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Approximation properties of β-expansions

by

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1. Introduction. Let $\beta \in (1,2)$ and $I_\beta := [0, 1/(\beta-1)]$. Given $x \in I_\beta$ we say that a sequence $(\epsilon_i)_{i=1}^{\infty} \in \{0,1\}^{\mathbb{N}}$ is a *β-expansion* for x if

$$x = \sum_{i=1}^{\infty} \frac{\epsilon_i}{\beta^i}. \tag{1.1}$$

It is a simple exercise to show that x has a β-expansion if and only if $x \in I_\beta$. Expansions of this form were pioneered in the papers of Parry [17] and Rényi [20]. One significant difference between integer base expansions and β-expansions is that almost every $x \in I_\beta$ has uncountably many β-expansions, unlike in the integer base case where every number has a unique expansion except for a countable set of exceptions which have precisely two. Whenever we use the phrase "almost every", we always mean with respect to Lebesgue measure. The fact that almost every $x \in I_\beta$ has uncountably many β-expansions is due to Sidorov [22].

We say that a finite sequence $(\epsilon_i)_{i=1}^{n} \in \{0,1\}^n$ is an *n-prefix* for x if there exists $(\epsilon_{n+i})_{i=1}^{\infty} \in \{0,1\}^{\mathbb{N}}$ such that

$$x = \sum_{i=1}^{n} \frac{\epsilon_i}{\beta^i} + \sum_{i=1}^{\infty} \frac{\epsilon_{n+i}}{\beta^{n+i}}.$$

So an n-prefix for x is simply any sequence of length n that can be extended to form a β-expansion for x. It is straightforward to show that a sequence $(\epsilon_i)_{i=1}^{n} \in \{0,1\}^n$ is an n-prefix for x if and only if

$$0 \leq x - \sum_{i=1}^{n} \frac{\epsilon_i}{\beta^i} \leq \frac{1}{\beta^n(\beta-1)}. \tag{1.2}$$

When $(\epsilon_i)_{i=1}^{n} \in \{0,1\}^n$ is an n-prefix for x, we also define the number $\sum_{i=1}^{n} \epsilon_i \beta^{-i}$ to be an n-prefix for x. Whether we are referring to a sequence

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or a number should be clear from the context. We refer to any number of the form $\sum_{i=1}^{n}\epsilon_i\beta^{-i}$ as a *level* n *sum*.

In this paper we study how well a typical $x\in I_\beta$ can be approximated by its prefixes. To this end we introduce the following general setup. Let $\Psi:\mathbb{N}\to\mathbb{R}_{\geq 0}$ and

$$W_\beta(\Psi):=\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}\bigcup_{(\epsilon_i)_{i=1}^n\in\{0,1\}^n}\left[\sum_{i=1}^{n}\frac{\epsilon_i}{\beta^i},\sum_{i=1}^{n}\frac{\epsilon_i}{\beta^i}+\Psi(n)\right].$$

Alternatively, $W_\beta(\Psi)$ is the set of $x\in\mathbb{R}$ such that for infinitely many $n\in\mathbb{N}$ there exists a level n sum satisfying the inequalities

$$0\leq x-\sum_{i=1}^{n}\frac{\epsilon_i}{\beta^i}\leq\Psi(n). \tag{1.3}$$

In (1.3) the approximation to x is given by a level n sum, not necessarily an n-prefix for x. However, as the following argument shows, if (1.3) is satisfied by a level n sum then it must also be satisfied by an n-prefix for x. For if $(\epsilon_i)_{i=1}^n$ satisfies (1.3) and $(\epsilon_i)_{i=1}^n$ is not an n-prefix for x, then $\Psi(n)>(\beta^n(\beta-1))^{-1}$ by (1.2). Every element of I_β has an n-prefix for each $n\in\mathbb{N}$. Let us denote the n-prefix for x by $(\epsilon_i')_{i=1}^n$. Applying (1.2) we see that

$$0\leq x-\sum_{i=1}^{n}\frac{\epsilon_i'}{\beta^i}\leq\frac{1}{\beta^n(\beta-1)}<\Psi(n).$$

Therefore, if $x\in W_\beta(\Psi)$ then there exist infinitely many n-prefixes for x satisfying (1.3).

If $\sum_{n=1}^{\infty}2^n\Psi(n)<\infty$, the Borel–Cantelli lemma tells us that $\lambda(W_\beta(\Psi))=0$. Here and throughout $\lambda(\cdot)$ denotes the Lebesgue measure. Motivated by observations and results from metric number theory, we expect that if $\sum_{n=1}^{\infty}2^n\Psi(n)=\infty$ and the level n sums are distributed sufficiently uniformly throughout I_β then $W_\beta(\Psi)$ is a set of full measure within I_β.

With the above in mind we introduce the following definition. We say that β is *approximation regular* if for each function $\Psi:\mathbb{N}\to\mathbb{R}_{\geq 0}$ satisfying $\sum_{n=1}^{\infty}2^n\Psi(n)=\infty$, the set $W_\beta(\Psi)$ is of full measure within I_β. We make the following conjecture.

Conjecture 1.1. *Almost every* $\beta\in(1,2)$ *is approximation regular.*

We cannot hope to extend this "almost every" statement to an "every" statement. For example, if we take β to be a *Pisot number*, i.e., a real algebraic integer strictly greater than 1 whose conjugates all have modulus strictly less than 1, then the cardinality of the set of level n sums is of the order β^n. This follows from Garsia's results [10]. If $\Psi(n)=2^{-n}$ it is clear that $\sum_{n=1}^{\infty} 2^n \Psi(n) = \infty$. However a simple covering argument appealing to the Borel–Cantelli lemma implies $\lambda(W_\beta(\Psi)) = 0$.

In this paper we fail to prove Conjecture 1.1. Instead we show that whenever β is a special type of algebraic integer known as a Garsia number then β is approximation regular. For our purposes a *Garsia number* is a positive real algebraic integer with norm ± 2, whose conjugates are all of modulus strictly greater than 1. Recall that the *norm* of an algebraic integer β is defined to be the product of β with all of its conjugates. The reader should be aware that in the literature Garsia numbers are not always defined to be positive, and in some cases are taken to be complex. Garsia numbers were first studied as a separate significant class of algebraic integers in a paper by Garsia [10]. For more on Garsia numbers we refer the reader to the paper of Hare and Panju [12] and the references therein.

Our main result is the following.

THEOREM 1.2. *Let $\beta \in (1,2)$ be a Garsia number. Then β is approximation regular.*

REMARK 1.3. It is worth commenting on the fact that throughout this paper we have imposed no restrictions on the monotonicity of Ψ. In classical Diophantine approximation, when $\Psi : \mathbb{N} \to \mathbb{R}_{\geq 0}$ is decreasing the set

$$W(\Psi) := \{x \in \mathbb{R} : \text{there exist infinitely many } (p,q) \in \mathbb{Z} \times \mathbb{N} \text{ such that } |x - p/q| \leq \Psi(q)\}$$

is either null or full with respect to Lebesgue measure depending on whether $\sum_{q=1}^{\infty} q\Psi(q)$ converges or diverges. In [6] Duffin and Schaeffer showed that it is not possible to relax the monotonicity assumption on Ψ. They constructed a function $\Psi : \mathbb{N} \to \mathbb{R}_{\geq 0}$ such that $\sum_{q=1}^{\infty} q\Psi(q) = \infty$ yet $\lambda(W(\Psi)) = 0$.

Suppose β is approximation regular and the function $\Psi : \mathbb{N} \to \mathbb{R}_{\geq 0}$ satisfies $\sum_{n=1}^{\infty} 2^n \Psi(n) = \infty$. For a Lebesgue generic $x \in I_\beta$ it is natural to ask whether x has a β-expansion $(\epsilon_i)_{i=1}^{\infty} \in \{0,1\}^{\mathbb{N}}$ such that the inequalities

$$0 \leq x - \sum_{i=1}^{n} \frac{\epsilon_i}{\beta^i} \leq \Psi(n)$$

are satisfied for infinitely many $n \in \mathbb{N}$. This turns out to be the case whenever Ψ satisfies a mild technical condition. We say that $\Psi : \mathbb{N} \to \mathbb{R}_{\geq 0}$ is *decaying regularly* if for each $m \in \mathbb{N}$ there exists $C_m \in \mathbb{N}$ such that

$$\frac{\Psi(n+m)}{\Psi(n)} \geq \frac{1}{C_m} \tag{1.4}$$

for every $n \in \mathbb{N}$. We emphasise that the constant C_m is allowed to depend on m. As an example, when $\Psi(n) = 2^{-n}$ then Ψ is decaying regularly. For each $m \in \mathbb{N}$ we can take $C_m = 2^m$.

Theorem 1.4. *Let β be approximation regular and suppose $\Psi : \mathbb{N} \to \mathbb{R}_{\geq 0}$ is decaying regularly and satisfies $\sum_{n=1}^{\infty} 2^n \Psi(n) = \infty$. Then for almost every $x \in I_\beta$ there exists a β-expansion for x satisfying the inequalities*

$$0 \leq x - \sum_{i=1}^{n} \frac{\epsilon_i}{\beta^i} \leq \Psi(n)$$

for infinitely many $n \in \mathbb{N}$.

As an application of Theorems 1.2 and 1.4 we have the following result.

Corollary 1.5. *Let $\beta \in (1,2)$ be a Garsia number. Then for almost every $x \in I_\beta$ there exists a β-expansion of x which satisfies the inequalities*

$$0 \leq x - \sum_{i=1}^{n} \frac{\epsilon_i}{\beta^i} \leq \frac{1}{n 2^n \log n}$$

for infinitely many $n \in \mathbb{N}$.

In Section 3 we prove Theorem 1.2 and in Section 4 we prove Theorem 1.4. In Section 5 we discuss the connection between the set $I_\beta \setminus W_\beta(\Psi)$ and the set of points with a unique β-expansion. We end our introduction by giving a summary of related work undertaken by other authors.

In two recent papers by Persson and Reeve [18, 19], the authors considered a setup similar to ours. Let

$$K_\beta(\Psi) := \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \bigcup_{(\epsilon_i)_{i=1}^n \in \{0,1\}^n} \Big[\sum_{i=1}^{n} \frac{\epsilon_i}{\beta^i} - \Psi(n), \sum_{i=1}^{n} \frac{\epsilon_i}{\beta^i} + \Psi(n) \Big].$$

Notice that $W_\beta(\Psi) \subseteq K_\beta(\Psi)$. In the definition of $K_\beta(\Psi)$ the level n sums form the centres of the significant intervals, whereas in the definition of $W_\beta(\Psi)$ the level n sums are the left endpoints of the significant intervals. The reason we have insisted on the level n sums being the left endpoints is because we are interested in the approximation provided by an n-prefix, rather than a general level n sum. It is an obvious consequence of (1.2) that if $x < \sum_{i=1}^{n} \epsilon_i \beta^{-i}$ then $(\epsilon_i)_{i=1}^n \in \{0,1\}^n$ cannot be an n-prefix for x.

Persson and Reeve studied the set $K_\beta(\Psi)$ when $\Psi(n) = 2^{-\alpha n}$ for some $\alpha \in (1, \infty)$. In this case $\sum_{n=1}^{\infty} 2^n \Psi(n)$ always converges. Motivated by Falconer [9], they studied the intersection properties of $K_\beta(\Psi)$. Falconer defined G^s to be the set of $A \subseteq \mathbb{R}$ which have the property that for any countable collection of similarities $\{f_j\}_{j=1}^{\infty}$,

$$\dim_H \Big(\bigcap_{j=1}^{\infty} f_j(A) \Big) \geq s.$$

Persson and Reeve generalised the definition of G^s to arbitrary intervals I by defining $G^s(I) := \{A \subseteq I : A + \mathrm{diam}(I)\mathbb{Z} \in G^s\}$. The main results of [18, 19] can be summarised in the following theorem.

THEOREM 1.6. *Let $\alpha \in (1,\infty)$ and $\Psi(n) = 2^{-\alpha n}$.*

- *For all $\beta \in (1,2)$, $\dim_H(K_\beta(\Psi)) \le 1/\alpha$.*
- *For almost every $\beta \in (1,2)$, $K_\beta(\Psi) \in G^s(I_\beta)$ for $s = 1/\alpha$.*
- *For a dense set of $\beta \in (1,2)$, $\dim_H(K_\beta(\Psi)) < 1/\alpha$.*
- *For all $\beta \in (1,2)$, $K_\beta(\Psi) \in G^s(I_\beta)$ for $s = \frac{\log\beta}{\alpha\log 2}$.*
- *For a countable set of $\beta \in (1,2)$, $\dim_H(K_\beta(\Psi)) = \frac{\log\beta}{\alpha\log 2}$.*

The approximation properties of β-expansions were also studied in a paper by Dajani, Komornik, Loreti, and de Vries [4]. Given $x \in I_\beta$ and $(\epsilon_i)_{i=1}^\infty$ a β-expansion for x, we say that $(\epsilon_i)_{i=1}^\infty$ is an *optimal expansion* if for any other β-expansion for x the following holds for all $n \in \mathbb{N}$:

$$x - \sum_{i=1}^n \frac{\epsilon_i}{\beta^i} \le x - \sum_{i=1}^n \frac{\epsilon_i'}{\beta^i}.$$

In other words, a β-expansion for x is an optimal expansion if for each $n \in \mathbb{N}$ the n-prefix $(\epsilon_i)_{i=1}^n$ always provides the closest approximation to x. Before we state the main result of [4] we recall a definition: a *multinacci number* is the unique root of an equation of the form $x^n = x^{n-1} + \cdots + x + 1$ lying in $(1,2)$, where $n \ge 2$. The golden ratio is a multinacci number; this is the case when $n = 2$. It can be shown that every multinacci number is a Pisot number. The main result of [4] is the following.

THEOREM 1.7.

- *Let β be a multinacci number. Then every $x \in I_\beta$ has an optimal expansion.*
- *If $\beta \in (1,2)$ is not a multinacci number, then the set of $x \in I_\beta$ with an optimal expansion is nowhere dense and has zero Lebesgue measure.*

2. Preliminaries. In this section we state the necessary background information from the theory of Bernoulli convolutions. Let $\beta \in (1,2)$. The *Bernoulli convolution* associated to β is defined to be the measure μ_β where

$$\mu_\beta(E) = \mathbb{P}\bigg(\bigg\{(\epsilon_i)_{i=1}^\infty \in \{0,1\}^{\mathbb{N}} : \sum_{i=1}^\infty \frac{\epsilon_i}{\beta^i} \in E\bigg\}\bigg)$$

for any Borel set $E \subseteq \mathbb{R}$. Here $\mathbb{P}$ is the $(1/2,1/2)$ probability measure on $\{0,1\}^{\mathbb{N}}$. It is a long-standing problem to determine precisely those β for which μ_β is absolutely continuous with respect to Lebesgue measure. When μ_β is absolutely continuous, we denote the density function by h_β. We emphasise that the density function is only defined almost everywhere.

Jessen and Wintner showed that μ_β is either absolutely continuous with respect to the Lebesgue measure or purely singular [13]. This was later improved upon by Simon and Mauldin [16], who showed that μ_β is either equivalent to the Lebesgue measure or purely singular. Erdős [7] showed that whenever β is a Pisot number then μ_β is purely singular. No other examples of $\beta \in (1,2)$ for which μ_β is singular are known. In a standout paper, Solomyak proved that for almost every $\beta \in (1,2)$ the Bernoulli convolution is absolutely continuous [23]. This was later improved upon in a paper of Shmerkin [21], where it was shown that the set of $\beta \in (1,2)$ for which μ_β is singular has Hausdorff dimension zero. Loosely speaking, it is believed that whenever the level n sums are distributed sufficiently uniformly throughout I_β, then the associated Bernoulli convolution will be absolutely continuous. Similarly, when the level n sums are distributed sufficiently uniformly throughout I_β, we expect β to be approximation regular. As such, the results of Shmerkin and Solomyak lend some weight to the validity of Conjecture 1.1.

The following theorem due to Garsia [10] will be essential in our later work.

Theorem 2.1. *If $\beta \in (1,2)$ is a Garsia number then μ_β is absolutely continuous. Moreover, the density of μ_β is bounded above by*

$$\frac{2}{\prod_{i=1}^{k}(\gamma_i - 1)}.$$

Here $\gamma_1, \ldots, \gamma_k$ are the conjugates of β.

Garsia numbers are the largest explicit class of real numbers for which it is known that μ_β is always absolutely continuous.

Our proof of Theorem 1.2 also requires the following results taken from Kempton [14]. They emphasise the connection between β-expansions and Bernoulli convolutions. Given $\beta \in (1,2)$ and $x \in I_\beta$, we denote the set of n-prefixes for x by $\Sigma_{\beta,n}(x)$. Kempton studied the growth rate of $|\Sigma_{\beta,n}(x)|$, in particular the following limits:

$$\underline{f}(x) := \liminf_{n\to\infty} \frac{(\beta-1)\beta^n}{2^n}|\Sigma_{\beta,n}(x)|, \qquad \overline{f}(x) := \limsup_{n\to\infty} \frac{(\beta-1)\beta^n}{2^n}|\Sigma_{\beta,n}(x)|.$$

The main results of [14] are the following two theorems.

Theorem 2.2. *The Bernoulli convolution μ_β is absolutely continuous if and only if*

$$0 < \int_{I_\beta} \overline{f}(x)\,dx < \infty.$$

In this case the density h_β of μ_β satisfies

$$h_\beta(x) = \frac{\overline{f}(x)}{\int_{I_\beta} \overline{f}(y)\,dy}.$$

THEOREM 2.3. *Suppose that*

$$0 < \int_{I_\beta} \underline{f}(x)\, dx < \infty.$$

Then μ_β is absolutely continuous with density function

$$h_\beta(x) = \frac{\underline{f}(x)}{\int_{I_\beta} \underline{f}(y)\, dy}.$$

Conversely, if μ_β is absolutely continuous with bounded density function h_β then

$$0 < \int_{I_\beta} \underline{f}(x)\, dx < \infty.$$

When $\beta \in (1, 2)$ is a Garsia number, Theorem 2.1 tells us that μ_β is absolutely continuous with bounded density function h_β. By combining Theorems 2.2 and 2.3 the following proposition is immediate.

PROPOSITION 2.4. *Let $\beta \in (1, 2)$ be a Garsia number and $x \in I_\beta$ be such that $h_\beta(x)$ is defined. Then there exist $K_1 > 1$ and $N(x) \in \mathbb{N}$ sufficiently large such that for all $n \geq N(x)$,*

$$\frac{h_\beta(x)}{K_1} \leq \frac{\beta^n}{2^n} |\Sigma_{\beta,n}(x)| \leq K_1 h_\beta(x).$$

Here K_1 only depends on β.

Proposition 2.4 will be a vital tool when it comes to proving Theorem 1.2.

3. Proof of Theorem 1.2. Our proof of Theorem 1.2 is inspired by the work of Beresnevich [1, 2]. However, it is not a simple case of swapping notation where appropriate—a much more delicate argument is required.

We start by proving several technical lemmas. The following one is due to Garsia [10].

LEMMA 3.1. *Let $\beta \in (1, 2)$ be a Garsia number and $(\epsilon_i)_{i=1}^n, (\epsilon_i')_{i=1}^n \in \{0, 1\}^n$. If $(\epsilon_i)_{i=1}^n \neq (\epsilon_i')_{i=1}^n$ then*

$$\left| \sum_{i=1}^n \frac{\epsilon_i}{\beta^i} - \sum_{i=1}^n \frac{\epsilon_i'}{\beta^i} \right| > \frac{K_2}{2^n}$$

for some strictly positive constant K_2 that only depends on β.

Lemma 3.1 is well known. However to keep our work as self-contained as possible we provide a short proof.

Proof. Let $(\epsilon_i)_{i=1}^n, (\epsilon_i')_{i=1}^n \in \{0, 1\}^n$ and assume $(\epsilon_i)_{i=1}^n \neq (\epsilon_i')_{i=1}^n$. We introduce the polynomials

$$P(z) = \epsilon_1 z^{n-1} + \cdots + \epsilon_{n-1} z + \epsilon_n, \qquad P'(z) = \epsilon_1' z^{n-1} + \cdots + \epsilon_{n-1}' z + \epsilon_n'.$$

Since β is an algebraic integer with norm ± 2 it is a zero of no polynomial with coefficients in $\{-1,0,1\}$. Therefore $P(\beta)-P'(\beta)\neq 0$. Moreover, if $\gamma_1,\ldots,\gamma_k$ are the conjugates of β then

$$(P(\beta)-P'(\beta))\prod_{i=1}^{k}(P(\gamma_i)-P'(\gamma_i))\in\mathbb{Z}\setminus\{0\}. \tag{3.1}$$

Taking the absolute value of (3.1) and applying a trivial lower bound, we see that (3.1) implies that

$$\begin{aligned}1 &\le \Big|(P(\beta)-P'(\beta))\prod_{i=1}^{k}(P(\gamma_i)-P'(\gamma_i))\Big|\\ &\le |P(\beta)-P'(\beta)|\prod_{i=1}^{k}(1+|\gamma_i|+\cdots+|\gamma_i^{n-1}|) < |P(\beta)-P'(\beta)|\prod_{i=1}^{k}\frac{|\gamma_i^n|}{|\gamma_i|-1}\\ &\le |P(\beta)-P'(\beta)|\frac{2^n}{\beta^n}\prod_{i=1}^{k}\frac{1}{|\gamma_i|-1} = 2^n\Big|\sum_{i=1}^{n}\frac{\epsilon_i}{\beta^i}-\sum_{i=1}^{n}\frac{\epsilon_i'}{\beta^i}\Big|\prod_{i=1}^{k}\frac{1}{|\gamma_i|-1}.\end{aligned}$$

This gives the required lower bound. In the above we have used the fact $\beta^n\prod_{i=1}^{k}|\gamma_i|^n=2^n$, which follows from the norm of β being ± 2. ∎

Recall the Lebesgue differentiation theorem: if $f\in L^1(\mathbb{R})$ then for almost every $x\in\mathbb{R}$,

$$\lim_{r\to 0}\frac{1}{2r}\int_{B_r(x)} f(y)\,d\lambda(y)=f(x). \tag{3.2}$$

Here $B_r(x)$ denotes the closed interval centred at x with radius r. Given $f\in L^1(\mathbb{R})$, we call any $x\in\mathbb{R}$ satisfying (3.2) a *Lebesgue differentiation point for f*. The Lebesgue differentiation theorem tells us that given $f\in L^1(\mathbb{R})$, almost every $x\in\mathbb{R}$ is a Lebesgue differentiation point for f. With this theorem in mind we establish the following lemma.

LEMMA 3.2. *Let $\beta\in(1,2)$ be a Garsia number, and let $x\in I_\beta$ be a Lebesgue differentiation point for h_β satisfying $h_\beta(x)>0$. Let $r^*(x)$ be such that*

$$\frac{h_\beta(x)}{2}\le\frac{1}{2r}\int_{B_r(x)} h_\beta(y)\,d\lambda(y)$$

for all $r\in(0,r^*(x))$. Then there exists $L\in\mathbb{N}$ and $\kappa\in(1,2)$ such that for all $r\in(0,r^*(x))$,

$$\lambda(\{y\in B_r(x):h_\beta(y)\le 1/L\})\le\kappa r.$$

Moreover, L and κ only depend upon β and x.

Proof. Fix β and x that satisfy the hypothesis of the lemma. We begin by relabelling the upper bound for the density provided by Theorem 2.1. Let

$$C := \frac{2}{\prod_{i=1}^{k}(\gamma_i - 1)}$$

where $\gamma_1, \ldots, \gamma_k$ are the conjugates of β. To each $L \in \mathbb{N}$ we associate

$$A_L := \{y \in B_r(x) : h_\beta(y) \le 1/L\}.$$

For $r \in (0, r^*(x))$, by trivial estimates,

$$\begin{aligned}\frac{h_\beta(x)}{2} &\le \frac{1}{2r}\Big(\int_{A_L} h_\beta(y)\,d\lambda(y) + \int_{B_r(x)\setminus A_L} h_\beta(y)\,d\lambda(y)\Big) \\ &\le \frac{1}{2r}\bigg(\frac{1}{L}\lambda(A_L) + (2r - \lambda(A_L))C\bigg).\end{aligned} \tag{3.3}$$

Manipulating (3.3) yields

$$\lambda(A_L)(C - 1/L) \le r(2C - h_\beta(x)). \tag{3.4}$$

We may assume that $L \in \mathbb{N}$ is sufficiently large that $C - L^{-1} > 0$. Hence

$$\lambda(A_L) \le r\bigg(\frac{2C - h_\beta(x)}{C - 1/L}\bigg). \tag{3.5}$$

It is obvious that as $L \to \infty$,

$$\frac{2C - h_\beta(x)}{C - 1/L} \to \frac{2C - h_\beta(x)}{C}.$$

Since $(2C - h_\beta(x))C^{-1} \in (1, 2)$, we conclude that there exist $L \in \mathbb{N}$ and $\kappa \in (1, 2)$ such that for all $r \in (0, r^*(x))$ we have $\lambda(A_L) \le \kappa r$. Moreover, both L and κ only depend upon x and β. ∎

We also make use of the following lemma due to Chung and Erdős [3].

LEMMA 3.3. *Let $(E_n)_{n=1}^{\infty}$ be a sequence of measurable sets contained in a bounded interval. If $\sum_{n=1}^{\infty} \lambda(E_n) = \infty$, then*

$$\lambda\Big(\limsup_{n\to\infty} E_n\Big) \ge \limsup_{k\to\infty} \frac{(\sum_{n=1}^{k} \lambda(E_n))^2}{\sum_{n=1}^{k}\sum_{m=1}^{k} \lambda(E_n \cap E_m)}.$$

We are now in a position to give our proof of Theorem 1.2.

Proof of Theorem 1.2. The proof depends on an application of the Lebesgue density theorem, which states that if $E \subseteq \mathbb{R}$ is a measurable set, then for almost every $x \in E$,

$$\lim_{r\to 0} \frac{\lambda(E \cap B_r(x))}{2r} = 1.$$

As a consequence, to show that $W_\beta(\Psi)$ is of full measure within I_β, it suffices to show that for almost every $x \in I_\beta$ there exists $\delta > 0$ such that

$$\lambda(W_\beta(\Psi) \cap B_r(x)) \geq \delta r \tag{3.6}$$

for all r sufficiently small. Here δ is allowed to depend on x but not on r. This will be the strategy we employ to show $W_\beta(\Psi)$ is of full measure. It is worth noting that the Lebesgue density theorem is simply the Lebesgue differentiation theorem when f is the indicator function on E.

For the rest of the proof we fix $x \in I_\beta$. We only need to show that (3.6) holds for almost every $x \in I_\beta$. We may therefore assume without loss of generality that: $h_\beta(x)$ exists, $h_\beta(x) > 0$, and x is a Lebesgue differentiation point for h_β. In this case, both Proposition 2.4 and Lemma 3.2 can be applied. The fact that we can take $h_\beta(x) > 0$ is a consequence of the aforementioned work of Simon and Mauldin [16], who showed that if μ_β is absolutely continuous with respect to the Lebesgue measure then it is in fact equivalent to the Lebesgue measure.

For ease of exposition we break what remains of our proof into three parts.

(1) *Replacing Ψ with $\tilde{\Psi}$.* Let K_2 be as in Lemma 3.1, so if $(\epsilon_i)_{i=1}^n \neq (\epsilon_i')_{i=1}^n$ then

$$\left|\sum_{i=1}^n \frac{\epsilon_i}{\beta^i} - \sum_{i=1}^n \frac{\epsilon_i'}{\beta^i}\right| > \frac{K_2}{2^n}. \tag{3.7}$$

Let $\tilde{\Psi}(n) = \min\{\Psi(n), K_2 2^{-n}\}$. Then $\sum_{n=1}^\infty 2^n \tilde{\Psi}(n) = \infty$. To see this, we remark that if $\sum_{n=1}^\infty 2^n \tilde{\Psi}(n) < \infty$ then there must exist infinitely many $n \in \mathbb{N}$ for which $\tilde{\Psi}(n) = K_2 2^{-n}$. This is a consequence of $\sum_{n=1}^\infty 2^n \Psi(n)$ diverging. However, this implies that for infinitely many $n \in \mathbb{N}$ the term $2^n \tilde{\Psi}(n)$ equals K_2, and as $K_2 > 0$ the sum must diverge.

Clearly $W_\beta(\tilde{\Psi}) \subseteq W_\beta(\Psi)$. Therefore, to show that (3.6) holds and $W_\beta(\Psi)$ is a set of full measure within I_β, it is sufficient to show that the following analogue of (3.6) holds for some $\delta > 0$ and for all r sufficiently small:

$$\lambda(W_\beta(\tilde{\Psi}) \cap B_r(x)) \geq \delta r. \tag{3.8}$$

The important feature of our new function $\tilde{\Psi}$ is that (3.7) implies that for $(\epsilon_i)_{i=1}^n \neq (\epsilon_i')_{i=1}^n$ we have

$$\left[\sum_{i=1}^n \frac{\epsilon_i}{\beta^i}, \sum_{i=1}^n \frac{\epsilon_i}{\beta^i} + \tilde{\Psi}(n)\right] \cap \left[\sum_{i=1}^n \frac{\epsilon_i'}{\beta^i}, \sum_{i=1}^n \frac{\epsilon_i'}{\beta^i} + \tilde{\Psi}(n)\right] = \emptyset. \tag{3.9}$$

This observation will prove useful later on in our proof.

(2) *Construction of the E_n.* Let $r \in (0, r^*(x))$ and $L \in \mathbb{N}$ be as in Lemma 3.2. Let

$$B_L := \{y \in B_r(x) : h_\beta(y) \geq 1/L\}.$$

Lemma 3.2 tells us that $\lambda(B_L) \geq \omega r$ where $\omega := 2 - \kappa > 0$. Importantly, ω only depends upon β and x.

Proposition 2.4 shows that for almost every $y \in I_\beta$ there exists $N(y) \in \mathbb{N}$ sufficiently large that

$$\frac{h_\beta(y)}{K_1} \leq \frac{\beta^n}{2^n}|\Sigma_{\beta,n}(y)| \leq h_\beta(y)K_1 \tag{3.10}$$

for all $n \geq N(y)$. Using the upper bound for the density provided by Theorem 2.1, we see that for almost every $y \in B_L$ there exists $N(y) \in \mathbb{N}$ such that

$$\frac{1}{LK_1} \leq \frac{\beta^n}{2^n}|\Sigma_{\beta,n}(y)| \leq \frac{2K_1}{\prod_{i=1}^k(\gamma_i - 1)} \tag{3.11}$$

for all $n \geq N(y)$. Now let us take $N^* \in \mathbb{N}$ to be sufficiently large that

(3.12)

$$\lambda\left(\left\{y \in B_L : \frac{1}{LK_1} \leq \frac{\beta^n}{2^n}|\Sigma_{\beta,n}(y)| \leq \frac{2K_1}{\prod_{i=1}^k(\gamma_i - 1)} \text{ for all } n \geq N^*\right\}\right) \geq \frac{\omega r}{2}.$$

Throughout our proof, N^* is allowed to depend on r. Let

$$C := \left\{y \in B_L : \frac{1}{LK_1} \leq \frac{\beta^n}{2^n}|\Sigma_{\beta,n}(y)| \leq \frac{2K_1}{\prod_{i=1}^k(\gamma_i - 1)} \text{ for all } n \geq N^*\right\}.$$

Upon relabelling, any $y \in C$ satisfies

$$\frac{1}{K_3} \leq \frac{\beta^n}{2^n}|\Sigma_{\beta,n}(y)| \leq K_3 \tag{3.13}$$

for all $n \geq N^*$, where K_3 is some positive constant depending only upon β and x. Importantly, K_3 does not depend on r.

We now focus our attention on the interval $B_r(x)$. Fix $n \geq N^*$ where N^* is as above. We fill $B_r(x)$ with closed intervals satisfying certain desirable properties. We may pick a set of closed intervals satisfying:

- Each interval is of width $(\beta^n(\beta - 1))^{-1}$.
- Each is strictly contained in $B_r(x)$.
- If they intersect it is only at a shared endpoint.
- They cover all of $B_r(x)$ except for a set of measure at most $\omega r/4$.

To assert that there exists a set of intervals with this covering property, it is necessary to assume that N^* is sufficiently large. This is permissible as N^* is allowed to depend on r. Let $\{I_j^n\}$ denote a set of intervals with the above properties. Then it is a consequence of (3.12) that

$$\lambda\Big(\bigcup_j I_j^n \cap C\Big) \geq \omega r/4. \tag{3.14}$$

Without loss of generality, we may assume that the enumeration of the set $\{I_j^n\}$ is such that I_1^n is the leftmost interval, then I_2^n sits immediately to the right of I_1^n, then I_3^n sits immediately to the right of I_2^n, and so on. This implies that for any two distinct intervals in $\{I_j^n\}$ whose subscripts have the same parity, there is at least one interval of size $(\beta^n(\beta-1))^{-1}$ sitting between them. We partition $\{I_j^n\}$ into two subsets, those with an odd subscript, $\{I_{j,\mathrm{odd}}^n\}$, and those with an even subscript, $\{I_{j,\mathrm{even}}^n\}$. It is a consequence of (3.14) that

$$\lambda\Big(\bigcup_j I_{j,\mathrm{odd}}^n \cap C\Big) \geq \omega r/8 \quad \text{or} \quad \lambda\Big(\bigcup_j I_{j,\mathrm{even}}^n \cap C\Big) \geq \omega r/8.$$

Without loss of generality we assume that $\lambda(\bigcup I_{j,\mathrm{odd}}^n \cap C) \geq \omega r/8$. Let

$$J := \{I_{j,\mathrm{odd}}^n : \mathrm{int}(I_{j,\mathrm{odd}}^n) \cap C \neq \emptyset\}.$$

Each $I_{j,\mathrm{odd}}^n$ is of width $(\beta^n(\beta-1))^{-1}$, therefore

$$|J| \geq [\beta^n(\beta-1)\omega r/8].$$

We pick a subset of J with cardinality precisely $[\beta^n(\beta-1)\omega r/8]$. Abusing notation we also denote this set by J.

For each $I_{j,\mathrm{odd}}^n \in J$ we choose a point $\alpha_j^n \in \mathrm{int}(I_{j,\mathrm{odd}}^n) \cap C$. Since $|J| = [\beta^n(\beta-1)\omega r/8]$ we have

$$|\{\alpha_j^n\}| = [\beta^n(\beta-1)\omega r/8]. \tag{3.15}$$

For each α_j^n, let $\{\nu_{s,j}^n\}$ be the set $\Sigma_{\beta,n}(\alpha_j^n)$ of n-prefixes. We are now in a position to define

$$E_n := \bigcup_{\alpha_j^n} \bigcup_{\nu_{s,j}^n \in \Sigma_{\beta,n}(\alpha_j^n)} [\nu_{s,j}^n, \nu_{s,j}^n + \tilde{\Psi}(n)]. \tag{3.16}$$

For distinct $\alpha_j^n, \alpha_{j'}^n$ we have $|\alpha_j^n - \alpha_{j'}^n| > (\beta^n(\beta-1))^{-1}$. This is because α_j^n and $\alpha_{j'}^n$ are in the interior of distinct I_j^n and $I_{j'}^n$, where j and j' have the same parity. Recall that it is as a consequence of our construction that for any two intervals of the same parity there exists an interval of width $(\beta^n(\beta-1))^{-1}$ sitting between them. By (1.2) each element of $\Sigma_{\beta,n}(\alpha_j^n)$ is contained in $[\alpha_j^n - 1/(\beta^n(\beta-1)), \alpha_j]$, and similarly each element of $\Sigma_{\beta,n}(\alpha_{j'}^n)$ is contained in $[\alpha_{j'}^n - 1/(\beta^n(\beta-1)), \alpha_{j'}^n]$. Therefore $\Sigma_{\beta,n}(\alpha_j^n) \cap \Sigma_{\beta,n}(\alpha_{j'}^n) = \emptyset$, and by (3.9) we may conclude that any two distinct intervals $[\nu_{s,j}^n, \nu_{s,j}^n + \tilde{\Psi}(n)]$

and $[\nu^n_{s',j'}, \nu^n_{s',j'} + \tilde{\Psi}(n)]$ appearing in (3.16) are disjoint. Making use of this fact, along with (3.13) and (3.15), we observe that

$$\left[\frac{\beta^n(\beta-1)\omega r}{8}\right]\frac{2^n}{\beta^n K_3}\tilde{\Psi}(n) \leq \lambda(E_n) \leq \left[\frac{\beta^n(\beta-1)\omega r}{8}\right]\frac{2^n K_3}{\beta^n}\tilde{\Psi}(n). \tag{3.17}$$

It is clear that (3.17) implies

$$\frac{2^n r}{K_4}\tilde{\Psi}(n) \leq \lambda(E_n) \leq 2^n r K_4 \tilde{\Psi}(n) \tag{3.18}$$

for some positive constant K_4 that only depends upon β and x.

Clearly $\limsup_{n\to\infty} E_n \subset W_\beta(\tilde{\Psi}) \cap B_r(x)$. Therefore to show that there exists $\delta > 0$ for which (3.8) holds, it suffices to show that there exists $\delta > 0$ such that

$$\lambda\Big(\limsup_{n\to\infty} E_n\Big) \geq \delta r. \tag{3.19}$$

Estimates (3.18) and our divergence assumption imply $\sum_{n=N^*}^{\infty} \lambda(E_n) = \infty$. Therefore we can apply Lemma 3.3. In the next part of our proof we obtain a lower bound for $\lambda(\limsup_{n\to\infty} E_n)$ using Lemma 3.3. As we will see, this lower bound yields a δ that satisfies (3.19).

(3) *Applying Lemma 3.3 to E_n.* To begin with, let $M_0 \in \mathbb{N}$ be sufficiently large that

$$\sum_{n=N^*}^{M_0} 2^n \tilde{\Psi}(n) > 1. \tag{3.20}$$

Let $m, n \geq N^*$. For any $\nu^m_{s,j}$, the number of $\nu^n_{s',j'}$ whose corresponding interval $[\nu^n_{s',j'}, \nu^n_{s',j'} + \tilde{\Psi}(n)]$ may intersect $[\nu^m_{s,j}, \nu^m_{s,j} + \tilde{\Psi}(m)]$ is at most

$$2 + \frac{\tilde{\Psi}(m)}{K_2 2^{-n}} = 2 + \frac{2^n \tilde{\Psi}(m)}{K_2},$$

by Lemma 3.1. Therefore

$$\lambda(E_n \cap [\nu^m_{s,j}, \nu^m_{s,j} + \tilde{\Psi}(m)]) \leq \tilde{\Psi}(n)\bigg(2 + \frac{2^n \tilde{\Psi}(m)}{K_2}\bigg). \tag{3.21}$$

Applying (3.13) and (3.15) it is clear that

$$\Big|\bigcup_{\alpha^m_j} \Sigma_{\beta,m}(\alpha^m_j)\Big| \leq \left[\frac{\beta^m(\beta-1)\omega r}{8}\right]\frac{2^m}{\beta^m}K_3.$$

Therefore

$$\Big|\bigcup_{\alpha^m_j} \Sigma_{\beta,m}(\alpha^m_j)\Big| \leq 2^m r K_5, \tag{3.22}$$

where K_5 is a positive constant depending only on β and x. Combining (3.21) with (3.22) we obtain the bound

$$\begin{aligned}\lambda(E_n\cap E_m)&\leq 2^m rK_5\Big(\tilde{\Psi}(n)\Big(2+\frac{2^n\tilde{\Psi}(m)}{K_2}\Big)\Big)\\&\leq 2rK_5\Big(2^m\tilde{\Psi}(n)+\frac{2^{n+m}\tilde{\Psi}(n)\tilde{\Psi}(m)}{K_2}\Big).\end{aligned}\tag{3.23}$$

We now give an upper bound for the double sum in the denominator in Lemma 3.3. First of all we split up the terms in this summation:

$$\sum_{n=N^*}^{M_0}\sum_{m=N^*}^{M_0}\lambda(E_n\cap E_m)=\sum_{n=N^*}^{M_0}\lambda(E_n)+2\sum_{n=N^*+1}^{M_0}\sum_{m=N^*}^{n-1}\lambda(E_n\cap E_m).\tag{3.24}$$

By (3.18) and (3.20) we obtain

$$\sum_{n=N^*}^{M_0}\lambda(E_n)\leq rK_4\sum_{n=N^*}^{M_0}2^n\tilde{\Psi}(n)\leq rK_4\Big(\sum_{n=N^*}^{M_0}2^n\tilde{\Psi}(n)\Big)^2\tag{3.25}$$

As a consequence of (3.23),

$$\begin{aligned}&\sum_{n=N^*+1}^{M_0}\sum_{m=N^*}^{n-1}\lambda(E_n\cap E_m)\\&\qquad\leq 2rK_5\sum_{n=N^*+1}^{M_0}\sum_{m=N^*}^{n-1}\Big(2^m\tilde{\Psi}(n)+\frac{2^{n+m}\tilde{\Psi}(n)\tilde{\Psi}(m)}{K_2}\Big).\end{aligned}\tag{3.26}$$

We now split the summation in (3.26) into two summations. For the first we have the bound

$$\sum_{n=N^*+1}^{M_0}\sum_{m=N^*}^{n-1}2^m\tilde{\Psi}(n)\leq\sum_{n=N^*+1}^{M_0}2^n\tilde{\Psi}(n)\leq\Big(\sum_{n=N^*}^{M_0}2^n\tilde{\Psi}(n)\Big)^2.\tag{3.27}$$

For the second summation in (3.26) we observe

$$\sum_{n=N^*+1}^{M_0}\sum_{m=N^*}^{n-1}2^{n+m}\tilde{\Psi}(n)\tilde{\Psi}(m)\leq\Big(\sum_{n=N^*}^{M_0}2^n\tilde{\Psi}(n)\Big)^2.\tag{3.28}$$

Combining (3.18), (3.24)–(3.28) we obtain

$$\begin{aligned}&\frac{(\sum_{n=N^*}^{M_0}\lambda(E_n))^2}{\sum_{n=N^*}^{M_0}\sum_{m=N^*}^{M_0}\lambda(E_n\cap E_m)}\\&\qquad\geq\frac{r^2K_4^{-2}(\sum_{n=N^*}^{M_0}2^n\tilde{\Psi}(n))^2}{r(K_4+4K_5+4K_2^{-1}K_5)(\sum_{n=N^*}^{M_0}2^n\tilde{\Psi}(n))^2}.\end{aligned}\tag{3.29}$$

Let

$$\delta := \frac{K_4^{-2}}{K_4 + 4K_5 + 4K_2^{-1}K_5};$$

it is clear that δ only depends on β and x. Combining Lemma 3.3 and (3.29) we obtain

$$\lambda\Big(\limsup_{n\to\infty} E_n\Big) \geq \delta r.$$

Therefore (3.19) holds and we may conclude that $W_\beta(\Psi)$ is a set of full measure within I_β. ■

4. Proof of Theorem 1.4. Our proof is straightforward and relies on basic properties of the Lebesgue measure. For ease of exposition we recall that Ψ is *decaying regularly* if for each $m \in \mathbb{N}$ there exists $C_m \in \mathbb{N}$ such that

$$\frac{\Psi(n+m)}{\Psi(n)} \geq \frac{1}{C_m} \tag{4.1}$$

for every $n \in \mathbb{N}$.

Suppose $\Psi : \mathbb{N} \to \mathbb{R}_{\geq 0}$ satisfies $\sum_{n=1}^{\infty} 2^n \Psi(n) = \infty$. Given $k \in \mathbb{N}$ let $\Psi_k : \mathbb{N} \to \mathbb{R}_{\geq 0}$ be defined via $\Psi_k(n) := \Psi(n)k^{-1}$. For each $k \in \mathbb{N}$ the sum $\sum_{n=1}^{\infty} 2^n \Psi_k(n)$ also diverges. If β is approximation regular then $W_\beta(\Psi_k)$ is a set of full measure within I_β for each $k \in \mathbb{N}$. Therefore

$$\Omega_\beta(\Psi) := \bigcap_{k=1}^{\infty} W_\beta(\Psi_k)$$

is also of full measure. Let

$$\Gamma_\beta(\Psi) := I_\beta \setminus \Omega_\beta(\Psi),$$

so if β is approximation regular then $\lambda(\Gamma_\beta(\Psi)) = 0$. We introduce the functions $T_0(x) = \beta x$ and $T_1(x) = \beta x - 1$. We will denote a typical element of $\{T_0, T_1\}^n$ by $a = (a_1, \ldots, a_n)$. Moreover, we let $a(x)$ denote $(a_n \circ \cdots \circ a_1)(x)$. By $\{T_0, T_1\}^0$ we denote the set consisting of the identity function. Let

$$\Delta_\beta(\Psi) := \bigcup_{n=0}^{\infty} \bigcup_{a \in \{T_0, T_1\}^n} a^{-1}(\Gamma_\beta(\Psi)).$$

Since T_0^{-1} and T_1^{-1} are both similitudes, it follows that $\lambda(\Delta_\beta(\Psi)) = 0$ whenever β is approximation regular. We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. Assume β is approximation regular, $\Psi : \mathbb{N} \to \mathbb{R}_{\geq 0}$ is decaying regularly and $\sum_{n=1}^{\infty} 2^n \Psi(n) = \infty$. Let $x \in I_\beta \setminus \Delta_\beta(\Psi)$. By the above, $I_\beta \setminus \Delta_\beta(\Psi)$ is a set of full Lebesgue measure within I_β. We now show that x has a β-expansion $(\epsilon_i)_{i=1}^{\infty}$ which satisfies

$$0 \leq x - \sum_{i=1}^{n} \frac{\epsilon_i}{\beta^i} \leq \Psi(n)$$

for infinitely many $n \in \mathbb{N}$. Since $x \in I_\beta \setminus \Delta_\beta(\Psi)$ it is clear that $x \in W_\beta(\Psi)$. Therefore there exist infinitely many solutions to the inequalities

$$0 \leq x - \sum_{i=1}^{n} \frac{\epsilon_i}{\beta^i} \leq \Psi(n).$$

Let $(\epsilon_i^1)_{i=1}^{n_1}$ be the first sequence whose level n_1 sum satisfies these inequalities. Without loss of generality we may assume $(\epsilon_i^1)_{i=1}^{n_1}$ is an n_1-prefix for x. In this case, multiplying through by β^{n_1} in (1.2) gives

$$(T_{\epsilon_{n_1}^1} \circ \cdots \circ T_{\epsilon_1^1})(x) = \beta^{n_1} x - \epsilon_1^1 \beta^{n_1 - 1} - \cdots - \epsilon_{n_1-1}^1 \beta - \epsilon_{n_1}^1 \in I_\beta.$$

Let $C^1 \in \mathbb{N}$ be sufficiently large that

$$\frac{\Psi_{C^1}(n)}{\beta^{n_1}} \leq \Psi(n + n_1) \tag{4.2}$$

for all $n \in \mathbb{N}$. Such a C^1 exists because Ψ is decaying regularly. Since $x \in I_\beta \setminus \Delta_\beta(\Psi)$ we have $(T_{\epsilon_{n_1}^1} \circ \cdots \circ T_{\epsilon_1^1})(x) \in W_\beta(\Psi_{C^1})$. Therefore there exists $(\epsilon_1^2, \ldots, \epsilon_{n_2}^2)$ such that

$$(T_{\epsilon_{n_1}^1} \circ \cdots \circ T_{\epsilon_1^1})(x) - \sum_{i=1}^{n_2} \frac{\epsilon_i^2}{\beta^i} \leq \Psi_{C^1}(n_2). \tag{4.3}$$

Dividing through by β^{n_1} in (4.3) and applying (4.2) yields

$$x - \sum_{i=1}^{n_1} \frac{\epsilon_i^1}{\beta^i} - \frac{1}{\beta^{n_1}} \sum_{i=1}^{n_2} \frac{\epsilon_i^2}{\beta^i} \leq \frac{\Psi_{C^1}(n_2)}{\beta^{n_1}} \leq \Psi(n_1 + n_2).$$

Without loss of generality we may assume that $(\epsilon_1^1, \ldots, \epsilon_{n_1}^1, \epsilon_1^2, \ldots, \epsilon_{n_2}^2)$ is an $(n_1 + n_2)$-prefix for x.

Since $x \in I_\beta \setminus \Delta_\beta(\Psi)$ we have $(T_{\epsilon_{n_2}^2} \circ \cdots \circ T_{\epsilon_1^2} \circ T_{\epsilon_{n_1}^1} \circ \cdots \circ T_{\epsilon_1^1})(x) \in W_\beta(\Psi_k)$ for each $k \in \mathbb{N}$. We choose $C^2 \in \mathbb{N}$ sufficiently large that

$$\frac{\Psi_{C^2}(n)}{\beta^{n_1+n_2}} \leq \Psi(n + n_1 + n_2)$$

for all $n \in \mathbb{N}$. We then repeat the above argument with C^1 replaced by C^2 and $(T_{\epsilon_{n_1}^1} \circ \cdots \circ T_{\epsilon_1^1})$ replaced by $(T_{\epsilon_{n_2}^2} \circ \cdots \circ T_{\epsilon_1^2} \circ T_{\epsilon_{n_1}^1} \circ \cdots \circ T_{\epsilon_1^1})$ to obtain a sequence $(\epsilon_1^3, \ldots, \epsilon_{n_3}^3)$ such that

$$x - \sum_{i=1}^{n_1} \frac{\epsilon_i}{\beta^i} - \frac{1}{\beta^{n_1}} \sum_{i=1}^{n_2} \frac{\epsilon_i^2}{\beta^i} - \frac{1}{\beta^{n_1+n_2}} \sum_{i=1}^{n_3} \frac{\epsilon_i^3}{\beta^i} \leq \Psi(n_1 + n_2 + n_3).$$

Again we may assume that the sequence $(\epsilon_1^1, \ldots, \epsilon_{n_1}^1, \epsilon_1^2, \ldots, \epsilon_{n_2}^2, \epsilon_1^3, \ldots, \epsilon_{n_3}^3)$ is an $(n_1 + n_2 + n_3)$-prefix for x.

Repeatedly applying the above procedure we obtain an infinite sequence $(\epsilon_i)_{i=1}^{\infty}$ which forms a β-expansion for x and satisfies

$$0 \leq x - \sum_{i=1}^{n} \frac{\epsilon_i}{\beta^i} \leq \Psi(n)$$

for infinitely many $n \in \mathbb{N}$. ■

5. Final comments. In this final section we make a few comments on the connection between the set of points with a unique β-expansion and $I_\beta \setminus W_\beta(\Psi)$. Let

$$U_\beta := \left\{ x \in \left(0, \frac{1}{\beta - 1}\right) : x \text{ has a unique } \beta\text{-expansion} \right\}.$$

The set U_β is a well studied object. It is a consequence of the work of Daróczy and Kátai [5] and Erdős, Joó and Komornik [8] that U_β is nonempty if and only if $\beta \in \left(\frac{1+\sqrt{5}}{2}, 2\right)$. Let $\beta_c \approx 1.78723$ be the Komornik–Loreti constant introduced in [15]. Glendinning and Sidorov [11] showed that U_β is countable if $\beta \in \left(\frac{1+\sqrt{5}}{2}, \beta_c\right)$, U_{β_c} is uncountable with zero Hausdorff dimension, and U_β has strictly positive Hausdorff dimension if $\beta \in (\beta_c, 2)$. Moreover, $\dim_H(U_\beta) \to 1$ as $\beta \to 2$.

The significance of U_β is that if $x \in U_\beta$ then

$$\frac{\kappa}{\beta^n(\beta - 1)} \leq x - \sum_{i=1}^{n} \frac{\epsilon_i}{\beta^i} \leq \frac{1}{\beta^n(\beta - 1)} \tag{5.1}$$

for all $n \in \mathbb{N}$, where $(\epsilon_i)_{i=1}^{\infty}$ is the unique β-expansion for x, and κ is some strictly positive constant that only depends on x. The existence of κ can be seen as a consequence of the symbolic interpretation of U_β provided by [11, Lemma 4]. Equation (5.1) then implies that for any $\Psi(n) = O(\gamma^{-n})$ where $\gamma > \beta$ there are finitely many solutions to the set of inequalities

$$0 \leq x - \sum_{i=1}^{n} \frac{\epsilon_i}{\beta^i} \leq \Psi(n).$$

Therefore if Ψ decays sufficiently quickly and $\beta \in \left(\frac{1+\sqrt{5}}{2}, 2\right)$ then $I_\beta \setminus W_\beta(\Psi)$ is always infinite. We finish with an example that emphasises the above.

Example 5.1. Take $\beta \approx 1.76929$, the appropriate root of $x^3 - 2x - 2 = 0$. Then β is a Garsia number and by Theorem 1.2 it is approximation regular. If we then take $\Psi(n) = 2^{-n}$ we see that $W_\beta(\Psi)$ is of full measure. Yet by the above $I_\beta \setminus W_\beta(\Psi)$ contains an infinite set.

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