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On zeros of approximate functions of the Rankin–Selberg L-functions

by

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Notations. As usual, $\mathbb{Z}$ is the ring of rational integers, $\mathbb{Z}_{>0}$ the set of positive integers, $\mathbb{C}$ the field of complex numbers. We denote by $\mathfrak{h}$ the upper half-plane, and by Γ the full modular group $\mathrm{PSL}_2(\mathbb{Z})$. For a complex variable s, we put $e(s) = e^{2\pi is}$, $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. We denote by $\zeta(s)$ and $\zeta^*(s) = \Gamma_{\mathbb{R}}(s)\zeta(s)$ the Riemann zeta-function and the completed Riemann zeta-function, respectively, and denote by $\sigma_\nu(n) = \sum_{d|n} d^\nu$ the divisor function. Throughout the paper, $z = x + iy$ ($x \in \mathbb{R}$, $y > 0$) is a variable on $\mathfrak{h}$, and $s = \sigma + it$ ($\sigma, t \in \mathbb{R}$) is a complex variable. A sum over the empty set is meant to be zero.

1. Introduction. Let $\mathrm{C}(s)$ be the trigonometric function

$$\mathrm{C}(s) := 2\cos(i(s-1/2)) = e^{s-1/2} + e^{-(s-1/2)}.$$

It satisfies the (trivial) functional equation $\mathrm{C}(s) = \mathrm{C}(1-s)$. A well-known but remarkable fact about $\mathrm{C}(s)$ is that it satisfies the Riemann hypothesis: *all zeros of* $\mathrm{C}(s)$ *lie on the central line* $\sigma = 1/2$ *of its functional equation.* We indicate how to prove the Riemann hypothesis for $\mathrm{C}(s)$. First, we note the (trivial) decomposition

$$\mathrm{C}(s) = \varphi(s) + \varphi(1-s), \qquad \varphi(s) = e^{s-1/2}.$$

Then we have

$$\mathrm{C}(s) = \varphi(s)\left(1 + \frac{\varphi(1-s)}{\varphi(s)}\right),$$

and find that

(A) $\varphi(s) \neq 0$ for $\sigma > 1/2$,

(B) $\left|\dfrac{\varphi(1-s)}{\varphi(s)}\right| < 1$ for $\sigma > 1/2$.

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Property (B) implies that

(C) $1+\dfrac{\varphi(1-s)}{\varphi(s)} \neq 0$ for $\sigma > 1/2$.

Therefore, $\mathrm{C}(s) \neq 0$ for $\sigma > 1/2$ by (A) and (C). The functional equation gives $\mathrm{C}(s) \neq 0$ if $\sigma \neq 1/2$. Hence we obtain the Riemann hypothesis for the function $\mathrm{C}(s)$. Note that $\mathrm{C}(s)$ has at least one zero.

Now let $L(s)$ be an entire function satisfying the functional equation

$$L(s) = L(1-s).$$

The above argument implies that if $L(s)$ has the decomposition

$$L(s) = \varphi(s) + \varphi(1-s) \tag{1.1}$$

such that $\varphi(s)$ satisfies (A) and (B), then the Riemann hypothesis holds for $L(s)$.

The study of zeros of entire functions along this line has a long history. The decomposition (1.1) with the function $\varphi(s)$ satisfying (A) and (B) is possible in several interesting cases.

Consider the case of the Riemann zeta function. Let

$$\phi(x) = 4\sum_{n=1}^{\infty}(2\pi^2 n^4 x^{9/2} - 3\pi n^2 x^{5/2})e^{-\pi n^2 x^2}.$$

Then we have

$$\xi(s) = s(s-1)\zeta^*(s) = \int_1^{\infty} \phi(x)(x^{s-1/2} + x^{-s+1/2})\,\frac{dx}{x}.$$

Replacing $\phi(x)$ by

$$\phi^*(x) = \pi^2(x^{9/2} + x^{-9/2})e^{-\pi(x^2+x^{-2})},$$

which is asymptotically equivalent to $\phi(x)$, we obtain

$$\xi^*(s) = \int_1^{\infty} \phi^*(x)(x^{s-1/2} + x^{-s+1/2})\,\frac{dx}{x}.$$

The function $\xi^*(s)$ is similar to $\xi(s)$ in a suitable sense, and has the decomposition (1.1) such that the corresponding $\varphi(s)$ satisfies (A) and (B) as well as $\mathrm{C}(s)$ [27, pp. 254–291]. For the decomposition of $\xi(s)$ as in (1.1) see Gonek [4] and Egorov [3].

Other interesting cases are the difference of two zeta functions, the constant term of the nonholomorphic Eisenstein series, Weng's zeta functions and a finite truncation of the Chowla–Selberg formula of Epstein zeta-functions etc. They were studied by several authors, e.g., Pólya [17], Taylor [26], Stark [20], Hejhal [6], Ki [9], Lagarias–Suzuki [10], Weng [31–33], Suzuki [22–24], Hayashi [5], Bauer [1], Müller [13], Velásquez [28] and Suzuki-Weng [25].

Can we find new examples of zeta- and L-functions $L(s)$ having (1.1) and satisfying (A) and (B)? In this paper, we show that the Rankin–Selberg L-function is one of such examples. More precisely, we derive a new formula (Theorem 1) for the Rankin–Selberg L-function attached to a pair of cusp forms on the full modular group by using the holomorphic projection of Sturm [21]. Then the well-known relation between the Rankin–Selberg L-function and the symmetric square L-function gives a new formula for the symmetric square L-function (Corollary 1). Using Theorem 1, we define a function which approximates the Rankin–Selberg L-function. We show that such an approximate function has a wide zero-free region (Theorem 2), and this uses the fact that it has the decomposition (1.1) with two properties similar to (A) and (B).

As a special case of Corollary 1, we obtain Noda's identity in [14] which relates the Fourier coefficients of the holomorphic cusp form f and the zeros of the Riemann zeta-function or the zeros of the symmetric square L-function of f. In addition, Theorem 1 gives an analytic series expansion of the central value $L(1/2, f \times g)$. Note that Mizumoto [12] showed that for every normalized Hecke eigen cusp form $f \in S_{k_1}$ and every even integer k_2 satisfying $k_2 \geq k_1$ and $k_2 \neq 14$, there exists a normalized Hecke eigen cusp form $g \in S_{k_2}$ such that $L(1/2, f \times g) \neq 0$.

There are nice results of Hoffstein–Lockhart [7], Hoffstein–Ramakrishnan [8] and Ramakrishnan–Wang [18] about the real zeros of the Rankin–Selberg L-function. They established the nonexistence of the Siegel zero of the Rankin–Selberg L-function attached to a pair of cusp forms on GL(2) and the symmetric square L-function of a cusp form on GL(2). Their results contain fairly good zero-free regions of the Rankin–Selberg L-function compared with the classical one. We expect that Theorem 1 and improving our proof of Theorem 2, should imply nice results on the distribution of complex zeros of the Rankin–Selberg L-function.

This paper is organized as follows. In Section 2, we state main results, Theorems 1 and 2. In Section 3, we apply the results of Section 2 to S_{12} and S_{24}. In Section 4, we review the theory of the Poincaré series, Eisenstein series, C^∞-modular forms and the Rankin–Selberg L-function as preliminaries for the proof of Theorems 1 and 2. In Section 5, we give a proof of Theorem 1. In Section 6, we prove Theorem 2. In Section 7, we interpret the argument in Section 5 from the viewpoint of the holomorphic projection of Sturm. In the Appendix, we give an asymptotic expansion of the associated Legendre function of the first kind according to Watson [29].

2. Statements of results. Let k be an even integer ≥ 12 and $\neq 14$. Let S_k be the vector space of all holomorphic cusp forms of weight k on Γ. We denote by $d = d_k$ the dimension of S_k. For two cusp forms $f(z) =$

$\sum_{n=1}^{\infty} a_f(n) n^{(k-1)/2} e(nz)$ and $g(z) = \sum_{n=1}^{\infty} a_g(n) n^{(k-1)/2} e(nz)$, the *Rankin–Selberg L-function* $L(s, f \otimes \overline{g})$ is defined by

$$L(s, f \otimes \overline{g}) = \sum_{n=1}^{\infty} a_f(n)\, \overline{a_g(n)}\, n^{-s}, \tag{2.1}$$

where bar means complex conjugation. The series on the right-hand side converges absolutely if the real part of s is sufficiently large. In addition, we define

$$L(s, f \times g) = \zeta(2s) L(s, f \otimes \overline{g})$$

and the completed function

$$\begin{aligned} L^*(s, f \times g) &= 2^{-k-1} \Gamma_{\mathbb{C}}(s+k-1) \Gamma_{\mathbb{C}}(s) L(s, f \times \overline{g}\,) \\ &= \pi^{-s} (4\pi)^{-s-k-1} \Gamma(s) \Gamma(s+k-1) L(s, f \times \overline{g}). \end{aligned}$$

Let $\mathcal{F} = \{f_1, \dots, f_d\}$ be an orthonormal basis of S_k and let $f_j(z) = \sum_{n=1}^{\infty} a_j(n) n^{(k-1)/2} e(nz)$ be the Fourier expansion of f_j $(1 \le j \le d)$ at the cusp $i\infty$. Let $\mathfrak{m} = (m_1, \dots, m_d) \in \mathbb{Z}_{>0}^d$ with $0 < m_1 < \dots < m_d$. Define

$$A_{\mathcal{F},\mathfrak{m}} = \begin{pmatrix} a_1(m_1) & \cdots & a_d(m_1) \\ \vdots & \ddots & \vdots \\ a_1(m_d) & \cdots & a_d(m_d) \end{pmatrix}. \tag{2.2}$$

In general, the matrix $A_{\mathcal{F},\mathfrak{m}}$ is *not* invertible. However, if the set of Poincaré series $\{P_{m_1}, \dots, P_{m_d}\} \subset S_k$ is a basis of S_k, then $A_{\mathcal{F},\mathfrak{m}}$ is invertible. In particular, for the vector $\mathfrak{m}_0 = (1, \dots, d)$, the matrix $A_{\mathcal{F},\mathfrak{m}_0}$ is invertible by the classical result of Petersson [15, 16] about the basis problem for elliptic modular forms. Thus we can always choose a vector $\mathfrak{m}$ such that $A_{\mathcal{F},\mathfrak{m}}$ is invertible.

THEOREM 1. *Let k be an even integer ≥ 12 and $\ne 14$. Let $\mathcal{F} = \{f_1, \dots, f_d\}$ be an orthonormal basis of S_k and let $f_j(z) = \sum_{n=1}^{\infty} a_j(n) n^{(k-1)/2} e(nz)$ be the Fourier expansion of f_j $(1 \le j \le d)$ at the cusp $i\infty$. Choose $\mathfrak{m} \in \mathbb{Z}_{>0}^d$ such that the matrix $A_{\mathcal{F},\mathfrak{m}}$ defined by* (2.2) *is invertible* $(\det A_{\mathcal{F},\mathfrak{m}} \ne 0)$. *Define the set of numbers $(\alpha_{ij})_{1 \le i,j \le d}$ by*

$$A_{\mathcal{F},\mathfrak{m}}^{-1} = (\alpha_{ij})_{1 \le i,j \le d}. \tag{2.3}$$

Then

$$\begin{aligned} &(4\pi)^{-k+1} \Gamma(k-1)\, L^*(s, f_i \times \overline{f}_j) \\ &\quad = (4\pi)^{-s-k+1} \Gamma(s+k-1)\, \zeta^*(2s) D_{\mathfrak{m},ij}(s) \\ &\qquad + (4\pi)^{s-k} \Gamma(k-s)\, \zeta^*(2s-1) D_{\mathfrak{m},ij}(1-s) \\ &\qquad + (4\pi)^{-k+1} \Gamma(s+k-1) \Gamma(k-s) \{W_{\mathfrak{m},ij}^{+}(s) + W_{\mathfrak{m},ij}^{-}(s)\} \end{aligned} \tag{2.4}$$

for all $1 \le i \le j \le d$ *in the vertical strip*

$$|\sigma - 1/2| < k/2 - 1 \tag{2.5}$$

except for the point $s = 1/2$*. Here*

$$D_{\mathfrak{m},ij}(s) = \sum_{h=1}^{d} \alpha_{jh}\, a_i(m_h)\, m_h^{-s}, \tag{2.6}$$

$$W^{+}_{\mathfrak{m},ij}(s) = \sum_{h=1}^{d}\sum_{n=1}^{\infty} \alpha_{jh} a_i(m_h+n)\, \frac{\tau_{s-1/2}(n)}{\sqrt{n}}\, P^{1-k}_{s-1}\biggl(\frac{2m_h+n}{n}\biggr), \tag{2.7}$$

$$W^{-}_{\mathfrak{m},ij}(s) = \sum_{h=1}^{d}\sum_{n=1}^{m_h-1} \alpha_{jh}\, a_i(m_h-n)\, \frac{\tau_{s-1/2}(n)}{\sqrt{n}}\, P^{1-k}_{s-1}\biggl(\frac{2m_h-n}{n}\biggr), \tag{2.8}$$

with $\tau_\nu(n) = n^\nu \sigma_{-2\nu}(n)$, *and* $P^\mu_\nu(z)$ *is the associated Legendre function of the first kind* (*see Appendix*)*. Further, at the point* $s = 1/2$,

$$\begin{aligned}
&(4\pi)^{-k+1}\Gamma(k-1)\, L^*(1/2, f_i \times \overline{f}_j)\\
&= (4\pi)^{-k+1/2}\Gamma\biggl(k-\frac{1}{2}\biggr)\sum_{h=1}^{d}\frac{\alpha_{jh}a_i(m_h)}{\sqrt{m_h}}\biggl\{\frac{\Gamma'}{\Gamma}\biggl(k-\frac{1}{2}\biggr) + \log\frac{e^\gamma}{16\pi^2 m_h}\biggr\}\\
&\quad + (4\pi)^{-k+1}\Gamma\biggl(k-\frac{1}{2}\biggr)^2 \{W^{+}_{\mathfrak{m},ij}(1/2) + W^{-}_{\mathfrak{m},ij}(1/2)\}.
\end{aligned}$$

The series $W^{+}_{\mathfrak{m},ij}(s)$ *converges absolutely and uniformly on every compact subset* K *in* (2.5), *and has the asymptotic expansion*

$$\begin{aligned}
W^{+}_{\mathfrak{m},ij}(s) = &\sum_{h=1}^{d}\sum_{n=1}^{N-1} \alpha_{jh} a_i(m_h+n)\, \frac{\tau_{s-1/2}(n)}{\sqrt{n}}\, P^{1-k}_{s-1}\biggl(\frac{2m_h+n}{n}\biggr)\\
&+ O(N^{|\sigma-1/2|-k/2+1+\varepsilon}),
\end{aligned}$$

where the implied constant depends on $\mathcal{F}$, $\mathfrak{m}$ *and* K*.*

Remark 1. By definition of α_{ij}, we have

$$D_{\mathfrak{m},ij}(0) = \sum_{h=1}^{d} \alpha_{jh} a_i(m_h) = \delta_{ij}.$$

Hence the poles of the first two terms of (2.4) at $s = 0, 1$ cancel out whenever $i \ne j$. This agrees with the fact that the residue of $L(s, f_i \times \overline{f}_j)$ at $s = 1$ is a multiple of the Petersson inner product (f_i, f_j).

Let $f(z) = 1 + \sum_{n=2}^{\infty} a_f(n) n^{(k-1)/2} e(nz) \in S_k$ be a normalized Hecke eigen cusp form. The *symmetric square L-function* $L(s, \mathrm{sym}^2 f)$ is defined by the Euler product

$$L(s, \mathrm{sym}^2 f) = \prod_p (1 - \alpha_p^2 p^{-s})^{-1} (1 - \alpha_p \beta_p p^{-s})^{-1} (1 - \beta_p^2 p^{-s})^{-1},$$

where α_p and β_p are determined by $\alpha_p + \beta_p = a_f(p)$ and $\alpha_p \beta_p = 1$. The right-hand side converges absolutely if the real part of s is sufficiently large. The completed L-function $L^*(s, \mathrm{sym}^2 f)$ is defined by

$$\begin{aligned} L^*(s, \mathrm{sym}^2 f) &= \pi^{-3s/2} \Gamma\Big(\frac{s+1}{2}\Big) \Gamma\Big(\frac{s+k-1}{2}\Big) \Gamma\Big(\frac{s+k}{2}\Big) L(s, \mathrm{sym}^2 f) \\ &= \pi^k \Gamma_{\mathbb{C}}(s+k-1) \Gamma_{\mathbb{C}}(s) \Gamma_{\mathbb{R}}(s)^{-1} L(s, \mathrm{sym}^2 f). \end{aligned}$$

It is known that $L(s, \mathrm{sym}^2 f)$ and $L(s, f \times \overline{f})$ are related via

$$\zeta(s) L(s, \mathrm{sym}^2 f) = L(s, f \times \overline{f}).$$

Therefore we have the equality

$$2^{-1} (2\pi)^{-k} \zeta^*(s) L^*(s, \mathrm{sym}^2 f) = L^*(s, f \times \overline{f}).$$

Corollary 1. *Let $\mathcal{F} = \{f_1, \ldots, f_d\}$ be the orthogonal basis of S_k consisting of normalized Hecke eigen cusp forms. Put $f_j^* = f_j / (f_j, f_j)^{1/2}$ and $\mathcal{F}^* = \{f_1^*, \ldots, f_d^*\}$. Choose $\mathfrak{m} \in \mathbb{Z}_{>0}^d$ such that $A_{\mathcal{F}^*, \mathfrak{m}}$ is invertible. Then*

$$\begin{aligned} (2.9) \qquad & 2^{-k} (2\pi)^{-2k+1} \frac{\Gamma(k-1)}{(f_j, f_j)} \zeta^*(s) L^*(s, \mathrm{sym}^2 f_j) \\ &= (4\pi)^{-s-k+1} \Gamma(s+k-1) \zeta^*(2s) D_{\mathfrak{m}, jj}(s) \\ &\quad + (4\pi)^{s-k} \Gamma(k-s) \zeta^*(2s-1) D_{\mathfrak{m}, jj}(1-s) \\ &\quad + (4\pi)^{-k+1} \Gamma(s+k-1) \Gamma(k-s) \{W_{\mathfrak{m}, jj}^+(s) + W_{\mathfrak{m}, jj}^-(s)\} \end{aligned}$$

for all $1 \le j \le d$ and all $s \ne 1/2$ in the vertical strip (2.5), *where $D_{\mathfrak{m}, jj}(s)$, $W_{\mathfrak{m}, jj}^+(s)$ and $W_{\mathfrak{m}, jj}^-(s)$ are defined by* (2.6)–(2.8) *for the basis $\mathcal{F}^*$ and the vector $\mathfrak{m}$.*

Remark 2. In the case $S_k = \mathbb{C} \Delta_k$ ($k = 12, 16, 18, 20, 22$ and 26), $D_{(m), 11}(s)$ is just m^{-s}. Hence, by taking s to be a zero of $\zeta(s)$ or a zero of $L(s, \mathrm{sym}^2 \Delta_k)$, we obtain a new proof of the result of Noda [14, Theorem]. His result is an equality which relates the zeros of the Riemann zeta function or the zeros of the symmetric square L-functions with the Fourier coefficients of the holomorphic cusp form Δ_k.

Corollary 2. *Under the notation of Theorem* 1, *we have the following formula for the central value*:

$$L(1/2, f_i \times \bar{f}_j) = \frac{(4\pi)^{k-1}}{\Gamma(k-1)} \sum_{h=1}^{d} \frac{\alpha_{jh} a_i(m_h)}{\sqrt{m_h}} \left\{ \frac{\Gamma'}{\Gamma}\left(k - \frac{1}{2}\right) + \log \frac{e^{\gamma}}{16\pi^2 m_h} \right\}$$
$$+ 4\pi^k \frac{\Gamma(2k-2)}{\Gamma(k-1)^2} \{W^{+}_{\mathfrak{m},ij}(1/2) + W^{-}_{\mathfrak{m},ij}(1/2)\}.$$

On the right-hand side we have

$$W^{+}_{\mathfrak{m},ij}(1/2) = \sum_{h=1}^{d} \sum_{n=1}^{N-1} \alpha_{jh} a_i(m_h + n) \frac{\sigma_0(n)}{\sqrt{n}} P^{1-k}_{-1/2}\left(\frac{2m_h + n}{n}\right)$$
$$+ O(N^{-k/2+1+\varepsilon})$$

for every positive integer N and every positive real number ε.

Considering equations (2.4) and (2.7), we define

$$W^{+,N}_{\mathfrak{m},ij}(s) = \sum_{h=1}^{d} \sum_{n=1}^{N} \alpha_{jh} a_i(m_h + n) \frac{\tau_{s-1/2}(n)}{\sqrt{n}} P^{1-k}_{s-1}\left(\frac{2m_h + n}{n}\right) \tag{2.10}$$

and

$$\begin{aligned} L^{N}_{\mathfrak{m},ij}(s) &= (4\pi)^{-s-k+1}\Gamma(s+k-1)\zeta^*(2s)D_{\mathfrak{m},ij}(s) \\ &+ (4\pi)^{s-k}\Gamma(k-s)\zeta^*(2s-1)D_{\mathfrak{m},ij}(1-s) \\ &+ (4\pi)^{-k+1}\Gamma(s+k-1)\Gamma(k-s)\{W^{+,N}_{\mathfrak{m},ij}(s) + W^{-}_{\mathfrak{m},ij}(s)\} \end{aligned} \tag{2.11}$$

for a positive integer N. In addition, we define

$$\begin{aligned} L^{0}_{\mathfrak{m},ij}(s) &= (4\pi)^{-s-k+1}\Gamma(s+k-1)\zeta^*(2s)D_{\mathfrak{m},ij}(s) \\ &+ (4\pi)^{s-k}\Gamma(k-s)\zeta^*(2s-1)D_{\mathfrak{m},ij}(1-s) \end{aligned}$$

for $N = 0$. The only difference between $L^{N}_{\mathfrak{m},ij}(s)$ and the right-hand side of (2.4) is in the bracketed expression $\{\cdots\}$. The functional equations $\tau_{s-1/2}(n) = \tau_{1/2-s}(n)$ and $P^{1-k}_{s-1}(z) = P^{1-k}_{-s}(z)$ imply that $L^{N}_{\mathfrak{m},ij}(s)$ satisfies the functional equation

$$L^{N}_{\mathfrak{m},ij}(s) = L^{N}_{\mathfrak{m},ij}(1-s). \tag{2.12}$$

THEOREM 2. *Let k be an even integer ≥ 12 and $\neq 14$. Let $\mathcal{F} = \{f_1, \ldots, f_d\}$ be an orthonormal basis of S_k. Choose $\mathfrak{m} \in \mathbb{Z}^d_{>0}$ such that $A_{\mathcal{F},\mathfrak{m}}$ is invertible. Further suppose that there exists $\delta = \delta_{\mathcal{F},\mathfrak{m}}$ such that $0 < \delta < 1/2$, and $D_{\mathfrak{m},ij}(s)$ has only finitely many zeros in the right half-plane $\sigma \geq 1/2 - \delta$. Then for every nonnegative integer N and every positive real number a there exists $C = C_{\mathfrak{m},N,a} > 0$ such that $L^{N}_{\mathfrak{m},ij}(s)$ has no zeros in the region*

$$\frac{\log\{C \log^{1/2}(|t|+1)\}}{\log(|t|+1)} < \left|\sigma - \frac{1}{2}\right| < a,$$

that is, all zeros of $L^N_{\mathfrak{m},ij}(s)$ *in the strip* $|\sigma-1/2|<a$ *are contained in*

$$\left|\sigma-\frac{1}{2}\right|\le\frac{\log\{C\log^{1/2}(|t|+1)\}}{\log(|t|+1)}.$$

In particular,

$$N(T,\sigma_1,\sigma_2)=O_{\sigma_1,\sigma_2}(1)$$

for all $0<\sigma_1<\sigma_2$, *where* $N(T,\sigma_1,\sigma_2)$ *is the number of zeros of* $L^N_{\mathfrak{m},ij}(s)$ *satisfying* $\sigma_1\le\sigma-1/2\le\sigma_2$ *and* $|t|\le T$ *counted with multiplicity.*

Remark 3. In the case of the Riemann zeta-function, Selberg established the estimate

$$N(T,1/2+4\delta)\ll T^{1-\delta}\log T$$

uniformly for $\delta\ge 0$ by using his mollification method. Here $N(T,a)$ is the number of zeros of $\zeta(s)$ satisfying $\sigma\ge a$ and $|t|\le T$ counted with multiplicity. Hence almost all zeros of $\zeta(s)$ lie in the region

$$\left|\sigma-\frac{1}{2}\right|\le\frac{\eta(t)}{\log(|t|+3)},$$

where $\eta(t)$ is any positive function which increases to infinity. Theorem 2 is an analogue of this result.

Remark 4. As in Remark 2, $D_{(m),11}(s)=m^{-s}$ if $\dim S_k=1$. Hence the assumption in Theorem 2 about the location of zeros of $D_{\mathfrak{m},ij}(s)$ is always satisfied if $\dim S_k=1$. However, in general the location of zeros of $D_{\mathfrak{m},ij}(s)$ strongly depends on the choice of the vector $\mathfrak{m}$ (see Section 3).

Remark 5. The existence of the vector $\mathfrak{m}$ such that $L^N_{\mathfrak{m},ij}(s)$ has no zeros in $0<|\sigma-1/2|<1/2$ for all sufficiently large N implies that the Riemann hypothesis for the Rankin–Selberg L-function $L(s,f_i\times\overline{f}_j)$ is true. Therefore such a result is desired for a pair of Hecke eigen cusp forms f_i and f_j. However, our proof of Theorem 2 in Section 6 does not need the condition that f_i and f_j are Hecke eigen cusp forms. Hence, a new idea using more precise arithmetic properties of the Fourier coefficients of f_i and f_j is needed in order to obtain results in the direction of the Riemann hypothesis.

3. Examples. In this section, we calculate the central values of L-functions by applying Corollary 2 to S_{12} and S_{24}. We calculate the value of the Petersson inner product according to Rankin [19].

3.1. *The case* $k=12$. In this case $\dim S_{12}=1$. As mentioned in Remark 2, we have $D_{(m),11}(s)=m^{-s}$ by definition (2.6). All members of S_{12} are constant multiples of the normalized Hecke eigen cusp form

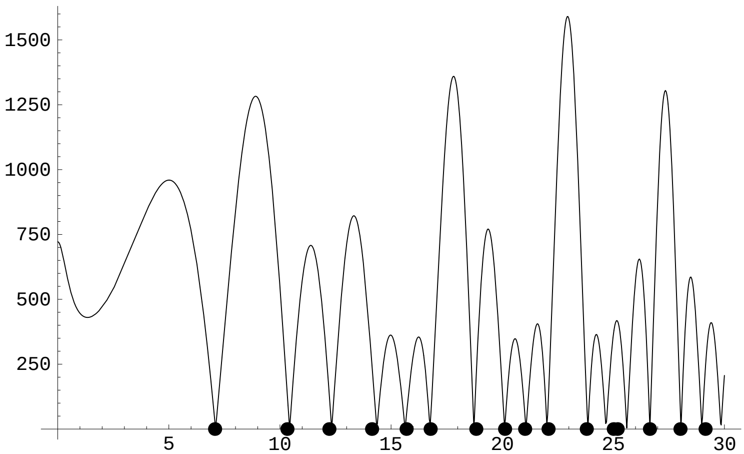

Fig. 1. $|L_0(1/2+it,\, \Delta\times\Delta)|$ for $0\le t\le 30$. Points • are zeros of $L(s,\Delta\times\Delta)$ on $\sigma=1/2$.

$\Delta(z) = e(z)\prod_{n=1}^{\infty}(1-e(nz))^{24} = \sum_{n=1}^{\infty}\tau(n)e(nz)$. Put $f=\Delta/(\Delta,\Delta)^{1/2}$, and choose $\mathfrak{m}=(m)=(1)$. Then we have $W^{-}_{(1),11}(s)\equiv 0$, and

$$\frac{L(1/2,\Delta\times\Delta)}{\sqrt{(\Delta,\Delta)}} = \frac{(4\pi)^{11}}{\Gamma(11)}\left\{\frac{\Gamma'}{\Gamma}\left(\frac{23}{2}\right)+\log\frac{e^{\gamma}}{16\pi^2}\right\} + \frac{4\pi^{12}\Gamma(22)}{\Gamma(11)^2\Gamma(12)}\sum_{n=1}^{\infty}\frac{\tau(n+1)}{(n+1)^{11}}\frac{\sigma_0(n)}{\sqrt{n}}\,F\left(\frac{1}{2},\frac{1}{2},12;-\frac{1}{n}\right).$$

Using the value $(\Delta,\Delta)=1.03536\ldots\times 10^{-6}$, we have

$$L(1/2,\Delta\times\Delta) = -7.25563\ldots\times 10^{2}.$$

Figure 1 is the graph of the absolute value of

$$L_0(s,\Delta\times\Delta) = \frac{\omega_{12}\sqrt{(\Delta,\Delta)}}{\pi^{-s}(4\pi)^{-s-11}\Gamma(s)\Gamma(s+11)}\,L^{0}_{(1),11}(s)$$

on the critical line $\sigma=1/2$, where $\omega_{12}=(4\pi)^{11}/\Gamma(11)$ and

$$L^{0}_{(1),11}(s) = (4\pi)^{-s-11}\Gamma(s+11)\zeta^*(2s) + (4\pi)^{s-12}\Gamma(12-s)\,\zeta^*(2s-1).$$

Figure 2 is the graph of the absolute value of

$$L_N(s,\Delta\times\Delta) = \frac{\omega_{12}\sqrt{(\Delta,\Delta)}}{\pi^{-s}(4\pi)^{-s-11}\Gamma(s)\Gamma(s+11)}\,L^{N}_{(1),11}(s)$$

for $N=10$ on the critical line $\sigma=1/2$, where

$$L^{N}_{(1),11}(s) = (4\pi)^{-s-11}\Gamma(s+11)\zeta^*(2s) + (4\pi)^{s-12}\Gamma(12-s)\zeta^*(2s-1)$$
$$+(4\pi)^{-11}\Gamma(s+11)\Gamma(12-s)\sum_{n=1}^{N}\frac{\tau(n+1)}{(n+1)^{11/2}}\frac{n^{s-1/2}\sigma_{1-2s}(n)}{\sqrt{n}}\,P^{-11}_{s-1}\left(1+\frac{2}{n}\right).$$

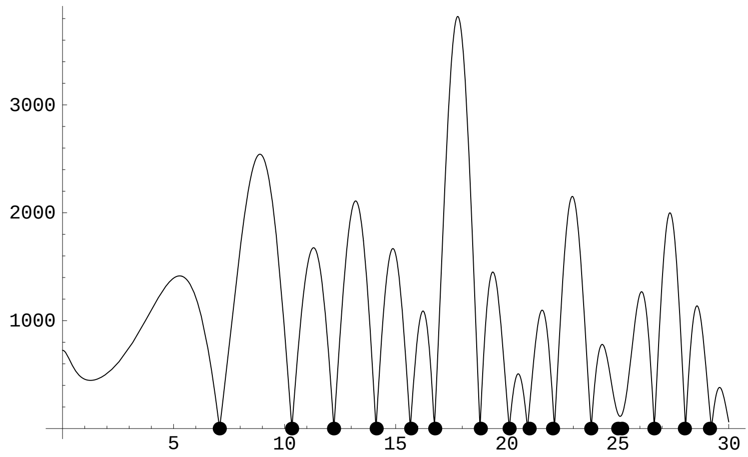

Fig. 2. $|L_{10}(1/2+it,\varDelta\times\varDelta)|$ for $0\le t\le 30$. Points $\bullet$ are zeros of $L(s,\varDelta\times\varDelta)$ on $\sigma=1/2$.

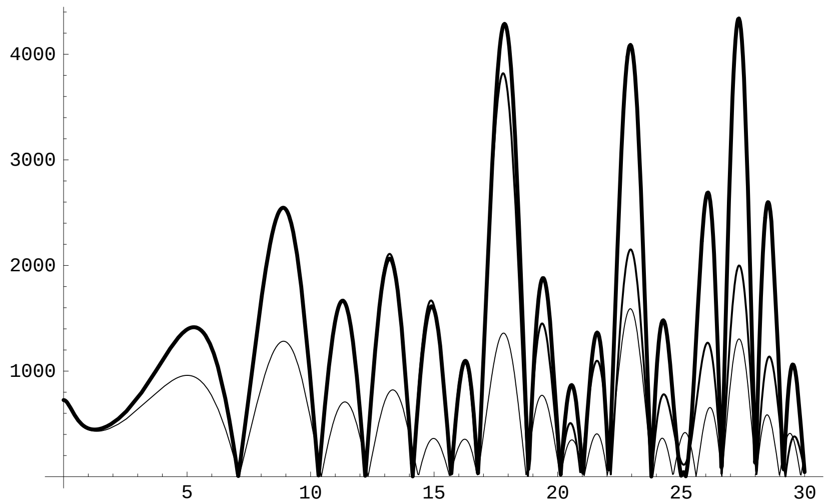

Fig. 3. The thin line is $|L_0(1/2+it,\varDelta\times\varDelta)|$ for $0\le t\le 30$, the line of medium thickness is $|L_{10}(1/2+it,\varDelta\times\varDelta)|$ for $0\le t\le 30$, and the thick line is $|L_{100}(1/2+it,\varDelta\times\varDelta)|$ for $0\le t\le 30$.

In Figures 1 and 2, dot points $\bullet$ are zeros of $L(s,\varDelta\times\varDelta)=\zeta(s)L(s,\mathrm{sym}^2\varDelta)$ on the critical line ([34, Table 3]). Interestingly, we observe that the lower zeros of $L(s,\varDelta\times\varDelta)$ on the critical line are approximated by zeros of the sum of the Riemann zeta-function $L_0(s,\varDelta\times\varDelta)$. Needless to say, this is not true for zeros of $L(s,\varDelta\times\varDelta)$ whose imaginary part becomes large. Figure 3 is the comparison of the absolute values $|L_0(s,\varDelta\times\varDelta)|$, $|L_{10}(s,\varDelta\times\varDelta)|$ and $|L_{100}(s,\varDelta\times\varDelta)|$ on the critical line. It shows that to know the value of $L(s,\varDelta\times\varDelta)$ for large $|t|$, we need many terms in $W^{\pm}_{\mathfrak{m},ij}(s)$ as large as $|t|$.

3.2. *The case* $k = 24$. This is the first case in which $d > 1$. We have $\dim S_{24} = 2$. Two functions f and g given by

$$\begin{aligned} f(z) &= E_{12}(z)\Delta(z) + 12\left(\frac{27017}{691} + \sqrt{144169}\right)\Delta^2(z) \\ &= \sum_{n=1}^{\infty} A_f(n)e(nz), \\ g(z) &= E_{12}(z)\Delta(z) + 12\left(\frac{27017}{691} - \sqrt{144169}\right)\Delta^2(z) \\ &= \sum_{n=1}^{\infty} A_g(n)e(nz) \end{aligned}$$

are distinct normalized Hecke eigen cusp forms of S_{24}, where

$$E_{12}(z) = 1 + \frac{65520}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)e(nz).$$

Put $\mathcal{F} = \{f/(f,f)^{1/2},\ g/(g,g)^{1/2}\}$. Then $\mathcal{F}$ is an orthonormal basis of S_{24}. Applying Corollary 2 to $\mathfrak{m} = (1,2)$, we obtain

$$\begin{aligned} \frac{L(1/2, f\times f)}{(f,f)} = \frac{1}{D}\,\frac{(4\pi)^{23}}{\Gamma(23)}&\left\{\frac{\Gamma'}{\Gamma}\left(\frac{47}{2}\right)\left(A_g(2) - \frac{A_f(2)}{\sqrt{2}}\right)\right. \\ &\left.+\left(A_g(2)\log\frac{e^\gamma}{16\pi^2 m} - \frac{A_f(2)}{\sqrt{2}}\log\frac{e^\gamma}{32\pi^2 m}\right)\right\} \\ + \frac{1}{D}\,\frac{4\pi^{24}\Gamma(46)}{\Gamma(23)^2\Gamma(24)}&\left\{\sum_{n=1}^{\infty} A_g(2)\,\frac{A_f(n+1)}{(n+1)^{23}}\,\frac{\sigma_0(n)}{\sqrt{n}}\,F\left(\frac12,\frac12,24;-\frac1n\right)\right. \\ - 2^{23}\sum_{n=1}^{\infty}\frac{A_f(n+2)}{(n+2)^{23}}\,&\left.\frac{\sigma_0(n)}{\sqrt{n}}\,F\left(\frac12,\frac12,24;-\frac2n\right) - 2^{23}F\left(\frac12,\frac12,24;-1\right)\right\}, \end{aligned}$$

$$\begin{aligned} \frac{L(1/2, f\times g)}{\sqrt{(f,f)(g,g)}} = \frac{A_f(2)}{D}\,\frac{\sqrt{(g,g)}}{\sqrt{(f,f)}}\,\frac{(4\pi)^{23}}{\Gamma(23)}&\left\{\frac{\Gamma'}{\Gamma}\left(\frac{47}{2}\right)\left(\frac{1-\sqrt{2}}{\sqrt{2}}\right)\right. \\ &\left.+\left(\frac{1}{\sqrt{2}}\log\frac{e^\gamma}{32\pi^2 m} - \log\frac{e^\gamma}{16\pi^2 m}\right)\right\} \\ - \frac{1}{D}\,\frac{4\pi^{24}\Gamma(46)}{\Gamma(23)^2\Gamma(24)}\,\frac{\sqrt{(g,g)}}{\sqrt{(f,f)}}&\left\{\sum_{n=1}^{\infty} A_f(2)\,\frac{A_f(n+1)}{(n+1)^{23}}\,\frac{\sigma_0(n)}{\sqrt{n}}\,F\left(\frac12,\frac12,24;-\frac1n\right)\right. \\ - 2^{23}\sum_{n=1}^{\infty}\frac{A_f(n+2)}{(n+2)^{23}}\,&\left.\frac{\sigma_0(n)}{\sqrt{n}}\,F\left(\frac12,\frac12,24;-\frac2n\right) - 2^{23}F\left(\frac12,\frac12,24;-1\right)\right\} \end{aligned}$$

and

$$\begin{aligned}\frac{L(1/2,g\times g)}{(g,g)} &= \frac{1}{D}\,\frac{(4\pi)^{23}}{\Gamma(23)}\left\{\frac{\Gamma'}{\Gamma}\left(\frac{47}{2}\right)\left(\frac{A_g(2)}{\sqrt{2}}-A_f(2)\right)\right.\\ &\qquad\left.+\left(\frac{A_g(2)}{\sqrt{2}}\log\frac{e^{\gamma}}{32\pi^2 m}-A_f(2)\log\frac{e^{\gamma}}{16\pi^2 m}\right)\right\}\\ &\quad-\frac{1}{D}\,\frac{4\pi^{24}\Gamma(46)}{\Gamma(23)^2\Gamma(24)}\left\{\sum_{n=1}^{\infty}A_f(2)\,\frac{A_g(n+1)}{(n+1)^{23}}\,\frac{\sigma_0(n)}{\sqrt{n}}\,F\left(\frac{1}{2},\frac{1}{2},24;-\frac{1}{n}\right)\right.\\ &\quad\left.-2^{23}\sum_{n=1}^{\infty}\frac{A_g(n+2)}{(n+2)^{23}}\,\frac{\sigma_0(n)}{\sqrt{n}}\,F\left(\frac{1}{2},\frac{1}{2},24;-\frac{2}{n}\right)-2^{23}F\left(\frac{1}{2},\frac{1}{2},24;-1\right)\right\},\end{aligned}$$

where $D=A_g(2)-A_f(2)$. As $(f,f)=1.28993\times10^{-4}$ and $(g,g)=1.07837\times10^{-4}$, we obtain the central values

$$\begin{aligned}L(1/2,f\times f)&=-3.07917\ldots,\\ L(1/2,f\times g)&=+9.79843\ldots\times10^{-3},\\ L(1/2,g\times g)&=-2.55952\ldots.\end{aligned}$$

Further, if $\det A_{\mathcal{F},\mathfrak{m}}\neq 0$, we have

$$\begin{aligned}D_{\mathfrak{m},11}(s)&=\frac{1}{D_{\mathfrak{m}}}\left\{\frac{A_f(m_1)A_g(m_2)}{m_1^s}-\frac{A_f(m_2)A_g(m_1)}{m_2^s}\right\},\\ D_{\mathfrak{m},12}(s)&=\frac{A_f(m_1)A_f(m_2)}{D_{\mathfrak{m}}}\,\frac{\sqrt{(g,g)}}{\sqrt{(f,f)}}\left\{\frac{1}{m_2^s}-\frac{1}{m_1^s}\right\},\\ D_{\mathfrak{m},21}(s)&=\frac{A_g(m_1)A_g(m_2)}{D_{\mathfrak{m}}}\,\frac{\sqrt{(f,f)}}{\sqrt{(g,g)}}\left\{\frac{1}{m_1^s}-\frac{1}{m_2^s}\right\},\\ D_{\mathfrak{m},22}(s)&=\frac{1}{D_{\mathfrak{m}}}\left\{\frac{A_f(m_1)A_g(m_2)}{m_2^s}-\frac{A_f(m_2)A_g(m_1)}{m_1^s}\right\},\end{aligned}\tag{3.1}$$

where $D_{\mathfrak{m}}=A_f(m_1)A_g(m_2)-A_f(m_2)A_g(m_1)$. We find that $A_{\mathcal{F},(1,2)}$, $A_{\mathcal{F},(2,3)}$ and $A_{\mathcal{F},(3,5)}$ are invertible by calculating their determinants directly. Using (3.1), we can determine the location of zeros of $D_{\mathfrak{m},11}(s)$ for a given vector $\mathfrak{m}$. For example, all zeros of $D_{\mathfrak{m},11}(s)$ lie on the line $\sigma=0.343579\ldots$ for $\mathfrak{m}=(1,2)$, $\sigma=-5.69519\ldots$ for $\mathfrak{m}=(2,3)$ and $\sigma=1.72665\ldots$ for $\mathfrak{m}=(3,5)$. These examples show that the location of zeros of $D_{\mathfrak{m},ij}(s)$ strongly depends on the choice of the vector $\mathfrak{m}$. It is not clear whether we can *always* choose a vector $\mathfrak{m}$ such that $D_{\mathfrak{m},ij}(s)$ satisfies the assumption of Theorem 2 in the case of large dimension of S_k.

4. Preliminaries

4.1. *Poincaré series.* Let m be a nonnegative integer. The mth Poincaré series $P_m(z)$ of weight k on Γ is defined by

$$P_m(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} j(\gamma, z)^{-k} e(m\gamma z),$$

where $\Gamma_\infty = \left\{ \pm \left(\begin{smallmatrix} 1 & n \\ & 1 \end{smallmatrix} \right) : n \in \mathbb{Z} \right\} \subset \Gamma$ and $j(\gamma, z) = cz + d$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$. If $k > 2$, the series on the right-hand side converges absolutely and uniformly on every compact subset of $\mathfrak{h}$. If $m \geq 1$, $P_m(z)$ is a cusp form, or may vanish identically. In particular, $P_m(z)$ vanishes identically for $k \leq 10$ and $k = 14$, since a cusp form of weight k on Γ exists only for $k = 12$ and $k \geq 16$. Petersson [15, 16] showed that a basis of S_k can be chosen from the Poincaré series $P_m(z)$, and the set $\{P_1(z), \ldots, P_d(z)\}$ $(d = \dim S_k)$ is a basis of S_k.

4.2. *Nonholomorphic Eisenstein series.* The *nonholomorphic Eisenstein series* $E(z, s)$ is defined by

$$E(z, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (\operatorname{Im} \gamma z)^s = \frac{1}{2} \sum_{\substack{(c,d) \in \mathbb{Z}^2 \\ (c,d)=1}} \frac{y^s}{|cz + d|^{2s}}.$$

The right-hand side converges absolutely for $\sigma > 1$. The modified function

$$E^*(z, s) = \zeta^*(2s) E(z, s)$$

is often called the *completed nonholomorphic Eisenstein series*. The function $E^*(z, s)$ is continued meromorphically to the whole s-plane, and is holomorphic except for simple poles at $s = 0$ and 1. It satisfies the functional equation $E^*(z, s) = E^*(z, 1 - s)$. On the other hand, $E((az + b)/(cz + d), s) = E(z, s)$ for every $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma$. Hence, in particular, $E^*(z, s)$ has the Fourier expansion

$$E^*(z, s) = \sum_{n=0}^{\infty} a_n(y, s) \cos(2\pi n x),$$

where

$$a_0(y, s) = \begin{cases} \zeta^*(2s) y^s + \zeta^*(2s - 1) y^{1-s}, & s \neq 0, 1/2, 1, \\ y^{1/2} \log y + (\gamma - \log 4\pi) y^{1/2}, & s = 1/2, \end{cases} \tag{4.1}$$

and

$$a_n(y, s) = 4\sqrt{y} \sum_{n=1}^{\infty} \tau_{s-1/2}(n) K_{s-1/2}(2\pi n y) \tag{4.2}$$

for $n \neq 0$. Here $\gamma = 0.57721\ldots$ is the Euler constant, $\tau_\nu(n) = n^\nu \sigma_{-2\nu}(n)$, $\sigma_\nu(n) = \sum_{d|n} d^\nu$ and $K_\nu(t)$ is the K-Bessel function.

4.3. *C^∞-modular forms.* A smooth function f on $\mathfrak{h}$ satisfying $f(\gamma z) = j(\gamma, z)^k f(z)$ for every $\gamma \in \Gamma$ is called a *C^∞-modular form of weight k*. The *Petersson inner product* (f, g) of C^∞-modular forms f and g is defined by

$$(f, g) := \int_{\Gamma \backslash \mathfrak{h}} f(z) \overline{g(z)} \, y^{k-2} \, dx \, dy,$$

if the right-hand side converges. In particular, (f, g) is defined if one of f and g belongs to M_k, and the other to S_k, where M_k is the space of all holomorphic modular forms of weight k on Γ. A C^∞-modular form f of weight k is called a C^∞-modular form of *bounded growth* if

$$\int_0^\infty \int_0^1 |f(z)| y^{k-2} e^{-\varepsilon y}\, dx\, dy < \infty \qquad \text{for every } \varepsilon > 0. \tag{4.3}$$

4.4. *Inner product with Poincaré series.* Let $f(z) = \sum_{n\in\mathbb{Z}} a_n(y)e(nx)$ be a C^∞-modular form of bounded growth. By the unfolding method we derive

$$(f, P_m) = \int_0^\infty \int_0^1 f(z) e(-m\overline{z}) y^{k-2}\, dx\, dy$$

for all $m \geq 0$. Substituting the Fourier expansion of f for the right-hand side, we obtain

$$(f, P_m) = \int_0^\infty a_m(y) e^{-2\pi m y} y^{k-2}\, dy \qquad (m \geq 0). \tag{4.4}$$

Interchanging integration and summation is justified by the growth condition (4.3) ([21, Proposition 1]). Hence, equality (4.4) holds for all C^∞-modular forms of bounded growth. Thus we have

$$\begin{aligned} (f, P_m) &= a_f(m) m^{(k-1)/2} \int_0^\infty e^{-4\pi m y} y^{k-2}\, dy \\ &= (4\pi)^{-k+1} \Gamma(k-1) a_f(m) m^{-(k-1)/2} \end{aligned}$$

for every nonnegative integer m, since the holomorphic cusp form $f(z) = \sum_{n=1}^\infty a_f(n) n^{(k-1)/2} e(nz)$ satisfies the condition (4.3).

4.5. *Rankin–Selberg L-functions.* Let $f(z) = \sum_{n=1}^\infty a_f(n) n^{(k-1)/2} e(nz)$ and $g(z) = \sum_{n=0}^\infty a_g(n) n^{(k-1)/2} e(nz)$ be modular forms in S_k and M_k, respectively. The Rankin–Selberg L-function $L(s, f \otimes \overline{g})$ is defined by (2.1) if the real part of s is sufficiently large. The function $F(z) = y^k f(z)\, \overline{g(z)}$ is a bounded Γ-invariant function on $\mathfrak{h}$ with rapid decay as $y \to +\infty$. Its Fourier expansion is

$$\begin{aligned} F(x+iy) &= y^k f(z)\, \overline{g(z)} \\ &= y^k \sum_{n\in\mathbb{Z}} \Big(\sum_{m=1-n}^\infty a_f(m+n)\, \overline{a_g(m)}\, (m+n)^{(k-1)/2} m^{(k-1)/2} e^{-2\pi(2m+n)y} \Big) e(nx). \end{aligned}$$

Therefore we obtain

$$\int_{\Gamma\backslash\mathfrak{h}} y^k f(z)\,\overline{g(z)}\,E(z,s)\,d\mu(z) = \int_0^\infty \Big(\sum_{n=1}^\infty a_f(n)\,\overline{a_g(n)}\,n^{k-1}e^{-4\pi ny}\Big)y^{s+k-1}\,\frac{dy}{y}$$

for $\sigma > 1$ by the unfolding method. The right-hand side is equal to

$$(4\pi)^{-s-k+1}\Gamma(s+k-1)\sum_{m=1}^\infty a_f(n)\,\overline{a_g(n)}\,n^{-s}$$

for $\sigma > k/2+1$, since the series converges absolutely there by the estimates $a_f(n) = O(n^{1/2})$ and $a_g(n) = O(n^{(k-1)/2})$. Hence we obtain

$$(fE_s^*,\,g) = \int_{\Gamma\backslash\mathfrak{h}} y^k f(z)\,\overline{g(z)}\,E^*(z,s)\,d\mu(z) = \pi^{-s}(4\pi)^{-s-k+1}\Gamma(s)\Gamma(s+k-1)L(s, f\times\overline{g}) \tag{4.5}$$

for $\sigma > k/2+1$, where $E_s^*(z) = E^*(z,s)$. The left-hand side is defined for all $s \in \mathbb{C}$ except for the poles of $E^*(z,s)$, since f is a cusp form. Therefore (4.5) gives the meromorphic continuation of $L(s, f\times\overline{g})$ to $\mathbb{C}$.

5. Proof of Theorem 1. Theorem 1 is a consequence of the following proposition.

PROPOSITION 1. *Let $\mathcal{F} = \{f_1,\ldots,f_d\}$ be an orthonormal basis of S_k, and let $f_j(z) = \sum_{n=1}^\infty a_j(n)n^{(k-1)/2}e(nz)$ be the Fourier expansion of f_j at $i\infty$. For every $f \in S_k$,*

$$\begin{aligned}(4\pi)^{-k+1}\Gamma(k-1)\sum_{j=1}^d a_j(m)L^*(s, f\times\overline{f}_j) &= a_f(m)[(4\pi)^{-s-k+1}\Gamma(s+k-1)\zeta^*(2s)m^{-s} \\ &\quad + (4\pi)^{s-k}\Gamma(k-s)\zeta^*(2s-1)m^{s-1}] \\ &\quad + (4\pi)^{-k+1}\Gamma(s+k-1)\Gamma(k-s) \\ &\quad \times \sum_{n=1}^{m-1} a_f(m-n)\,\frac{\tau_{s-1/2}(n)}{\sqrt{n}}\,P_{s-1}^{1-k}\Big(\frac{2m-n}{n}\Big) \\ &\quad + (4\pi)^{-k+1}\Gamma(s+k-1)\Gamma(k-s) \\ &\quad \times \sum_{n=1}^{\infty} a_f(m+n)\,\frac{\tau_{s-1/2}(n)}{\sqrt{n}}\,P_{s-1}^{1-k}\Big(\frac{2m+n}{n}\Big)\end{aligned} \tag{5.1}$$

in the strip (2.5) *if the first term* $a_f(m)[\cdots]$ *in* (5.1) *is replaced by*

$$a_f(m)(4\pi)^{-k+1/2}\Gamma\left(k-\frac{1}{2}\right)\left\{\frac{\Gamma'}{\Gamma}\left(k-\frac{1}{2}\right)+\log\frac{e^\gamma}{16\pi^2 m}\right\}\frac{1}{\sqrt{m}}$$

at the point $s=1/2$. *The series on the right-hand side of* (5.1) *converges absolutely and uniformly on every compact subset of the vertical strip* (2.5).

Proof. We denote $E^*(z,s)$ by $E^*_s(z)$. Calculating the Petersson inner product (fE^*_s,P_m) in two ways, we will obtain Proposition 1.

Let m be a positive integer, and let $\mathcal{F}=\{f_1,\ldots,f_d\}$ be an orthonormal basis of S_k. Expanding $P_m(z)$ with respect to the basis $\mathcal{F}$, we have

$$P_m(z)=(4\pi)^{-k+1}\Gamma(k-1)m^{-(k-1)/2}\sum_{j=1}^{d}\overline{a_j(m)}\,f_j(z),$$

where $a_j(m)$ is the mth Fourier coefficient of f_j. Using this expansion, we obtain the first formula

$$(5.2)\qquad (fE^*_s,P_m)=(4\pi)^{-k+1}\Gamma(k-1)m^{-(k-1)/2}\sum_{j=1}^{d}a_j(m)L^*(s,f\times\overline{f}_j)$$

for $\sigma>1$. Further (5.2) holds for all $s\in\mathbb{C}$, since f is a cusp form. By Lemma 1 of [14], the product $f(z)E(z,s)$ is a C^∞-modular form of bounded growth for $0<\sigma<1$. Hence, by (4.4), we have

$$(5.3)\quad (fE^*_s,P_m)=\int_0^\infty\Big(\sum_{n=1}^{\infty}a_f(n)a_{m-n}(y,s)n^{(k-1)/2}e^{-2\pi ny}\Big)e^{-2\pi my}y^{k-2}\,dy$$

for $0<\sigma<1$, where $a_n(y,s)$ is the nth Fourier coefficient of $E^*(z,s)$ given in (4.1) and (4.2). Formally, the right-hand side of (5.3) is equal to

$$\sum_{n=0}^{m-1}a_f(m-n)(m-n)^{(k-1)/2}\int_0^\infty a_n(y,s)e^{-2\pi(2m-n)y}y^{k-2}\,dy$$
$$+\sum_{n=1}^{\infty}a_f(m+n)(m+n)^{(k-1)/2}\int_0^\infty a_n(y,s)e^{-2\pi(2m+n)y}y^{k-2}\,dy.$$

This formal calculation is justified, since interchanging summation and integration is allowed by the estimates

$$|a_0(y,s)| \ll y^{\sigma} + y^{1-\sigma},$$
$$|a_n(y,s)| \ll y^{\sigma}|\sigma_{1-2s}(n)|e^{-\pi ny/2} \quad (n \neq 0),$$

and Fubini's theorem. For $n = 0$ and $s \neq 0, 1/2, 1$, we have

$$(5.4)\qquad \int_0^\infty a_0(y,s)e^{-4\pi my}y^{k-2}\,dy$$
$$= \zeta^*(2s)\int_0^\infty e^{-4\pi my}y^{k+s-2}\,dy + \zeta^*(2s-1)\int_0^\infty e^{-4\pi my}y^{k-s-1}\,dy$$
$$= (4\pi m)^{-s-k+1}\Gamma(s+k-1)\zeta^*(2s) + (4\pi m)^{s-k}\Gamma(k-s)\zeta^*(2s-1).$$

For $n = 0$ and $s = 1/2$, we have

$$(5.5)\qquad \int_0^\infty a_0(y,1/2)e^{-4\pi my}y^{k-2}\,dy$$
$$= \int_0^\infty e^{-4\pi my}y^{k-3/2}\log y\,dy + (\gamma - \log 4\pi)\int_0^\infty e^{-4\pi my}y^{k-3/2}\,dy$$
$$= (4\pi m)^{-k+1/2}\Gamma\Big(k-\frac{1}{2}\Big)\Big\{\frac{\Gamma'}{\Gamma}\Big(k-\frac{1}{2}\Big) + \log\frac{e^{\gamma}}{16\pi^2 m}\Big\}.$$

For $n \geq 1$, we have

$$(5.6)\qquad \int_0^\infty a_n(y,s)e^{-2\pi(2m\pm n)y}y^{k-2}\,dy$$
$$= 2\tau_{s-1/2}(n)\int_0^\infty K_{s-1/2}(2\pi ny)e^{-2\pi(2m\pm n)y}y^{k-3/2}\,dy$$
$$= (4\pi)^{-k+1}m^{-k+1}\Gamma(s+k-1)\Gamma(k-s)$$
$$\times\frac{\tau_{s-1/2}(n)}{\sqrt{n}}\Big(\frac{m}{m\pm n}\Big)^{(k-1)/2}P_{s-1}^{1-k}\Big(\frac{2m\pm n}{n}\Big)$$

by using the formula

$$\int_0^\infty K_\nu(x)e^{-ax}x^{\mu-1}\,dx = \sqrt{\frac{\pi}{2}}\,\frac{\Gamma(\mu+\nu)\Gamma(\mu-\nu)}{(a^2-1)^{\mu/2-1/4}}\,P_{\nu-1/2}^{-\mu+1/2}(a)$$

for $\mathrm{Re}(a) > -1$ and $\mathrm{Re}(\mu) > |\mathrm{Re}(\nu)|$ ([30, p. 388]). By (5.3), (5.4) and (5.6), we obtain the second formula

$$
\begin{aligned}
(5.7)\quad (fE_s^*, P_m) = m^{-(k-1)/2} a_f(m)[(4\pi)^{-s-k+1} m^{-s} \Gamma(s+k-1)\zeta^*(2s) \\
+ (4\pi)^{s-k} m^{s-1} \Gamma(k-s)\zeta^*(2s-1)] \\
+ m^{-(k-1)/2}(4\pi)^{-k+1}\Gamma(s+k-1)\Gamma(k-s) \\
\times \sum_{n=1}^{m-1} a_f(m-n)\, \frac{\tau_{s-1/2}(n)}{\sqrt{n}}\, P_{s-1}^{1-k}\Big(\frac{2m-n}{n}\Big) \\
+ m^{-(k-1)/2}(4\pi)^{-k+1}\Gamma(s+k-1)\Gamma(k-s) \\
\times \sum_{n=1}^{\infty} a_f(m+n)\, \frac{\tau_{s-1/2}(n)}{\sqrt{n}}\, P_{s-1}^{1-k}\Big(\frac{2m+n}{n}\Big).
\end{aligned}
$$

Combining (5.2) and (5.7), we obtain (5.1) for $0 < \sigma < 1$ except for $s = 1/2$. For $s = 1/2$, we use (5.5) instead of (5.4).

To complete the proof of Proposition 1, it suffices to show that the series on the right-hand side of (5.1) converges absolutely in the vertical strip (2.5), since the left-hand side of (5.1) is defined for all $s \in \mathbb{C}$ except for the possible poles at $s = 1$ and 0. Moreover, it suffices to show that the series

$$
(5.8)\qquad \sum_{n=1}^{\infty} a_f(m+n)\, \frac{\tau_{s-1/2}(n)}{\sqrt{n}}\, P_{s-1}^{k-1}\Big(\frac{2m+n}{n}\Big)
$$

converges absolutely in the strip (2.5), since

$$
P_{s-1}^{1-k}(z) = \frac{\Gamma(s-k+1)}{\Gamma(s+k-1)}\, P_{s-1}^{k-1}(z)
$$

for every positive integer $k \ge 2$. Suppose that $|a_f(n)| \ll n^{1/2-\alpha+\varepsilon}$ for some real number $0 \le \alpha \le 1/2$. Then

$$
\begin{aligned}
\sum_{n=1}^{\infty} |a_f(m+n)|\, \frac{|\tau_{s-1/2}(n)|}{\sqrt{n}} \left| P_{s-1}^{k-1}\Big(\frac{2m+n}{n}\Big)\right| \\
\ll_m \sum_{n=1}^{\infty} n^{|\sigma-1/2|-\alpha+\varepsilon} \left| P_{s-1}^{k-1}\Big(\frac{2m+n}{n}\Big)\right|,
\end{aligned}
$$

since $|\tau_{s-1/2}(n)| = |n^{s-1/2}\sigma_{1-2s}(n)| \ll_\varepsilon n^{|\sigma-1/2|+\varepsilon}$. Using the formula

$$
\begin{aligned}
P_{s-1}^{k-1}(z) = \frac{\Gamma(s+k-1)(z^2-1)^{(k-1)/2}}{2^{k-1}\sqrt{\pi}\,\Gamma(k-1/2)\Gamma(s-k+1)} \\
\times \int_0^{\pi} (z + \sqrt{z^2-1}\cos\theta)^{s-k} \sin^{2k-2}\theta\, d\theta
\end{aligned}
$$

for $\mathrm{Re}(z) > 0$ and $k \ge 1$ ([11, p. 199]), we have

$$
\left| P_{s-1}^{k-1}\Big(\frac{2m+n}{n}\Big)\right| \ll_m n^{-(k-1)/2}.
$$

Hence we obtain

$$|\text{series (5.8)}| \ll_m \sum_{n=1}^{\infty} n^{|\sigma-1/2|-(k-1)/2-\alpha+\varepsilon}.$$

The right-hand side converges absolutely for $2-k/2-\alpha < \mathrm{Re}(s) < k/2+\alpha-1$. Hence the Ramanujan–Deligne estimate $|a_f(n)| \ll_\varepsilon n^\varepsilon$ implies that the series on the right-hand side of (5.1) converges absolutely in the vertical strip (2.5). ∎

Proof of Theorem 1. Let $\mathcal{F} = \{f_1, \dots, f_d\}$, $\mathfrak{m} = (m_1, \dots, m_d) \in \mathbb{Z}^d_{>0}$ with $0 < m_1 < \dots < m_d$ and $\{\alpha_{ij}\}$ be as in the statement of the theorem. By Proposition 1,

$$(4\pi)^{-k+1}\Gamma(k-1)A_{\mathcal{F},\mathfrak{m}}\mathcal{L}_{\mathcal{F},f}(s) = \mathcal{N}_{\mathfrak{m},f}(s), \tag{5.9}$$

where

$$\mathcal{L}_{\mathcal{F},f}(s) = \begin{pmatrix} L^*(s, f\times \overline{f}_1) \\ \vdots \\ L^*(s, f\times \overline{f}_d)\end{pmatrix}, \quad \mathcal{N}_{\mathfrak{m},f}(s) = \begin{pmatrix} N_f(s, m_1) \\ \vdots \\ N_f(s, m_d)\end{pmatrix},$$

and

$$\begin{aligned} N_f(s, m_h) = a_f(m_h)[(4\pi)^{-s-k+1}&\Gamma(s+k-1)\zeta^*(2s)m_h^{-s} \\ &+ (4\pi)^{s-k}\Gamma(k-s)\zeta^*(2s-1)m_h^{s-1}] \\ + (4\pi)^{-k+1}&\Gamma(s+k-1)\Gamma(k-s) \\ \times &\sum_{n=1}^{m_h-1} a_f(m_h-n)\,\frac{\tau_{s-1/2}(n)}{\sqrt{n}}\,P^{1-k}_{s-1}\Big(\frac{2m_h-n}{n}\Big) \\ + (4\pi)^{-k+1}&\Gamma(s+k-1)\Gamma(k-s) \\ \times &\sum_{n=1}^{\infty} a_f(m_h+n)\,\frac{\tau_{s-1/2}(n)}{\sqrt{n}}\,P^{1-k}_{s-1}\Big(\frac{2m_h+n}{n}\Big). \end{aligned}$$

Multiplying (5.9) by the inverse matrix $A^{-1}_{\mathcal{F},\mathfrak{m}}$, we have

$$(4\pi)^{-k+1}\Gamma(k-1)\mathcal{L}_{\mathcal{F},f}(s) = A^{-1}_{\mathcal{F},\mathfrak{m}}\mathcal{N}_{\mathfrak{m},f}(s).$$

Comparing the jth components of both sides, we obtain

$$(4\pi)^{-k+1}\Gamma(k-1)L^*(s, f\times \overline{f}_j) = \sum_{h=1}^{d} \alpha_{jh} N_f(s, m_h).$$

Taking $f = f_i$, we obtain equality (2.4) of Theorem 1. ∎

6. Proof of Theorem 2. It suffices to investigate the zeros of $L^N_{\mathfrak{m},ij}(s)$ in $\sigma \ge 1/2$, because of the functional equation (2.12) of $L^N_{\mathfrak{m},ij}(s)$. By definition

(2.11) of $L^N_{\mathfrak{m},ij}(s)$, we have

$$L^N_{\mathfrak{m},ij}(s) = (4\pi)^{-s-k+1}\Gamma(s+k-1)\zeta^*(2s)D_{\mathfrak{m},ij}(s)\{1+R^N_{\mathfrak{m},ij}(s)\}, \tag{6.1}$$

where

$$R^N_{\mathfrak{m},ij}(s) = (4\pi)^{2s-1}\frac{\Gamma(k-s)\zeta^*(2s-1)}{\Gamma(s+k-1)\zeta^*(2s)}\frac{D_{\mathfrak{m},ij}(1-s)}{D_{\mathfrak{m},ij}(s)} + (4\pi)^s\frac{\Gamma(k-s)\{W^{+,N}_{\mathfrak{m},ij}(s)+W^{-}_{\mathfrak{m},ij}(s)\}}{\zeta^*(2s)D_{\mathfrak{m},ij}(s)}.$$

By the assumption on the location of zeros of $D_{\mathfrak{m},ij}(s)$ in Theorem 2, the factor $\zeta^*(2s)D_{\mathfrak{m},ij}(s)$ in (6.1) has only finitely many zeros in $\sigma \geq 1/2$. Hence, if the inequality

$$|R^N_{\mathfrak{m},ij}(s)| < 1$$

is valid for $1/2 < \sigma \leq a$ and sufficiently large $|t|$, then $L^N_{\mathfrak{m},ij}(s) \neq 0$ in that region. Now we show that there exists $T_{N,a,\varepsilon} > 1$ such that

$$|R^N_{\mathfrak{m},ij}(\sigma+it)| \ll |t|^{1-2\sigma}\log|t| \tag{6.2}$$

for $1/2 \leq \sigma \leq a$ and $|t| \geq T_{N,a,\varepsilon}$. We define

$$I_{\mathfrak{m},ij}(s) = (4\pi)^{2s-1}\frac{\Gamma(k-s)\zeta^*(2s-1)}{\Gamma(s+k-1)\zeta^*(2s)}\frac{D_{\mathfrak{m},ij}(1-s)}{D_{\mathfrak{m},ij}(s)}, \tag{6.3}$$

$$J^N_{\mathfrak{m},ij}(s) = (4\pi)^s\frac{\Gamma(k-s)\{W^{+,N}_{\mathfrak{m},ij}(s)+W^{-}_{\mathfrak{m},ij}(s)\}}{\zeta^*(2s)D_{\mathfrak{m},ij}(s)} \tag{6.4}$$

so that

$$R^N_{\mathfrak{m},ij}(s) = I_{\mathfrak{m},ij}(s) + J^N_{\mathfrak{m},ij}(s). \tag{6.5}$$

For $I_{\mathfrak{m},ij}(s)$ and $J^N_{\mathfrak{m},ij}(s)$, we obtain the following estimates.

Lemma 1. *There exists $T_1 > 0$ such that*

$$|I_{\mathfrak{m},ij}(s)| = O(|t|^{1-2\sigma})$$

for $1/2 \leq \sigma \leq a$ and $|t| \geq T_1$, where the implied constant depends on $\mathfrak{m}$, i and j.

Lemma 2. *There exists $T_2 > 0$ such that*

$$|J^N_{\mathfrak{m},ij}(s)| = O(|t|^{1-2\sigma}\log|t|)$$

for $1/2 \leq \sigma \leq a$ and $|t| \geq T_2$, where the implied constant depends on N, $\mathfrak{m}$, i and j.

Lemma 1, Lemma 2 and (6.5) imply (6.2). Hence the proof of Theorem 2 will be completed if we prove Lemmas 1 and 2. To do that, we use the following lemma.

LEMMA 3. *Let $g(s)$ be an exponential polynomial having the form*

$$g(s)=\sum_{j=1}^{n} p_j e^{\beta_j s}, \qquad 0=\beta_0<\beta_1<\cdots<\beta_n,$$

where $0\neq p_j\in\mathbb{C}$ $(0\le j\le n)$. Then $|g(s)|$ is uniformly bounded away from zero if s is uniformly separated from the zeros of $g(s)$.

Proof. See Theorem 12.6 of [2]. ∎

Proof of Lemma 1. Let $\xi(s)=s(s-1)\zeta^*(s)$. We have

$$\left|(4\pi)^{2s-1}\frac{\Gamma(k-s)\zeta^*(2s-1)}{\Gamma(s+k-1)\zeta^*(2s)}\right|=\left|\frac{t}{4\pi}\right|^{1-2\sigma}\frac{1+O(|t|^{-1})}{1+O(|t|^{-1})}\left|\frac{s}{s-1}\right|\left|\frac{\xi(2s-1)}{\xi(2s)}\right|$$

for $1/2\le\sigma\le a$ and $|t|\ge 1$ by using Stirling's formula

$$|\Gamma(\sigma+it)|=\sqrt{2\pi}\,|t|^{\sigma-1/2}e^{-(\pi/2)|t|}(1+O(|t|^{-1}))$$

for $\sigma_1\le\sigma\le\sigma_2$ and $|t|\ge 1$. By the proof of Theorem 2 in [10], we have

$$\left|\frac{\xi(2s-1)}{\xi(2s)}\right|\le 1$$

for $\sigma\ge 1/2$. Hence, we obtain

$$\left|(4\pi)^{2s-1}\frac{\Gamma(k-s)\zeta^*(2s-1)}{\Gamma(s+k-1)\zeta^*(2s)}\right|=O(|t|^{1-2\sigma}) \tag{6.6}$$

for $1/2\le\sigma\le a$ and $|t|\ge t_1$ (>1). By Lemma 3 and the assumption on the location of zeros of $D_{\mathfrak{m},ij}(s)$, we have

$$\left|\frac{D_{\mathfrak{m},ij}(1-s)}{D_{\mathfrak{m},ij}(s)}\right|=O(1) \tag{6.7}$$

for $1/2\le\sigma\le a$ and $|t|\ge t_2$. By (6.3), (6.6) and (6.7), we obtain the estimate in Lemma 1. ∎

Proof of Lemma 2. The asymptotic formula (A.1) of the Appendix yields

$$\begin{aligned}(4\pi)^s\frac{\Gamma(k-s)}{\zeta^*(2s)}&P_{s-1}^{1-k}(\cosh\zeta)\\ &=\frac{(2\pi)^{2s}}{\sqrt{\pi}}\frac{\Gamma(k-s)}{\Gamma(s+k-1)}\frac{1}{\zeta(2s)}\frac{1}{(s-1)^{1/2}}\frac{e^{-\zeta/2}}{\sqrt{1-e^{-2\zeta}}}\\ &\quad\times[e^{(s-1/2)\zeta}+e^{\pm\pi i(k-1/2)}e^{(-s+1/2)\zeta}+O(|s-1|^{-1})],\end{aligned}$$

where the implied constant depends on $\zeta > 0$. Therefore,

$$\left|(4\pi)^s \frac{\Gamma(k-s)}{\zeta^*(2s)} P_{s-1}^{1-k}(\cosh\zeta)\right| = \frac{(2\pi)^{2\sigma}}{\sqrt{\pi}} \left|\frac{\Gamma(k-s)}{\Gamma(s+k-1)}\right| \frac{1}{|\zeta(2s)|} \frac{1}{\sqrt{|s-1|}} \times \frac{e^{-\zeta/2}}{\sqrt{1-e^{-2\zeta}}} [e^{(\sigma-1/2)\zeta} + e^{(-\sigma+1/2)\zeta} + O(|s-1|^{-1})].$$

Using Stirling's formula, we have

$$\left|\frac{\Gamma(k-s)}{\Gamma(s+k-1)}\right| = |t|^{1-2\sigma} \frac{1+O(|t|^{-1})}{1+O(|t|^{-1})} \ll |t|^{1-2\sigma}$$

for $1/2 \le \sigma < a$ and $|t| \ge t_3$. On the other hand,

$$\frac{1}{|\zeta(s)|} = O(\log(|t|+2))$$

for $\sigma \ge 1 - A/\log(|t|+2)$ ([27, p. 60]). Hence we have

$$\left|(4\pi)^s \frac{\Gamma(k-s)}{\zeta^*(2s)} P_{s-1}^{1-k}(\cosh\zeta)\right| = O(|t|^{1-2\sigma}\log|t|) \tag{6.8}$$

for $1/2 - A'/\log|t| \le \sigma \le a$ and $|t| \ge t_4$. By Lemma 3 and the assumption on the location of zeros of $D_{\mathfrak{m},ij}(s)$, we have

$$\left|\frac{1}{D_{\mathfrak{m},ij}(s)}\right| = O(1) \tag{6.9}$$

for $1/2 \le \sigma \le a$ and $|t| \ge t_5$. Here we note that

$$\begin{aligned} 1+\frac{2}{m_h-1} < \frac{2m_h-n}{n} < 2m_h-1 \quad (1 \le n \le m_h-1,\ 1 \le h \le d), \\ 1+\frac{2m_h}{N} < \frac{2m_h+n}{n} < 2m_h+1 \quad (1 \le n \le N,\ 1 \le h \le d) \end{aligned} \tag{6.10}$$

for fixed $\mathfrak{m} = (m_1, \ldots, m_d)$. Combining (2.10), (6.4), (6.8), (6.9) and (6.10), we obtain Lemma 2. ■

7. Relation with the holomorphic projection. In this section, we reconsider the argument of Section 5 from the viewpoint of the holomorphic projection of Sturm [21]. Let $\mathcal{F} = \{f_1, \ldots, f_d\}$ be an orthonormal basis of S_k. Define

$$K(z,w) = \sum_{i=1}^{d} f_i(z)\,\overline{f_i(w)}. \tag{7.1}$$

Then $K(z,w)$ belongs to S_k as a function of $z \in \mathfrak{h}$ for every fixed $w \in \mathfrak{h}$, and has the reproducing property:

$$(g(z), K(z,w)) = g(w) \quad \text{for any } g \in S_k. \tag{7.2}$$

For a C^∞-modular form F of bounded growth, we define

$$\pi(F)(w) := (F(z), K(z,w)).$$

Then $\pi(F)(w)$ belongs to S_k, and is called the *holomorphic projection* of F. Using the formula

$$K(z,w) = \sum_{m=1}^{\infty} \frac{(4\pi m)^{k-1}}{\Gamma(k-1)} P_m(z) e(-m\overline{w}) \tag{7.3}$$

([21, p. 333]), we obtain

$$\pi(F)(w) = \sum_{m=1}^{\infty} \frac{(4\pi m)^{k-1}}{\Gamma(k-1)} (F, P_m) e(mw), \tag{7.4}$$

where the inner product (F, P_m) is given by (4.4). Using (7.2), we have

$$\big((F(z), K(z,w)), g(w)\big) = \big(F(z), (g(w), K(w,z))\big) = (F(z), g(z)).$$

Hence, we obtain

$$(F,\, g) = (\pi(F),\, g). \tag{7.5}$$

Applying (7.5) to $F(z) = (fE_s^*)(z) := f(z)E^*(z,s)$, we have

$$L^*(s, f \times \overline{g}) = (\pi(fE_s^*), g) \tag{7.6}$$

by (4.5) (compare (7.6) with (2.10) of [12]). By (7.3) and (7.5), we have

$$(F,g) = (\pi(F), g) = \sum_{m=1}^{\infty} \frac{(4\pi m)^{k-1}}{\Gamma(k-1)} \phi_m(g)(F, P_m), \tag{7.7}$$

where

$$\phi_m(g) = \int_{\Gamma\backslash\mathfrak{h}} \overline{g(w)}\, e(mw)\, d\mu(w).$$

Applying (7.7) to $F = fE_s^*$, we obtain

$$L^*(s, f \times \overline{g}) = \sum_{m=1}^{\infty} \frac{(4\pi m)^{k-1}}{\Gamma(k-1)} \phi_m(g)(fE_s^*, P_m) \tag{7.8}$$

by (7.6). However, this formula for $L(s, f \times \overline{g})$ is not useful for application, because each $\phi_m(g)$ depends on a choice of a fundamental domain of Γ.

To improve formula (7.8) of $L(s, f \times \overline{g})$, we consider the Fourier coefficients of $\pi(fE_s^*)$. Let $\mathcal{F} = \{f_1, \ldots, f_d\}$ be an orthogonal basis of S_k. Applying (7.4) to $F = fE_s^*$, we have

$$\pi(fE_s^*)(z) = \sum_{m=1}^{\infty} \frac{(4\pi m)^{k-1}}{\Gamma(k-1)} (fE_s^*, P_m) e(mz). \tag{7.9}$$

Because $\pi(fE_s^*) \in S_k$, there exist functions $C_j(s)$ of s such that

$$\pi(fE_s^*)(z) = \sum_{j=1}^{d} C_j(s) f_j(z). \tag{7.10}$$

By (7.1) and (7.6), we have

$$C_j(s) = \frac{1}{(f_j, f_j)} (\pi(fE_s^*), f_j) = \frac{1}{(f_j, f_j)} L^*(s, f \times \overline{f}_j). \tag{7.11}$$

Here we have used the Fourier expansion $f_j(z) = \sum_{n=1}^{\infty} a_j(n) n^{(k-1)/2} e(nz)$. Combining (7.9)–(7.11), and comparing the mth Fourier coefficients of both sides, we obtain

$$\begin{aligned}
\sum_{j=1}^{d} C_j(s) a_j(m) &= \sum_{j=1}^{d} \frac{a_j(m)}{(f_j, f_j)} L^*(s, f \times \overline{f}_j) = \frac{(4\pi m)^{k-1}}{\Gamma(k-1)} (fE_s^*, P_m) \\
&= \frac{a_f(m)}{(f,f)} \bigg\{ (4\pi m)^{-s} \frac{\Gamma(s+k-1)}{\Gamma(k-1)} \zeta^*(2s) + (4\pi m)^{s-1} \frac{\Gamma(k-s)}{\Gamma(k-1)} \zeta^*(2s-1) \\
&\quad + \frac{\Gamma(s+k-1)\Gamma(k-s)}{\Gamma(k-1)} \sum_{n=1}^{m-1} \frac{a_f(m-n)}{a_f(m)} \frac{\tau_s(n)}{\sqrt{n}} P_{s-1}^{1-k}\left(\frac{2m-n}{n}\right) \\
&\quad + \frac{\Gamma(s+k-1)\Gamma(k-s)}{\Gamma(k-1)} \sum_{n=1}^{\infty} \frac{a_f(m+n)}{a_f(m)} \frac{\tau_s(n)}{\sqrt{n}} P_{s-1}^{1-k}\left(\frac{2m+n}{n}\right) \bigg\}.
\end{aligned}$$

This is nothing other than equality (5.1).

Appendix. Asymptotic expansion of $P_\nu^\mu(z)$. In this section, we give an asymptotic expansion of the associated Legendre functions $P_\nu^\mu(z)$ for large $|\nu|$ according to Watson [29], where ν and μ do not have to be integers. The associated Legendre function $P_\nu^\mu(z)$ of the first kind is defined by

$$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)} \left(\frac{z+1}{z-1}\right)^{\mu/2} F\left(-\nu, \nu+1, 1-\mu; \frac{1-z}{2}\right)$$

for $z - 1 \in \mathbb{C} \setminus (-\infty, 0]$. We write $z = \cosh\zeta$, $\zeta = \xi + i\eta$ $(\xi, \eta \in \mathbb{R})$ for $z - 1 \in \mathbb{C} \setminus (-\infty, 0]$, and define the values $\omega_i = \omega_i(z)$ $(i = 1, 2)$ by

$$\omega_1 = -\arctan\left(\frac{\eta - \pi}{\xi}\right), \qquad \omega_2 = \arctan\left(\frac{\eta}{\xi}\right)$$

if $\eta \geq 0$, and by

$$\omega_1 = -\arctan\left(\frac{\eta}{\xi}\right), \qquad \omega_2 = -\arctan\left(\frac{\eta + \pi}{\xi}\right)$$

if $\eta \le 0$. In each case arctan denotes an acute angle, positive or negative. Define

$$\tau = \log\left(\frac{t-z}{t^2-1}\right) + \log(2e^{\zeta}).$$

We define the numbers c_n and d_n by using the expansion

$$(1-t)^{\mu}(1+t)^{-\mu}(z-t)^{-1}\frac{dt}{d\tau} = \pm C\sum_{n=0}^{\infty} c_n\tau^{n-1/2} + \sum_{n=0}^{\infty} d_n\tau^n,$$

where $C = 2^{-1}(1-e^{\zeta})^{\mu+1/2}(1+e^{\zeta})^{1/2-\mu}(z-e^{\zeta})^{-1}$ and multiple-valued functions are specified by the conventions

$$|\arg(1-e^{\zeta})| < \pi, \qquad |\arg(1+e^{\zeta})| < \pi.$$

In particular,

$$c_0 = 1, \qquad c_1 = \frac{8\mu^2 - 3 + 3e^{2\zeta}}{4(1-e^{2\zeta})}.$$

We define the numbers c'_n from c_n by changing the sign of ζ. In particular,

$$c'_0 = 1, \qquad c'_1 = \frac{8\mu^2 - 3 + 3e^{-2\zeta}}{4(1-e^{-2\zeta})}.$$

Proposition 2 (Watson). *Let z be a complex number such that $z-1 \in \mathbb{C}\backslash(-\infty, 0]$. In the range of* $\arg\nu$ *depending on z and given by*

$$-\frac{\pi}{2} - \omega_2 + \delta \le \arg\nu \le \frac{\pi}{2} + \omega_1 + \delta,$$

the associated Legendre function $P^{\mu}_{\nu}(z)$ has the asymptotic expansion

$$\text{(A.1)}\quad P^{\mu}_{\nu}(z) = \frac{\Gamma(\nu+1)}{\Gamma(\nu-\mu+1)}\frac{e^{-\zeta/2}}{(\nu\pi)^{1/2}(1-e^{-2\zeta})^{1/2}}$$
$$\times\left[e^{(\nu+1/2)\zeta}\sum_{n=0}^{N-1}\frac{\Gamma(n+1/2)}{\Gamma(1/2)}c_n\nu^{-n}\right.$$
$$\left.+ e^{\mp\pi i(\mu-1/2)}e^{-(\nu+1/2)\zeta}\sum_{n=0}^{N-1}\frac{\Gamma(n+1/2)}{\Gamma(1/2)}c'_n\nu^{-n} + O(|\nu|^{-N})\right]$$

as $|\nu| \to +\infty$, where the implied constant depends on z and μ.

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