from (3.1) we get

$$\int_0^1 \Big(\sum_{k=1}^N c_k f_\alpha(n_k x)\Big)^2\,dx = \frac{\zeta(2\alpha)}{2}\sum_{k,\ell=1}^N c_k c_\ell \frac{(\gcd(n_k,n_\ell))^{2\alpha}}{(n_k n_\ell)^\alpha}, \tag{3.8}$$

as will also be established in Lemma 1 below.

Estimates (3.4) and (3.7), as well as the identities (3.3) and (3.8), show that to understand the convergence of (1.3) and (1.5) for f in C_α it is important to have good upper and lower bounds for sums of the form

$$\sum_{k,\ell=1}^{N} c_k c_\ell \frac{(\gcd(k,\ell))^{2\alpha}}{(k\ell)^\alpha} \quad \text{and} \quad \sum_{k,\ell=1}^{N} c_k c_\ell \frac{(\gcd(n_k,n_\ell))^{2\alpha}}{(n_k n_\ell)^\alpha}. \tag{3.9}$$

Now let $G_N^{(\alpha)}$ be the $N \times N$ matrix with entries

$$g_{k\ell} = \frac{(\gcd(k,\ell))^{2\alpha}}{(k\ell)^\alpha}, \tag{3.10}$$

and $H_N^{(\alpha)}$ the $N \times N$ matrix with entries

$$h_{k\ell} = \frac{(\gcd(n_k,n_\ell))^{2\alpha}}{(n_k n_\ell)^\alpha}.$$

It is well-known that both of these matrices are positive definite (see e.g. [25]). Thus for the largest eigenvalue $\Lambda(G_N^{(\alpha)})$ of $G_N^{(\alpha)}$ we have

$$\Lambda(G_N^{(\alpha)}) = \max_{\substack{c_1,\dots,c_N:\\ c_1^2+\dots+c_N^2=1}} \sum_{k,\ell=1}^{N} c_k c_\ell \frac{(\gcd(k,\ell))^{2\alpha}}{(k\ell)^\alpha}, \tag{3.11}$$

and the largest eigenvalue $\Lambda(H_N^{(\alpha)})$ of $H_N^{(\alpha)}$ satisfies

$$\Lambda(H_N^{(\alpha)}) = \max_{\substack{c_1,\dots,c_N:\\ c_1^2+\dots+c_N^2=1}} \sum_{k,\ell=1}^{N} c_k c_\ell \frac{(\gcd(n_k,n_\ell))^{2\alpha}}{(n_k n_\ell)^\alpha}. \tag{3.12}$$

Consequently, by (3.7) and (3.8), the problem of finding upper and lower bounds for the largest eigenvalue (or the square-root of the spectral norm) of $G_N^{(\alpha)}$ and $H_N^{(\alpha)}$ is precisely the problem of finding general upper bounds for respectively

$$\int_0^1 \Big(\sum_{k=1}^{N} c_k f(kx)\Big)^2 dx \quad \text{and} \quad \int_0^1 \Big(\sum_{k=1}^{N} c_k f(n_k x)\Big)^2 dx \tag{3.13}$$

when f is in C_α, and of finding lower bounds for these integrals in the special case when $f = f_\alpha$.

The problem of calculating $\Lambda(G_N^{(\alpha)})$, and accordingly the problem of estimating the left-hand integral in (3.13), was solved by Hilberdink [15], who proved that

$$\Lambda(G_N^{(\alpha)}) = \frac{1}{\zeta(2)}(e^\gamma \log\log N + \mathcal{O}(1))^2 \quad \text{for } \alpha = 1, \tag{3.14}$$

$$\Lambda(G_N^{(\alpha)}) \ll \exp\bigg(K \frac{(\log N)^{1-\alpha}}{\log\log N}\bigg) \quad \text{for } 1/2 < \alpha < 1. \tag{3.15}$$

In (3.15) the constant K depends on α, and (3.15) is optimal except for the precise value of K. For $H_N^{(\alpha)}$, in [3, Lemma 4 and Theorem 5] it was shown that

$$\Lambda(H_N^{(\alpha)}) \ll (\log\log N)^4 \qquad \text{for } \alpha = 1, \tag{3.16}$$

$$\Lambda(H_N^{(\alpha)}) \ll \exp\biggl(K\frac{(\log N)^{1-\alpha}}{(\log\log N)^{\alpha}}\biggr) \qquad \text{for } 1/2 < \alpha < 1, \tag{3.17}$$

where the constant K depends on α. Here (3.17) is optimal except for the precise value of K, but it remains a profound problem to decide whether the exponent 4 of $\log\log N$ on the right-hand side of (3.16) is optimal. By a classical theorem of Gál [13], it is known that this exponent cannot be smaller than 2 ([2]).

As noted above, the results (3.14)–(3.17) imply corresponding upper bounds for the integrals in (3.13) when $f \in C_\alpha$, and the optimality of (3.14), (3.15), and (3.17) implies corresponding lower bounds for the integrals in (3.13) in the special case when $f = f_\alpha$; this is the reason why the exponential factor from (3.15) appears in Theorem 1, and that from (3.17) appears in Theorem 2. Note the difference between the powers of $\log\log N$ on the right-hand side of (3.15) and (3.17): it is $\log\log N$ in the first case and $(\log\log N)^\alpha$ in the second one. Since both results are optimal, this shows that in the case $1/2 < \alpha < 1$ there is a significant difference between the spectral norms of $G_N^{(\alpha)}$ and $H_N^{(\alpha)}$, and accordingly also a difference between the convergence problems for (1.3) and (1.5). In [15], a connection is established between the spectral norm of $G_N^{(\alpha)}$ and the maximal order of magnitude of the Riemann zeta function along vertical lines, using Soundararajan's "resonance method" from [30]. However, Hilberdink's results cannot reach the stronger lower bounds of Montgomery [28], which in turn bear a striking resemblance to the bounds for the spectral norm of $H_N^{(\alpha)}$ in [3] ([3]).

We close this section by making an observation on our extremal functions f_α and $\bar{f}_\alpha$ in (3.1), which will be needed in what follows. We note first that they are, up to normalization, the even and odd parts of the Hurwitz zeta function. In fact, from the Fourier series representation in (3.6) it is easily seen that

$$\frac{\zeta(1-\alpha, x) - \zeta(1-\alpha, 1-x)}{2} = \frac{2\Gamma(\alpha)}{(2\pi)^\alpha}\sin(\pi\alpha/2) f_\alpha(x), \tag{3.18}$$

$$\frac{\zeta(1-\alpha, x) + \zeta(1-\alpha, 1-x)}{2} = \frac{2\Gamma(\alpha)}{(2\pi)^\alpha}\cos(\pi\alpha/2) \bar{f}_\alpha(x). \tag{3.19}$$

These representations can be used to describe the rate with which $f_\alpha(x)$ and

([2]) Cf. footnote 1.

([3]) See [2] for proving Montgomery's original bound via the GCD sum connection.

$\bar{f}_\alpha(x)$ tend to infinity as $x \to 0$. Mikolás proved that for fixed $\alpha \in (1/2, 1)$,

$$\lim_{x \to 0+} x^{1-\alpha} \zeta(1-\alpha, x) = 1$$

(this is equation (12) in [27]). Consequently, since $\lim_{x\to 0+} \zeta(1-\alpha, 1-x) = \zeta(1-\alpha, 1) = \zeta(1-\alpha)$ is a constant, we have

$$\lim_{x \to 0+} x^{1-\alpha} f_\alpha(x) = \frac{(2\pi)^\alpha}{\Gamma(\alpha)\sin(\pi\alpha/2)}.$$

In particular this implies that

$$f_\alpha \in L^p(0,1) \quad \text{for } p < \frac{1}{1-\alpha}, \tag{3.20}$$

which will be a crucial ingredient in the proof of the necessary condition for almost everywhere convergence of (1.5). More precisely, (3.20) implies that for any $\alpha \in (1/2, 1)$ the function f_α is in $L^{2+\delta}$ for some $\delta = \delta(\alpha) > 0$, which will allow us to apply Lyapunov's central limit theorem (which requires the existence of an absolute moment of order $2+\delta$ for some $\delta > 0$). Similar results hold if f_α is replaced by $\bar{f}_\alpha$.

4. Auxiliary results. Throughout the rest of this paper, we will use the notation $\|\cdot\|$ for the $L^2(0,1)$ norm. Moreover, we will always assume that $\alpha \in (1/2, 1)$.

LEMMA 1. *Assume that $f \in C_\alpha$. Then*

$$\int_0^1 \Big(\sum_{k=1}^N c_k f(n_k x)\Big)^2 dx \ll \sum_{k=1}^N |c_k c_\ell| \frac{(\gcd(n_k, n_\ell))^{2\alpha}}{(n_k n_\ell)^\alpha}.$$

For the particular function f_α from (3.1) *we have*

$$\int_0^1 \Big(\sum_{k=1}^N c_k f_\alpha(n_k x)\Big)^2 dx = \frac{\zeta(2\alpha)}{2} \sum_{k,\ell=1}^N c_k c_\ell \frac{(\gcd(n_k, n_\ell))^{2\alpha}}{(n_k n_\ell)^\alpha}. \tag{4.1}$$

Note that a special case of Lemma 1 is

$$\int_0^1 \Big(\sum_{k=1}^N c_k f(kx)\Big)^2 dx \ll \sum_{k,\ell=1}^N |c_k c_\ell| \frac{(\gcd(k, \ell))^{2\alpha}}{(k\ell)^\alpha}. \tag{4.2}$$

Proof of Lemma 1. The argument is a simple generalization of those leading to (3.3) and (3.4), respectively. We write

$$f(x) \sim \sum_{j=1}^\infty a_j \sin 2\pi j x,$$

assuming, to shorten formulas, that f is an odd function; the proof in the general case is exactly the same. Then, by the orthogonality of the trigonometric system, for arbitrary positive integers m, n we have

$$\int_0^1 f(mx)f(nx)\,dx = \frac{1}{2}\sum_{j_1,j_2=1}^{\infty} a_{j_1}a_{j_2}\mathbb{1}(j_1 m = j_2 n) \tag{4.3}$$

$$= \frac{1}{2}\sum_{j=1}^{\infty} a_{jm/\gcd(m,n)}a_{jn/\gcd(m,n)} \tag{4.4}$$

$$\ll \sum_{j=1}^{\infty}\left(\frac{\gcd(m,n)}{jm}\right)^{\alpha}\left(\frac{\gcd(m,n)}{jn}\right)^{\alpha}$$

$$\ll \left(\frac{(\gcd(m,n))^2}{mn}\right)^{\alpha}.$$

In (4.3), we have used the fact that $j_1 m = j_2 n$ if and only if

$$j_1 = jn/\gcd(m,n) \quad \text{and} \quad j_2 = jm/\gcd(m,n)$$

for some positive integer j. Applying the above inequality for all pairs (n_k, n_ℓ) gives the first part of the lemma.

For $f = f_\alpha$ we have $a_j = j^{-\alpha}$, $j \geq 1$. Inserting this into (4.4) we get

$$\int_0^1 f(mx)f(nx)\,dx = \frac{1}{2}\sum_{j=1}^{\infty}\left(\frac{\gcd(m,n)}{jm}\right)^{\alpha}\left(\frac{\gcd(m,n)}{jn}\right)^{\alpha}$$

$$= \frac{\zeta(2\alpha)}{2}\left(\frac{(\gcd(m,n))^2}{mn}\right)^{\alpha}.$$

Again we obtain the desired result by summing over all pairs (n_k, n_ℓ). ∎

LEMMA 2. *Assume that $f \in C_\alpha$. Then there exist constants K_1, K_2 such that*

$$\Big\|\sum_{k=1}^{N} c_k f(kx)\Big\|^2 \ll \exp\left(\frac{K_1(\log N)^{1-\alpha}}{\log\log N}\right)\sum_{k=1}^{N} c_k^2,$$

$$\Big\|\sum_{k=1}^{N} c_k f(n_k x)\Big\|^2 \ll \exp\left(\frac{K_2(\log N)^{1-\alpha}}{(\log\log N)^{\alpha}}\right)\sum_{k=1}^{N} c_k^2.$$

We can choose K_1, K_2 such that

$$K_1 < 3/(1-\alpha) + 4/\sqrt{2\alpha-1}, \qquad K_2 < 6/(1-\alpha) + 7(|\log(2\alpha-1)|^{1/2}+1).$$

By Lemma 1 and (3.11) and (3.12), the estimates in Lemma 2 follow from corresponding upper bounds for the largest eigenvalues of the matrices $G_N^{(\alpha)}$ and $H_N^{(\alpha)}$, respectively, which were already stated in (3.15) and (3.17). The given value for K_1 is a rough estimate for that stated in a more precise form in the proof of [15, Theorem 2.3] and at the end of [15, Section 3]; the value for K_2 is obtained by using the method of the recent paper [8], which improves in a significant way the arguments from [3].

Using the same method as in the proof of the Rademacher–Menshov inequality, we easily obtain the following lemma, which is a maximal version of Lemma 2. Note that the proof of the Rademacher–Menshov inequality gives an additional logarithmic factor, which however in our case can be included in the exponential term if we slightly increase the value of the constants.

LEMMA 3. *Assume that $f \in C_\alpha$. Then there exist constants K_1, K_2 such that*

$$\Big\| \max_{1\le M\le N} \Big| \sum_{k=1}^{M} c_k f(kx) \Big| \Big\|^2 \ll \exp\Big(\frac{K_1 (\log N)^{1-\alpha}}{\log\log N}\Big) \sum_{k=1}^{N} c_k^2,$$

$$\Big\| \max_{1\le M\le N} \Big| \sum_{k=1}^{M} c_k f(n_k x) \Big| \Big\|^2 \ll \exp\Big(\frac{K_2 (\log N)^{1-\alpha}}{(\log\log N)^{\alpha}}\Big) \sum_{k=1}^{N} c_k^2.$$

We can choose K_1, K_2 such that

$$K_1 < 3/(1-\alpha) + 4/\sqrt{2\alpha - 1}, \qquad K_2 < 6/(1-\alpha) + 7(|\log(2\alpha-1)|^{1/2} + 1).$$

LEMMA 4 ([1, Lemma 6]). *Assume that for every given $\varepsilon > 0$ there exists an $M_0(\varepsilon)$ such that*

$$\Big\| \sup_{M > M_0} \Big| \sum_{k=M_0+1}^{M} c_k f(kx) \Big| \Big\| \le \varepsilon. \tag{4.5}$$

Then $\sum_{k=1}^{\infty} c_k f(kx)$ is almost everywhere convergent.

For the formulation of the following lemma we note that the unit interval, equipped with Borel sets and Lebesgue measure, is a probability space. Throughout the rest of this paper, the symbols $\mathbb{P}$ and $\mathbb{E}$ will refer to this probability space. Furthermore, $\log_2 x$ will mean $\max\{1, \log_2 x\}$, where the latter $\log_2$ is the dyadic logarithm.

The following lemma is a variant of [3, Lemma 5].

LEMMA 5. *For given $\alpha \in (1/2, 1)$, set $\eta = 12/(2\alpha - 1)$ and let $1 \le S_1 < T_1 < S_2 < T_2 < \cdots$ be integers such that*

$$S_{i+1} \ge T_i + \eta \log_2 i.$$

Furthermore, let $\Delta_1, \Delta_2, \ldots$ be sets of integers such that $\Delta_i \subset [2^{S_i}, 2^{T_i}]$ and each element of Δ_i is divisible by 2^{S_i}. For $i \ge 1$ and $x \in (0,1)$ set

$$X_i = X_i(x) := \sum_{k \in \Delta_i} f_\alpha(kx).$$

Then there exist independent *random variables* $Y_1, Y_2, \ldots$ *on the probability space* $((0,1), \mathcal{B}, \mathbb{P})$ *such that* $\mathbb{E}Y_i = 0$ *and*

$$\|X_i - Y_i\| \ll i^{-2} \cdot \#\Delta_i.$$

For the proof, we need the following lemma, which is [4, Lemma 3.1]. Here, given an integrable function $g(x)$ on $[0,1]$ and an arbitrary integer m, we write $[g]_m$ for the function which takes the constant value

$$m \int_{k/m}^{(k+1)/m} g(x)\,dx$$

in the intervals $[k/m, (k+1)/m)$, for $k = 0, \ldots, m-1$.

LEMMA 6 ([4, Lemma 3.1]). *Assume that* $f \in C_\alpha$. *Let* $k \geq 1$ *be a positive integer, and write* $g(x) = f(kx)$. *Then for any integer* $m \geq k$ *we have*

$$\|g - [g]_m\| \ll \left(\frac{k}{m}\right)^{(2\alpha-1)/6}.$$

Proof of Lemma 5. Let $\mathcal{F}_i$ denote the σ-field generated by the dyadic intervals

$$U_j := [j2^{-S_{i+1}}, (j+1)2^{-S_{i+1}}), \quad 0 \leq j < 2^{S_{i+1}}, \tag{4.6}$$

and set

$$\xi_k = \xi_k(\cdot) = \mathbb{E}(f_\alpha(k\,\cdot)|\mathcal{F}_i), \quad k \in \Delta_i, \quad \text{and} \quad Y_i = Y_i(x) = \sum_{k\in\Delta_i} \xi_k(x).$$

Then clearly $\mathbb{E}\xi_k = 0$, which implies $\mathbb{E}Y_i = 0$. By Lemma 6 and (4.5) for every $k \in \Delta_i$ we have

$$\|\xi_k(\cdot) - f_\alpha(k\cdot)\| \ll \left(\frac{k}{2^{S_{i+1}}}\right)^{(2\alpha-1)/6} \ll \left(\frac{2^{T_i}}{2^{T_i+\eta\log_2 i}}\right)^{(2\alpha-1)/6}$$
$$\ll i^{-\eta(2\alpha-1)/6} \ll i^{-2},$$

which implies that

$$\|X_i - Y_i\| \ll i^{-2} \cdot \#\Delta_i.$$

Since by assumption every $k \in \Delta_{i+1}$ is a multiple of $2^{S_{i+1}}$, each interval U_j in (4.6) is a period interval of $f_\alpha(kx)$ for all $k \in \Delta_{i+1}$, and therefore also for ξ_k for all $k \in \Delta_{i+1}$. Consequently, Y_{i+1} is independent of the σ-field $\mathcal{F}_i$. Since $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$ and since Y_i is $\mathcal{F}_i$-measurable, the random variables $Y_1, Y_2, \ldots$ are independent. ∎

The following lemma can be deduced from [15, Proposition 3.1], just as [15, (3.2)].

LEMMA 7. *We have*

$$\sum_{k,\ell=1}^{N} c_k c_\ell \frac{(\gcd(k,\ell))^{2\alpha}}{(k\ell)^\alpha} \le \sum_{k=1}^{N^2} b_k^2, \quad \textit{where} \quad b_k = \frac{1}{k^\alpha} \sum_{d|k} d^\alpha |c_d|.$$

5. Proofs. *Proof of the convergence part of Theorem 1.* Throughout this proof, we will write K_1 for the constant in Theorem 1, and K_2 for the constant in the first part of Lemma 2. Note that we can assume that $K_1 > K_2$. Relation (2.1) implies that

$$\sum_{k=e^m+1}^{e^{m+1}} c_k^2 \exp\left(\frac{K_1(\log k)^{1-\alpha}}{\log\log k}\right) \ll 1 \quad \text{for } m \ge 1,$$

which also yields

$$\sum_{k=e^m+1}^{e^{m+1}} c_k^2 \ll \exp\left(\frac{-K_1 m^{1-\alpha}}{\log m}\right) \quad \text{for } m \ge 1.$$

Here and below, a sum $\sum_{k=A}^{B}$ with non-integral A, B is meant as $\sum_{A \le k \le B}$. Consequently, by Lemma 2, for any M, N satisfying $e^m < M < N < e^{m+1}$, we have

$$\left\|\sum_{k=M}^{N} c_k f(kx)\right\|^2 \ll \exp\left(\frac{K_2(m+1)^{1-\alpha}}{\log(m+1)}\right) \exp\left(\frac{-K_1 m^{1-\alpha}}{\log m}\right) \ll \exp\left(\frac{-\varepsilon m^{1-\alpha}}{\log m}\right) \tag{5.1}$$

for some $\varepsilon > 0$, since $K_1 > K_2$. For given $M < N$, let $\hat{m}$ denote the integer for which $M \in (e^{\hat{m}}, e^{\hat{m}+1}]$, and $\hat{n}$ the integer for which $N \in (e^{\hat{n}}, e^{\hat{n}+1}]$. If $\hat{m} = \hat{n}$, then by (5.1) we have

$$\left\|\sum_{k=M}^{N} c_k f(kx)\right\| \ll \exp\left(\frac{-\varepsilon \hat{m}^{1-\alpha}}{2\log \hat{m}}\right). \tag{5.2}$$

If $\hat{m} < \hat{n}$, then by (5.1) and Minkowski's inequality we have

$$\begin{aligned} \left\|\sum_{k=M}^{N} c_k f(kx)\right\| &\ll \left\|\sum_{k=M}^{e^{\hat{m}+1}} c_k f(kx)\right\| + \sum_{m=\hat{m}+1}^{\hat{n}-1} \left\|\sum_{k=e^m+1}^{e^{m+1}} c_k f(kx)\right\| + \left\|\sum_{k=e^{\hat{n}}+1}^{N} c_k f(kx)\right\| \\ &\ll \sum_{m=\hat{m}}^{\infty} \exp\left(\frac{-\varepsilon m^{1-\alpha}}{2\log m}\right). \end{aligned} \tag{5.3}$$

The upper bounds in (5.2) and (5.3) can be made arbitrarily small if $\hat{m}$ is sufficiently large and thus the series $\sum_{k=1}^{\infty} c_k f(kx)$ is convergent in L^2. In a similar way, using Lemma 3 instead of Lemma 2, for any $M < N$ we obtain

$$\Big\| \max_{M<L\leq N} \Big| \sum_{k=M}^{L} c_k f(kx) \Big| \Big\| \ll \sum_{m=\hat{m}}^{\infty} \exp\bigg(\frac{-\varepsilon m^{1-\alpha}}{2 \log m} \bigg),$$

where $\hat{m}$ is defined as before. Again the right-hand side can be made arbitrarily small if M is sufficiently large. Thus the monotone convergence theorem and Lemma 4 imply that $\sum_{k=1}^{\infty} c_k f(kx)$ is almost everywhere convergent. ∎

Proof of the optimality of Theorem 1. For given $\alpha \in (1/2, 1)$, we will show that there exists a sequence $(c_k)_{k\geq 1}$ satisfying (2.1) for a "small" value of K, for which for the function $f(x) = f_\alpha(x)$ from (3.1) the series $\sum_{k=1}^{\infty} c_k f_\alpha(kx)$ is divergent in L^2. We will construct $(c_k)_{k\geq 1}$ so that it is supported on a set of indices which have a small number of prime factors; this idea already appears in [3, 13, 15] and other places. However, there it is only used to construct a finite sequence, whereas we have to construct an infinite one. Note that by (3.5), (3.18), and (3.19) the L^2 norms of sums of dilated functions $f_\alpha(x)$, $\bar{f}_\alpha(x)$, and $\zeta(1-\alpha, x)$ are the same, up to a multiplicative constant, and consequently we could also use the functions $\bar{f}_\alpha(x)$ or $\zeta(1-\alpha, x)$ instead of $f_\alpha(x)$.

We write $(p_r)_{r\geq 1}$ for the sequence of primes in increasing order. We define sets Δ_i in the following way: for given $i \geq 1$, the set Δ_i contains those positive integers which are of the form

$$2^{2i} p_1^{w_1} \ldots p_i^{w_i} \quad \text{for } (w_1, \ldots, w_i) \in \{0,1\}^i.$$

By construction the sets Δ_i, $i \geq 1$, are mutually disjoint (since all numbers in Δ_i are multiples of either 2^{2i} or 2^{2i+1}, but not of 2^{2i+2}). Note that the number of elements of Δ_i is 2^i.

Let $\varepsilon > 0$ be fixed, and set $\eta = (1-2\varepsilon)/(1+\varepsilon)$. We define

$$c_k = \begin{cases} 2^{-i/2} i^{-1} \exp\Big(-\dfrac{\eta (\log k)^{1-\alpha}}{2(1-\alpha) \log\log k} \Big) & \text{if } k \in \Delta_i \text{ for some } i \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\sum_{k=1}^{\infty} c_k^2 \exp\bigg(\frac{\eta (\log k)^{1-\alpha}}{(1-\alpha)\log\log k} \bigg) = \sum_{i=1}^{\infty} \sum_{k\in\Delta_i} 2^{-i} i^{-2} = \sum_{i=1}^{\infty} i^{-2} < \infty.$$

By the prime number theorem, for all sufficiently large i and all $k \in \Delta_i$ we have

$$k \leq 2^{2i} \prod_{r=1}^{i} p_r \leq 2^{2i} \big(((1+\varepsilon) i \log i)^i \big),$$

and consequently, for sufficiently large i and all $k \in \Delta_i$,

$$\frac{(\log k)^{1-\alpha}}{\log\log k} \le (1+\varepsilon)(i\log i)^{1-\alpha}(\log i)^{-1} = (1+\varepsilon)i^{1-\alpha}(\log i)^{-\alpha}.$$

Thus for $i \ge 1$ for all $k \in \Delta_i$ we have

$$c_k \gg 2^{-i/2} i^{-1} \exp\left(-\frac{\eta(1+\varepsilon)}{2(1-\alpha)} i^{1-\alpha}(\log i)^{-\alpha}\right). \tag{5.4}$$

Using the second part of Lemma 1 and the facts that f_α has only positive Fourier coefficients and that all coefficients c_k are non-negative, we obtain

$$\begin{aligned}
\lim_{N\to\infty} \Big\|\sum_{k=1}^{N} c_k f_\alpha(kx)\Big\|^2 &\ge \lim_{M\to\infty} \Big\|\sum_{i=1}^{M}\sum_{k\in\Delta_i} c_k f_\alpha(kx)\Big\|^2 \\
&\ge \lim_{M\to\infty} \sum_{i=1}^{M} \Big\|\sum_{k\in\Delta_i} c_k f_\alpha(kx)\Big\|^2 = \sum_{i=1}^{\infty}\sum_{k,\ell\in\Delta_i} c_k c_\ell \frac{(\gcd(k,\ell))^{2\alpha}}{(k\ell)^\alpha}.
\end{aligned} \tag{5.5}$$

By the structure of the set Δ_i for any fixed $k \in \Delta_i$ we have

$$\sum_{\ell\in\Delta_i} \frac{(\gcd(k,\ell))^{2\alpha}}{(k\ell)^\alpha} = \prod_{r=1}^{i}(1+p_r^{-\alpha}),$$

which implies that

$$\sum_{k,\ell\in\Delta_i} \frac{(\gcd(k,\ell))^{2\alpha}}{(k\ell)^\alpha} = 2^i \prod_{r=1}^{i}(1+p_r^{-\alpha}) \tag{5.6}$$

(an argument of this type already appears in Gál's paper [13]). By the prime number theorem we have

$$\prod_{r=1}^{i}(1+p_r^{-\alpha}) \gg \exp\left(\frac{1-\varepsilon}{1-\alpha} i^{1-\alpha}(\log i)^{-\alpha}\right).$$

Combining (5.4)–(5.6) we get

$$\begin{aligned}
\lim_{N\to\infty} \Big\|\sum_{k=1}^{N} c_k f_\alpha(kx)\Big\|^2 & \\
&\gg \sum_{i=1}^{\infty} i^{-2} \exp\left(\frac{(1-\varepsilon)-\eta(1+\varepsilon)}{1-\alpha} i^{1-\alpha}(\log i)^{-\alpha}\right).
\end{aligned} \tag{5.7}$$

Note that $(1-\varepsilon)-\eta(1+\varepsilon) = \varepsilon$, and thus the series on the right-hand side of (5.7) is divergent. Consequently, the series $\sum_{k=1}^{\infty} c_k f_\alpha(kx)$ is divergent in L^2, although $(c_k)_{k\ge 1}$ satisfies the extra convergence condition (2.1) for $K = \eta/(1-\alpha)$. Note that by choosing ε small, η can be moved arbitrarily close to 1. This proves the optimality of Theorem 1, apart from the precise optimal value of the constant K in (2.1). ∎

Proof of the convergence part of Theorem 2. The proof is exactly the same as the proof of the convergence part of Theorem 1 above, using the second parts of Lemmas 2 and 3 instead of the first parts. ∎

Proof of the optimality of Theorem 2. The optimality of condition (2.2) in the case of L^2 convergence can be shown in much the same way as the optimality of condition (2.1) in Theorem 1. Again we construct a set of integers which is composed of a relatively small number of prime factors. In particular, again we will use an equality similar to (5.6), which allows a precise computation of the corresponding GCD sum. Again we choose $f = f_\alpha$, but as in the proof of the optimality of Theorem 1 we could also use the functions $\bar{f}_\alpha$ or $\zeta(1-\alpha,\cdot)$ instead. The main difference between the present case and the proof of Theorem 1 is that we can make the sequence $(n_k)_{k\geq 1}$ grow as fast as we wish. Together with the well-established principle that lacunary sequences of functions show almost independent behavior, this is the reason why for Theorem 2 we can also prove optimality with respect to almost everywhere convergence (which was not possible for Theorem 1).

First we recall that $f_\alpha \in L^p(0,1)$ for $p < (1-\alpha)^{-1}$, which was established in (3.20). Thus we can choose $\delta \in (0,1)$ such that $2+\delta < (1-\alpha)^{-1}$. Furthermore, we can find $\beta \in (0,1)$ which satisfies

$$\beta < \frac{\delta}{2+\delta}.$$

Then

$$\left(-\frac{1}{2}+\frac{\beta}{2}\right)(2+\delta) < -1. \tag{5.8}$$

Let $(p_r)_{r\geq 1}$ denote the sequence of primes in increasing order. We set $A(1)=1$ and

$$A(i) = \lceil \beta \log_2 i \rceil, \qquad i \geq 2.$$

We define numbers S_i and T_i recursively in the following way:

- $S_1 = 2$,
- $T_i = S_i + \left\lceil \log_2\left(\prod_{r=1}^{A(i)} p_r\right)\right\rceil$, $i \geq 1$,
- $S_{i+1} = T_i + \lceil \eta \log_2 i \rceil$, $i \geq 1$, where $\eta = 12/(2\alpha-1)$.

Then obviously $(S_i)_{i\geq 1}$ and $(T_i)_{i\geq 1}$ satisfy the conditions of Lemma 5. For $i \geq 1$, we define Δ_i to be the set of all numbers k of the form

$$k = 2^{S_i} \prod_{r=1}^{A(i)} p_r^{w_r}, \qquad \text{where } (w_1,\ldots,w_{A(i)}) \in \{0,1\}^{A(i)}.$$

Clearly, all elements of Δ_i are divisible by 2^{S_i}, and $\Delta_i \subset [2^{S_i}, 2^{T_i}]$; that is, the sets Δ_i also satisfy the assumptions of Lemma 5.

Let $(n_k)_{k\geq 1}$ denote the sequence consisting of all elements of $\bigcup_{i\geq 1} \Delta_i$ in increasing order. Note that by definition we have

$$\#\Delta_i = 2^{A(i)} \in [i^\beta, 2i^\beta].$$

Furthermore we define sets Γ_i, $i \geq 1$, of integers such that

$$k \in \Gamma_i \quad \text{if and only if} \quad n_k \in \Delta_i.$$

Then $(\Gamma_i)_{i\geq 1}$ is a decomposition of $\mathbb{N}$.

Let K_1 denote a "small" constant to be specified later. For every $k \geq 1$ there is an i such that $k \in \Gamma_i$, and we define

$$c_k = i^{-\beta/2-1/2}(\log i)^{-1} \exp\biggl(-\frac{K_1(\log i)^{1-\alpha}}{2(\log\log i)^\alpha}\biggr).$$

Note that c_k only depends on the index i for which $k \in \Gamma_i$. Thus we can also define numbers $(d_i)_{i\geq 1}$ such that

$$d_i = c_k \quad \text{whenever} \quad k \in \Gamma_i, \qquad \text{for } i, k \geq 1,$$

which implies that

$$\sum_{k\in\Delta_i} c_k f_\alpha(kx) = d_i \sum_{k\in\Gamma_i} f_\alpha(n_k x).$$

Furthermore,

$$\sum_{k=1}^{\infty} c_k^2 \exp\biggl(\frac{K_1(\log i)^{1-\alpha}}{(\log\log i)^\alpha}\biggr) = \sum_{i\geq 1}\sum_{k\in\Gamma_i} i^{-\beta-1}(\log i)^{-2} \cdot \underbrace{\#\Gamma_i}_{\leq 2i^\beta} \tag{5.9}$$

$$\leq 2\sum_{i\geq 1} i^{-1}(\log i)^{-2}. \tag{5.10}$$

Since the series in (5.10) is convergent, so is the left-hand side of (5.9). Furthermore, since $k \ll i^{\beta+1}$ for $k \in \Gamma_i$, the convergence of the left-hand side of (5.9) implies that there exists a positive constant K_2 (depending on K_1) such that

$$\sum_{k=1}^{\infty} c_k^2 \exp\bigl(K_2(\log k)^{1-\alpha}(\log\log k)^{-\alpha}\bigr) < \infty.$$

As in the lines following (5.6) we get

$$\sum_{k,\ell\in\Gamma_i} \frac{(\gcd(n_k,n_\ell))^{2\alpha}}{(n_k n_\ell)^\alpha} = \sum_{k,\ell\in\Delta_i} \frac{(\gcd(k,\ell))^{2\alpha}}{(k\ell)^\alpha} = \#\Delta_i \prod_{r=1}^{A(i)} (1+p_r^{-\alpha}) \tag{5.11}$$

$$\gg i^\beta \exp\bigl(K_3(\log i)^{1-\alpha}(\log\log i)^{-\alpha}\bigr)$$

for some positive constant K_3. Together with the second part of Lemma 1 this implies that

$$(5.12)\qquad \Big\|\sum_{k\in\Gamma_i} c_k f_\alpha(n_k x)\Big\|^2 \gg i^{-1}(\log i)^{-2}\exp\big((K_3-K_1)(\log i)^{1-\alpha}(\log\log i)^{-\alpha}\big).$$

Since all coefficients $(c_k)_{k\geq 1}$ are non-negative, we have

$$\lim_{N\to\infty}\Big\|\sum_{k=1}^{N} c_k f_\alpha(n_k x)\Big\|^2 \geq \lim_{M\to\infty}\sum_{i=1}^{M}\Big\|\sum_{k\in\Delta_i} c_k f_\alpha(n_k x)\Big\|^2.$$

Combining this with (5.12) we arrive at

$$(5.13)\qquad \lim_{N\to\infty}\Big\|\sum_{k=1}^{N} c_k f_\alpha(n_k x)\Big\|^2 \gg \lim_{M\to\infty}\sum_{i=1}^{M} i^{-1}(\log i)^{-2}\exp\big((K_3-K_1)(\log i)^{1-\alpha}(\log\log i)^{-\alpha}\big).$$

We can assume that K_1 was chosen so small that $K_1 < K_3$. Since the right-hand side of (5.13) is then divergent, the series $\sum_{k=1}^{\infty} c_k f_\alpha(n_k x)$ is divergent in L^2. This proves the optimality of Theorem 2 for L^2 convergence (except for the exact value of the constant K in the extra divergence condition).

To show that Theorem 2 is also optimal with respect to almost everywhere convergence, we apply Lemma 5. As noted before, Lemma 5 can be used for $S_i,\ T_i,\ \Delta_i$ as defined above. Consequently, there exist *independent* random variables $Y_1, Y_2, \ldots$ on $((0,1),\mathcal{B},\mathbb{P})$ such that

$$(5.14)\qquad \Big\|d_i Y_i - \sum_{k\in\Gamma_i} c_k f_\alpha(n_k x)\Big\| \ll d_i\Big\|Y_i - \sum_{k\in\Delta_i} f_\alpha(kx)\Big\| \ll i^{-\beta/2-1/2} i^{-2}\cdot \#\Delta_i \ll i^{-5/2+\beta/2}.$$

The proof of Lemma 5 shows that the random variables Y_i are constructed as the conditional expectation of $\sum_{k\in\Delta_i} f_\alpha(n_k x)$ with respect to some appropriate σ-fields. Thus the conditional form of Jensen's inequality (see for example [22, Theorem 13.3]) implies that

$$(5.15)\qquad \mathbb{E}(|d_i Y_i|^{2+\delta}) \leq d_i^{2+\delta}\,\mathbb{E}\Big(\Big(\sum_{k\in\Gamma_i} f_\alpha(n_k\,\cdot)\Big)^{2+\delta}\Big).$$

We have chosen δ so that $f_\alpha\in L^{2+\delta}(0,1)$. Thus by Minkowski's inequality,

$$\Big\|\sum_{k\in\Gamma_i} f_\alpha(n_k\,\cdot)\Big\|_{2+\delta} \leq \|f_\alpha\|_{2+\delta}\cdot \underbrace{\#\Gamma_i}_{=\#\Delta_i} \ll i^{\beta},$$

which together with (5.15) implies that

$$\mathbb{E}(|d_i Y_i|^{2+\delta}) \ll i^{(\beta/2-1/2)(2+\delta)}. \tag{5.16}$$

On the other hand, by (5.12) and (5.14) we have

$$\mathbb{E}((d_i Y_i)^2) \gg i^{-1} (\log i)^{-2} \exp\big(K_4 (\log i)^{1-\alpha} (\log\log i)^{-\alpha}\big), \tag{5.17}$$

where $K_4 := K_3 - K_1$ is a positive constant (again we assume that K_1 was chosen sufficiently small). Let

$$B_M = \sum_{i=1}^{M} \mathbb{E}((d_i Y_i)^2), \qquad D_M = \sum_{i=1}^{M} \mathbb{E}(|d_i Y_i|^{2+\delta}),$$

and

$$F_M(t) = \mathbb{P}\Big(x \in (0,1) : \sum_{i=1}^{M} d_i Y_i < t\sqrt{B_M}\Big).$$

By (5.8) and (5.16) the sequence $(D_M)_{M\ge 1}$ is bounded. On the other hand, by (5.17) we have

$$B_M \gg \exp\big(K_5 (\log M)^{1-\alpha} (\log\log M)^{-\alpha}\big) \tag{5.18}$$

for some positive constant K_5, so in particular $B_M \to \infty$ as $M \to \infty$. Thus, the so-called Lyapunov condition for the central limit theorem is satisfied, which implies that

$$\sup_{t\in\mathbb{R}} |F_M(t) - \Phi(t)| \ll L_M \qquad \text{as } M \to \infty,$$

where

$$L_M := D_M / B_M^{1+\delta/2}$$

and Φ is the standard normal distribution. (For Lyapunov's central limit theorem, see for example [29, p. 126].) Consequently,

$$\mathbb{P}\bigg(\Big|\sum_{i=1}^{M} d_i Y_i\Big| \ge \frac{\sqrt{B_M}}{\log M}\bigg) \to 1 \qquad \text{as } M \to \infty,$$

which together with (5.18) implies that

$$\limsup_{M\to\infty} \Big|\sum_{i=1}^{M} d_i Y_i\Big| = \infty \qquad \text{a.e.}$$

Now (5.14) and the first Borel–Cantelli lemma yield

$$\limsup_{M\to\infty} \Big|\sum_{i=1}^{M} \sum_{k\in\Gamma_i} c_k f_\alpha(n_k x)\Big| = \infty \qquad \text{a.e.},$$

hence

$$\limsup_{N\to\infty}\Big|\sum_{k=1}^{N} c_k f_\alpha(n_k x)\Big| = \infty \qquad \text{a.e.}$$

This proves the optimality of Theorem 2 for almost everywhere convergence.

A more detailed analysis shows that a possible choice for the constant K_1, and accordingly also for $\hat{K}(\alpha)$ in Theorem 2, is

$$K_1 = \big((2\alpha-1)/(2\alpha\log 2)\big)^{1-\alpha}(1-\alpha)^{-1} - \varepsilon$$

for an arbitrary $\varepsilon > 0$. Consequently, the "blowup" of the constant in the extra convergence condition is of order $(1-\alpha)^{-1}$ as $\alpha \to 1$, both in the sufficiency condition and in the optimality result. ∎

Proof of Theorem 3. By (4.2) and Lemma 7 for any real sequence $(c_k)_{k\ge1}$ and for any M, N satisfying $1 \le M < N$ we have

$$\int_0^1 \Big(\sum_{k=M}^{N} c_k f(kx)\Big)^2 dx \ll \sum_{k,\ell=M}^{N} |c_k c_\ell| \frac{(\gcd(k,\ell))^{2\alpha}}{(k\ell)^\alpha} \ll \sum_{k\le N^2} \hat{b}_k^2, \tag{5.19}$$

where the numbers $\hat{b}_k$ are defined by

$$\hat{b}_k = \frac{1}{k^\alpha}\sum_{d|k,\, d\ge M} d^\alpha |c_d|. \tag{5.20}$$

Let $\varepsilon > 0$ be so small that $1 - 2\alpha + \varepsilon < 0$ and that (2.3) holds. For the simplicity of formulas, we will write $\sigma(k)$ for $\sigma_{1-2\alpha+\varepsilon}(k)$. By (5.20) and the Cauchy–Schwarz inequality we have

$$\begin{aligned}
\hat{b}_k^2 &= \Big(\sum_{d|k,\, d\ge M} |c_d|(d/k)^\alpha\Big)^2 = \Big(\sum_{d|k,\, d\ge M} |c_d|(d/k)^{1/2+\varepsilon/2}(k/d)^{-\alpha+1/2+\varepsilon/2}\Big)^2\\
&\le \sum_{d|k,\, d\ge M} c_d^2 (d/k)^{1+\varepsilon} \sum_{d|k,\, d\ge M} (k/d)^{1-2\alpha+\varepsilon}\\
&= \sum_{d|k,\, d\ge M} c_d^2 (d/k)^{1+\varepsilon} \sum_{h|k,\, h\le k/M} h^{1-2\alpha+\varepsilon} \le \sum_{d|k,\, d\ge M} c_d^2 (d/k)^{1+\varepsilon}\sigma(k).
\end{aligned}$$

Thus

$$\sum_{k=M}^{N^2} \hat{b}_k^2 \le \sum_{k=M}^{N^2}\sum_{d|k,\, d\ge M} c_d^2 (d/k)^{1+\varepsilon}\sigma(k) \le \sum_{d=M}^{N^2} c_d^2 d^{1+\varepsilon} \sum \sigma(k) k^{-(1+\varepsilon)}, \tag{5.21}$$

where the inner sum is taken over all k of the form $k = jd$, $j = 1, 2, \ldots$. But

$\sigma(jd) \leq \sigma(d)\sigma(j)$, and thus this inner sum is bounded by

$$\sum_{j=1}^{\infty} \sigma(d)\sigma(j)(dj)^{-(1+\varepsilon)} \ll \sigma(d)d^{-(1+\varepsilon)} \underbrace{\sum_{j=1}^{\infty} \sigma(j)j^{-(1+\varepsilon)}}_{\ll 1} \ll \sigma(d)d^{-(1+\varepsilon)},$$

where we have used the fact that $\sigma(j) \leq d(j) = O(j^\eta)$ for any $\eta > 0$. Substituting this into (5.21) and taking into account (5.19), we get

$$\int_0^1 \Big(\sum_{k=M}^{N} c_k f(kx)\Big)^2 dx \ll \sum_{k=M}^{N^2} c_k^2 \sigma(k). \tag{5.22}$$

By (2.3) the right-hand side of (5.22) can be made arbitrarily small if M is chosen sufficiently large. Thus by the Cauchy convergence test the series (1.3) is convergent in L^2.

To prove the second part of Theorem 3, let $\alpha \in (1/2, 1)$, $0 < \beta < 1$, and choose $\delta > 0$ so small that $\beta(1+\delta) < 1$. Then by the second statement of Theorem 1 there exist $f \in C_\alpha$ and $(c_k)_{k\geq 1}$ such that

$$\sum_{k=1}^{\infty} c_k^2 \exp\left(\frac{\beta(1+\delta)}{1-\alpha} \frac{(\log k)^{1-\alpha}}{\log\log k}\right) < \infty \tag{5.23}$$

but the series (1.3) does not converge in L^2 norm. In view of (2.6), the terms of the sum in (2.4) are smaller than those of (5.23) for sufficiently large k, and thus the sum (2.4) converges. ∎

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