On zeros of approximate functions of the Rankin–Selberg \(L\)-functions

by

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Notations. As usual, \(\mathbb{Z}\) is the ring of rational integers, \(\mathbb{Z}_{>0}\) the set of positive integers, \(\mathbb{C}\) the field of complex numbers. We denote by \(\mathbb{H}\) the upper half-plane, and by \(\Gamma\) the full modular group \(\text{PSL}_2(\mathbb{Z})\). For a complex variable \(s\), we put \(e(s) = e^{2\pi is}, \Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)\) and \(\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)\). We denote by \(\zeta(s)\) and \(\zeta^*(s) = \Gamma_{\mathbb{R}}(s)\zeta(s)\) the Riemann zeta-function and the completed Riemann zeta-function, respectively, and denote by \(\sigma_{\nu}(n) = \sum_{d|n} d^\nu\) the divisor function. Throughout the paper, \(z = x + iy\) \((x \in \mathbb{R}, y > 0)\) is a variable on \(\mathbb{H}\), and \(s = \sigma + it\) \((\sigma, t \in \mathbb{R})\) is a complex variable. A sum over the empty set is meant to be zero.

1. Introduction. Let \(C(s)\) be the trigonometric function
\[ C(s) := 2\cos(i(s - 1/2)) = e^{s-1/2} + e^{-(s-1/2)}. \]
It satisfies the (trivial) functional equation \(C(s) = C(1 - s)\). A well-known but remarkable fact about \(C(s)\) is that it satisfies the Riemann hypothesis: all zeros of \(C(s)\) lie on the central line \(\sigma = 1/2\) of its functional equation. We indicate how to prove the Riemann hypothesis for \(C(s)\). First, we note the (trivial) decomposition
\[ C(s) = \varphi(s) + \varphi(1 - s), \quad \varphi(s) = e^{s-1/2}. \]
Then we have
\[ C(s) = \varphi(s) \left( 1 + \frac{\varphi(1 - s)}{\varphi(s)} \right), \]
and find that
\[ (A) \ \varphi(s) \neq 0 \text{ for } \sigma > 1/2, \]
\[ (B) \ \left| \frac{\varphi(1 - s)}{\varphi(s)} \right| < 1 \text{ for } \sigma > 1/2. \]


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Property (B) implies that
\[ C(1 + \frac{\varphi(1 - s)}{\varphi(s)}) \neq 0 \text{ for } \sigma > 1/2. \]

Therefore, \( C(s) \neq 0 \) for \( \sigma > 1/2 \) by (A) and (C). The functional equation gives \( C(s) \neq 0 \) if \( \sigma \neq 1/2 \). Hence we obtain the Riemann hypothesis for the function \( C(s) \). Note that \( C(s) \) has at least one zero.

Now let \( L(s) \) be an entire function satisfying the functional equation
\[ L(s) = L(1 - s). \]
The above argument implies that if \( L(s) \) has the decomposition
\[ L(s) = \varphi(s) + \varphi(1 - s) \]
such that \( \varphi(s) \) satisfies (A) and (B), then the Riemann hypothesis holds for \( L(s) \).

The study of zeros of entire functions along this line has a long history. The decomposition (1.1) with the function \( \varphi(s) \) satisfying (A) and (B) is possible in several interesting cases.

Consider the case of the Riemann zeta function. Let
\[ \phi(x) = 4 \sum_{n=1}^{\infty} (2\pi^2 n^4 x^{9/2} - 3\pi n^2 x^{5/2}) e^{-\pi n^2 x^2}. \]

Then we have
\[ \xi(s) = s(s-1)\zeta^*(s) = \int_1^\infty \phi(x)(x^{s-1/2} + x^{-s+1/2}) \frac{dx}{x}. \]
Replacing \( \phi(x) \) by
\[ \phi^*(x) = \pi^2 (x^{9/2} + x^{-9/2}) e^{-\pi(x^2 + x^{-2})}, \]
which is asymptotically equivalent to \( \phi(x) \), we obtain
\[ \xi^*(s) = \int_1^\infty \phi^*(x)(x^{s-1/2} + x^{-s+1/2}) \frac{dx}{x}. \]
The function \( \xi^*(s) \) is similar to \( \xi(s) \) in a suitable sense, and has the decomposition (1.1) such that the corresponding \( \varphi(s) \) satisfies (A) and (B) as well as \( C(s) \) [27, pp. 254–291]. For the decomposition of \( \xi(s) \) as in (1.1) see Gonek [4] and Egorov [3].

Other interesting cases are the difference of two zeta functions, the constant term of the nonholomorphic Eisenstein series, Weng’s zeta functions and a finite truncation of the Chowla–Selberg formula of Epstein zeta-functions etc. They were studied by several authors, e.g., Pólya [17], Taylor [26], Stark [20], Hejhal [6], Ki [9], Lagarias–Suzuki [10], Weng [31–33], Suzuki [22–24], Hayashi [5], Bauer [1], Müller [13], Velásquez [28] and Suzuki-Weng [25].
Can we find new examples of zeta- and \( L \)-functions \( L(s) \) having (1.1) and satisfying (A) and (B)? In this paper, we show that the Rankin–Selberg \( L \)-function is one of such examples. More precisely, we derive a new formula (Theorem 1) for the Rankin–Selberg \( L \)-function attached to a pair of cusp forms on the full modular group by using the holomorphic projection of Sturm [21]. Then the well-known relation between the Rankin–Selberg \( L \)-function and the symmetric square \( L \)-function gives a new formula for the symmetric square \( L \)-function (Corollary 1). Using Theorem 1, we define a function which approximates the Rankin–Selberg \( L \)-function. We show that such an approximate function has a wide zero-free region (Theorem 2), and this uses the fact that it has the decomposition (1.1) with two properties similar to (A) and (B).

As a special case of Corollary 1, we obtain Noda’s identity in [14] which relates the Fourier coefficients of the holomorphic cusp form \( f \) and the zeros of the Riemann zeta-function or the zeros of the symmetric square \( L \)-function of \( f \). In addition, Theorem 1 gives an analytic series expansion of the central value \( L(1/2, f \times g) \). Note that Mizumoto [12] showed that for every normalized Hecke eigen cusp form \( f \in S_{k_1} \) and every even integer \( k_2 \) satisfying \( k_2 \geq k_1 \) and \( k_2 \neq 14 \), there exists a normalized Hecke eigen cusp form \( g \in S_{k_2} \) such that \( L(1/2, f \times g) \neq 0 \).

There are nice results of Hoffstein–Lockhart [7], Hoffstein–Ramakrishnan [8] and Ramakrishnan–Wang [18] about the real zeros of the Rankin–Selberg \( L \)-function. They established the nonexistence of the Siegel zero of the Rankin–Selberg \( L \)-function attached to a pair of cusp forms on GL(2) and the symmetric square \( L \)-function of a cusp form on GL(2). Their results contain fairly good zero-free regions of the Rankin–Selberg \( L \)-function compared with the classical one. We expect that Theorem 1 and improving our proof of Theorem 2, should imply nice results on the distribution of complex zeros of the Rankin–Selberg \( L \)-function.

This paper is organized as follows. In Section 2, we state main results, Theorems 1 and 2. In Section 3, we apply the results of Section 2 to \( S_{12} \) and \( S_{24} \). In Section 4, we review the theory of the Poincaré series, Eisenstein series, \( C^\infty \)-modular forms and the Rankin–Selberg \( L \)-function as preliminaries for the proof of Theorems 1 and 2. In Section 5, we give a proof of Theorem 1. In Section 6, we prove Theorem 2. In Section 7, we interpret the argument in Section 5 from the viewpoint of the holomorphic projection of Sturm. In the Appendix, we give an asymptotic expansion of the associated Legendre function of the first kind according to Watson [29].

2. Statements of results. Let \( k \) be an even integer \( \geq 12 \) and \( \neq 14 \). Let \( S_k \) be the vector space of all holomorphic cusp forms of weight \( k \) on \( \Gamma \). We denote by \( d = d_k \) the dimension of \( S_k \). For two cusp forms \( f(z) = \)
\[ \sum_{n=1}^{\infty} a_f(n)n^{(k-1)/2}e(nz) \text{ and } g(z) = \sum_{n=1}^{\infty} a_g(n)n^{(k-1)/2}e(nz), \] the Rankin–Selberg \( L \)-function \( L(s, f \otimes g) \) is defined by

\[ L(s, f \otimes g) = \sum_{n=1}^{\infty} a_f(n)\overline{a_g(n)}n^{-s}, \tag{2.1} \]

where \( \bar{\text{means}} \) complex conjugation. The series on the right-hand side converges absolutely if the real part of \( s \) is sufficiently large. In addition, we define

\[ L(s, f \times g) = \zeta(2s)L(s, f \otimes g) \]

and the completed function

\[ L^*(s, f \times g) = 2^{-k-1}\Gamma(s + k - 1)\Gamma(s)\Gamma(s + k - 1)L(s, f \times g) \]

\[ = \pi^{-s}(4\pi)^{-s-k-1}\Gamma(s)\Gamma(s + k - 1)L(s, f \times g). \]

Let \( \mathcal{F} = \{f_1, \ldots, f_d\} \) be an orthonormal basis of \( S_k \) and let \( f_j(z) = \sum_{n=1}^{\infty} a_j(n)n^{(k-1)/2}e(nz) \) be the Fourier expansion of \( f_j \) \((1 \leq j \leq d)\) at the cusp \( i\infty \). Let \( \mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{Z}_{>0}^d \) with \( 0 < m_1 < \cdots < m_d \). Define

\[ A_{\mathcal{F}, \mathbf{m}} = \begin{pmatrix} a_1(m_1) & \cdots & a_d(m_1) \\ \vdots & \ddots & \vdots \\ a_1(m_d) & \cdots & a_d(m_d) \end{pmatrix}. \tag{2.2} \]

In general, the matrix \( A_{\mathcal{F}, \mathbf{m}} \) is not invertible. However, if the set of Poincaré series \( \{P_{m_1}, \ldots, P_{m_d}\} \subset S_k \) is a basis of \( S_k \), then \( A_{\mathcal{F}, \mathbf{m}} \) is invertible. In particular, for the vector \( \mathbf{m}_0 = (1, \ldots, d) \), the matrix \( A_{\mathcal{F}, \mathbf{m}_0} \) is invertible by the classical result of Petersson [15, 16] about the basis problem for elliptic modular forms. Thus we can always choose a vector \( \mathbf{m} \) such that \( A_{\mathcal{F}, \mathbf{m}} \) is invertible.

**Theorem 1.** Let \( k \) be an even integer \( \geq 12 \) and \( \neq 14 \). Let \( \mathcal{F} = \{f_1, \ldots, f_d\} \) be an orthonormal basis of \( S_k \) and let \( f_j(z) = \sum_{n=1}^{\infty} a_j(n)n^{(k-1)/2}e(nz) \) be the Fourier expansion of \( f_j \) \((1 \leq j \leq d)\) at the cusp \( i\infty \). Choose \( \mathbf{m} \in \mathbb{Z}_{>0}^d \) such that the matrix \( A_{\mathcal{F}, \mathbf{m}} \) defined by (2.2) is invertible \((\det A_{\mathcal{F}, \mathbf{m}} \neq 0)\). Define the set of numbers \((\alpha_{ij})_{1 \leq i,j \leq d}\) by

\[ A_{\mathcal{F}, \mathbf{m}}^{-1} = (\alpha_{ij})_{1 \leq i,j \leq d}. \tag{2.3} \]

Then

\[ (4\pi)^{-k+1}\Gamma(k-1)L^*(s, f_i \times \overline{f_j}) \]

\[ = (4\pi)^{-s-k+1}\Gamma(s + k - 1)\zeta^*(2s)D_{\mathbf{m},ij}(s) \]

\[ + (4\pi)^{s-k}\Gamma(k-s)\zeta^*(2s-1)D_{\mathbf{m},ij}(1-s) \]

\[ + (4\pi)^{-k+1}\Gamma(s + k - 1)\Gamma(k-s)\{W^+_{\mathbf{m},ij}(s) + W^-_{\mathbf{m},ij}(s)\} \tag{2.4} \]
for all \(1 \leq i \leq j \leq d\) in the vertical strip
\[
|\sigma - 1/2| < k/2 - 1
\]
except for the point \(s = 1/2\). Here
\[
D_{m,ij}(s) = \sum_{h=1}^{d} \alpha_{jh} a_i(m_h) m_h^{-s},
\]
\[
W^+_{m,ij}(s) = \sum_{h=1}^{d} \sum_{n=1}^{\infty} \alpha_{jh} a_i(m_h + n) \frac{\tau_{s-1/2}(n)}{\sqrt{n}} P^{1-k}_{s-1} \left( \frac{2m_h + n}{n} \right),
\]
\[
W^-_{m,ij}(s) = \sum_{h=1}^{d} \sum_{n=1}^{m_h-1} \alpha_{jh} a_i(m_h - n) \frac{\tau_{s-1/2}(n)}{\sqrt{n}} P^{1-k}_{s-1} \left( \frac{2m_h - n}{n} \right),
\]
with \(\tau_\nu(n) = n^\nu \sigma - 2\nu(n)\), and \(P^\mu_\nu(z)\) is the associated Legendre function of the first kind (see Appendix). Further, at the point \(s = 1/2\),
\[
(4\pi)^{-k+1} \Gamma(k - 1) L^*(1/2, f_i \times \bar{f}_j)
\]
\[
= (4\pi)^{-k+1/2} \left\{ \frac{\Gamma'}{\Gamma} \left( k - \frac{1}{2} \right) + \log \frac{e^\gamma}{16\pi^2 m_h} \right\}
\]
\[
+ (4\pi)^{-k+1} \Gamma \left( k - \frac{1}{2} \right)^2 \left\{ W^+_{m,ij}(1/2) + W^-_{m,ij}(1/2) \right\}.
\]
The series \(W^+_{m,ij}(s)\) converges absolutely and uniformly on every compact subset \(K\) in (2.5), and has the asymptotic expansion
\[
W^+_{m,ij}(s) = \sum_{h=1}^{d} \sum_{n=1}^{N-1} \alpha_{jh} a_i(m_h + n) \frac{\tau_{s-1/2}(n)}{\sqrt{n}} P^{1-k}_{s-1} \left( \frac{2m_h + n}{n} \right)
\]
\[
+ O(N^{1|\sigma-1/2|+k/2+1+\varepsilon}),
\]
where the implied constant depends on \(\mathcal{F}, m\) and \(K\).

**Remark 1.** By definition of \(\alpha_{ij}\), we have
\[
D_{m,ij}(0) = \sum_{h=1}^{d} \alpha_{jh} a_i(m_h) = \delta_{ij}.
\]
Hence the poles of the first two terms of (2.4) at \(s = 0, 1\) cancel out whenever \(i \neq j\). This agrees with the fact that the residue of \(L(s, f_i \times \bar{f}_j)\) at \(s = 1\) is a multiple of the Petersson inner product \((f_i, f_j)\).
Let \( f(z) = 1 + \sum_{n=2}^{\infty} a_f(n)n^{(k-1)/2}e(nz) \) be a normalized Hecke cusp form. The symmetric square \( L \)-function \( L(s, \text{sym}^2 f) \) is defined by the Euler product

\[
L(s, \text{sym}^2 f) = \prod_p (1 - \alpha_p^2 p^{-s})^{-1}(1 - \alpha_p \beta_p p^{-s})^{-1}(1 - \beta_p^2 p^{-s})^{-1},
\]

where \( \alpha_p \) and \( \beta_p \) are determined by \( \alpha_p + \beta_p = a_f(p) \) and \( \alpha_p \beta_p = 1 \). The right-hand side converges absolutely if the real part of \( s \) is sufficiently large.

The completed \( L \)-function \( L^*(s, \text{sym}^2 f) \) is defined by

\[
L^*(s, \text{sym}^2 f) = \pi^{-3s/2} \Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s+k-1}{2}\right)\Gamma\left(\frac{s+k}{2}\right)L(s, \text{sym}^2 f)
\]

\[
= \pi^k \Gamma_C(s+k-1)\Gamma_C(s)\Gamma_R(s)^{-1}L(s, \text{sym}^2 f).
\]

It is known that \( L(s, \text{sym}^2 f) \) and \( L(s, f \times \bar{f}) \) are related via

\[
\zeta(s)L(s, \text{sym}^2 f) = L(s, f \times \bar{f}).
\]

Therefore we have the equality

\[
2^{-1}(2\pi)^{-k} \zeta^*(s)L^*(s, \text{sym}^2 f) = L^*(s, f \times \bar{f}).
\]

**Corollary 1.** Let \( \mathcal{F} = \{f_1, \ldots, f_d\} \) be the orthogonal basis of \( S_k \) consisting of normalized Hecke cusp forms. Put \( f_j^* = f_j/(f_j; f_j)^{1/2} \) and \( \mathcal{F}^* = \{f_1^*, \ldots, f_d^*\} \). Choose \( m \in \mathbb{Z}_{>0}^d \) such that \( A_{\mathcal{F}^*, m} \) is invertible. Then

\[
(2.9) \quad 2^{-k}(2\pi)^{-2k+1} \frac{\Gamma(k-1)}{(f_j; f_j)} \zeta^*(s)L^*(s, \text{sym}^2 f_j)
\]

\[
= (4\pi)^{-s-k+1} \Gamma(s+k-1)\zeta^*(2s)D_{m,jj}(s)
\]

\[
+ (4\pi)^{s-k} \Gamma(k-s)\zeta^*(2s-1)D_{m,jj}(1-s)
\]

\[
+ (4\pi)^{-k+1} \Gamma(s+k-1)\Gamma(k-s)\{W_{m,jj}^+(s) + W_{m,jj}^-(s)\}
\]

for all \( 1 \leq j \leq d \) and all \( s \neq 1/2 \) in the vertical strip (2.5), where \( D_{m,jj}(s) \), \( W_{m,jj}^+(s) \) and \( W_{m,jj}^-(s) \) are defined by (2.6)–(2.8) for the basis \( \mathcal{F}^* \) and the vector \( m \).

**Remark 2.** In the case \( S_k = \mathbb{C} \Delta_k \) (\( k = 12, 16, 18, 20, 22 \) and 26), \( D_{(m),11}(s) \) is just \( m^{-s} \). Hence, by taking \( s \) to be a zero of \( \zeta(s) \) or a zero of \( L(s, \text{sym}^2 \Delta_k) \), we obtain a new proof of the result of Noda [14, Theorem]. His result is an equality which relates the zeros of the Riemann zeta function or the zeros of the symmetric square \( L \)-functions with the Fourier coefficients of the holomorphic cusp form \( \Delta_k \).

**Corollary 2.** Under the notation of Theorem 1, we have the following formula for the central value:
\[
L(1/2, f_i \times \vec{f}_j) = \frac{(4\pi)^{k-1}}{\Gamma(k-1)} \sum_{h=1}^{d} \alpha_{jh}a_i(m_h) \left\{ \frac{\Gamma'(k - \frac{1}{2})}{\Gamma} + \log \frac{e^y}{16\pi^2m_h} \right\} \\
+ 4\pi^k \frac{\Gamma(2k-2)}{\Gamma(k-1)^2} \{W^+_{m,ij}(1/2) + W^-_{m,ij}(1/2)\}. 
\]

On the right-hand side we have

\[
W^+_{m,ij}(1/2) = \sum_{h=1}^{d} \sum_{n=1}^{N-1} \alpha_{jh}a_i(m_h + n) \frac{\sigma_0(n)}{\sqrt{n}} P^{1-k}_{s-1/2} \left( \frac{2m_h + n}{n} \right) \\
+ O(N^{-k/2+1+\varepsilon}) 
\]

for every positive integer \(N\) and every positive real number \(\varepsilon\).

Considering equations (2.4) and (2.7), we define

\[
W^{+,N}_{m,ij}(s) = \sum_{h=1}^{d} \sum_{n=1}^{N} \alpha_{jh}a_i(m_h + n) \frac{\tau_{s-1/2}(n)}{\sqrt{n}} P^{1-k}_{s-1} \left( \frac{2m_h + n}{n} \right) 
\]

and

\[
L^N_{m,ij}(s) = (4\pi)^{-s-k+1} \Gamma(s + k - 1)\zeta^*(2s)D_{m,ij}(s) \\
+ (4\pi)^{s-k} \Gamma(k - s)\zeta^*(2s - 1)D_{m,ij}(1 - s) \\
+ (4\pi)^{-k+1} \Gamma(s + k - 1)\Gamma(k - s) \{W^{+,N}_{m,ij}(s) + W^-_{m,ij}(s)\} 
\]

for a positive integer \(N\). In addition, we define

\[
L^0_{m,ij}(s) = (4\pi)^{-s-k+1} \Gamma(s + k - 1)\zeta^*(2s)D_{m,ij}(s) \\
+ (4\pi)^{s-k} \Gamma(k - s)\zeta^*(2s - 1)D_{m,ij}(1 - s) 
\]

for \(N = 0\). The only difference between \(L^N_{m,ij}(s)\) and the right-hand side of (2.4) is in the bracketed expression \(\cdots\). The functional equations \(\tau_{s-1/2}(n) = \tau_{1/2-s}(n)\) and \(P^{1-k}_{s-1}(z) = P^{1-k}_{s-1}(z)\) imply that \(L^N_{m,ij}(s)\) satisfies the functional equation

\[
(2.12) \quad L^N_{m,ij}(s) = L^N_{m,ij}(1 - s). 
\]

**Theorem 2.** Let \(k\) be an even integer \(\geq 12\) and \(\neq 14\). Let \(F = \{f_1, \ldots, f_d\}\) be an orthonormal basis of \(S_k\). Choose \(m \in \mathbb{Z}_{>0}^d\) such that \(A_{F,m}\) is invertible. Further suppose that there exists \(\delta = \delta_{F,m}\) such that \(0 < \delta < 1/2\), and \(D_{m,ij}(s)\) has only finitely many zeros in the right half-plane \(\sigma \geq 1/2 - \delta\). Then for every nonnegative integer \(N\) and every positive real number \(a\) there exists \(C = C_{m,N,a} > 0\) such that \(L^N_{m,ij}(s)\) has no zeros in the region

\[
\frac{\log\{C \log^{1/2}(|t| + 1)\}}{\log(|t| + 1)} < \left| \sigma - \frac{1}{2} \right| < a, 
\]
that is, all zeros of \( L_{m,ij}^N(s) \) in the strip \(|\sigma - 1/2| < a\) are contained in
\[
\left|\sigma - \frac{1}{2}\right| \leq \log\left\{ C\log^{1/2}(|t| + 1) \right\} \log(|t| + 1).
\]

In particular,
\[
N(T, \sigma_1, \sigma_2) = O_{\sigma_1, \sigma_2}(1)
\]
for all \( 0 < \sigma_1 < \sigma_2 \), where \( N(T, \sigma_1, \sigma_2) \) is the number of zeros of \( L_{m,ij}^N(s) \) satisfying \( \sigma_1 \leq \sigma - 1/2 \leq \sigma_2 \) and \(|t| \leq T\) counted with multiplicity.

**Remark 3.** In the case of the Riemann zeta-function, Selberg established the estimate
\[
N(T, 1/2 + 4\delta) \ll T^{1-\delta} \log T
\]
uniformly for \( \delta \geq 0 \) by using his mollification method. Here \( N(T, a) \) is the number of zeros of \( \zeta(s) \) satisfying \( \sigma \geq a \) and \(|t| \leq T\) counted with multiplicity. Hence almost all zeros of \( \zeta(s) \) lie in the region
\[
\left|\sigma - \frac{1}{2}\right| \leq \frac{\eta(t)}{\log(|t| + 3)},
\]
where \( \eta(t) \) is any positive function which increases to infinity. Theorem 2 is an analogue of this result.

**Remark 4.** As in Remark 2, \( D_{(m),11}(s) = m^{-s} \) if \( \dim S_k = 1 \). Hence the assumption in Theorem 2 about the location of zeros of \( D_{m,ij}(s) \) is always satisfied if \( \dim S_k = 1 \). However, in general the location of zeros of \( D_{m,ij}(s) \) strongly depends on the choice of the vector \( m \) (see Section 3).

**Remark 5.** The existence of the vector \( m \) such that \( L_{m,ij}^N(s) \) has no zeros in \( 0 < |\sigma - 1/2| < 1/2 \) for all sufficiently large \( N \) implies that the Riemann hypothesis for the Rankin–Selberg \( L \)-function \( L(s, f_i \times \bar{f}_j) \) is true. Therefore such a result is desired for a pair of Hecke eigen cusp forms \( f_i \) and \( f_j \). However, our proof of Theorem 2 in Section 6 does not need the condition that \( f_i \) and \( f_j \) are Hecke eigen cusp forms. Hence, a new idea using more precise arithmetic properties of the Fourier coefficients of \( f_i \) and \( f_j \) is needed in order to obtain results in the direction of the Riemann hypothesis.

### 3. Examples

In this section, we calculate the central values of \( L \)-functions by applying Corollary 2 to \( S_{12} \) and \( S_{24} \). We calculate the value of the Petersson inner product according to Rankin [19].

#### 3.1. The case \( k = 12 \)

In this case \( \dim S_{12} = 1 \). As mentioned in Remark 2, we have \( D_{(m),11}(s) = m^{-s} \) by definition (2.6). All members of \( S_{12} \) are constant multiples of the normalized Hecke eigen cusp form
Fig. 1. $|L_0(1/2 + it, \Delta \times \Delta)|$ for $0 \leq t \leq 30$. Points $\bullet$ are zeros of $L(s, \Delta \times \Delta)$ on $\sigma = 1/2$.

$$\Delta(z) = e(z) \prod_{n=1}^{\infty} (1 - e(nz))^{24} = \sum_{n=1}^{\infty} \tau(n)e(nz).$$

Put $f = \Delta/(\Delta, \Delta)^{1/2}$, and choose $m = (m) = (1)$. Then we have $W_{(1),11}(s) \equiv 0$, and

$$\frac{L(1/2, \Delta \times \Delta)}{\sqrt{(\Delta, \Delta)}} = \frac{(4\pi)^{11}}{\Gamma(11)} \left\{ \frac{\Gamma'(23/2)}{\Gamma(23/2)} + \log \frac{e^{\gamma}}{16\pi^2} \right\}$$

$$+ \frac{4\pi^{12}\Gamma(22)}{\Gamma(11)^2\Gamma(12)} \sum_{n=1}^{\infty} \frac{\tau(n+1)}{(n+1)^{11/2}} \frac{\sigma_0(n)}{\sqrt{n}} F\left(\frac{1}{2}, \frac{1}{2}; 12; -\frac{1}{n}\right).$$

Using the value $(\Delta, \Delta) = 1.03536 \ldots \times 10^{-6}$, we have

$$L(1/2, \Delta \times \Delta) = -7.25563 \ldots \times 10^2.$$

Figure 1 is the graph of the absolute value of

$$L_0(s, \Delta \times \Delta) = \frac{\omega_{12}\sqrt{(\Delta, \Delta)}}{\pi^{-s}(4\pi)^{-s-11}\Gamma(s)\Gamma(s+11)} L_{(1),11}^0(s)$$

on the critical line $\sigma = 1/2$, where $\omega_{12} = (4\pi)^{11}/\Gamma(11)$ and

$$L_{(1),11}^0(s) = (4\pi)^{-s-11}\Gamma(s+11)\zeta^*(2s) + (4\pi)^{s-12}\Gamma(12-s)\zeta^*(2s-1).$$

Figure 2 is the graph of the absolute value of

$$L_N(s, \Delta \times \Delta) = \frac{\omega_{12}\sqrt{(\Delta, \Delta)}}{\pi^{-s}(4\pi)^{-s-11}\Gamma(s)\Gamma(s+11)} L_{(1),11}^N(s)$$

for $N = 10$ on the critical line $\sigma = 1/2$, where

$$L_{(1),11}^N(s) = (4\pi)^{-s-11}\Gamma(s+11)\zeta^*(2s) + (4\pi)^{s-12}\Gamma(12-s)\zeta^*(2s-1)$$

$$+ (4\pi)^{-11}\Gamma(s+11)\Gamma(12-s) \sum_{n=1}^{N} \frac{\tau(n+1)}{(n+1)^{11/2}} \frac{n^{s-1/2}\sigma_1-2s(n)}{\sqrt{n}} P_{s-1}^{-11} \left(1 + \frac{2}{n}\right).$$
Fig. 2. $|L_{10}(1/2 + it, \Delta \times \Delta)|$ for $0 \leq t \leq 30$. Points • are zeros of $L(s, \Delta \times \Delta)$ on $\sigma = 1/2$.

Fig. 3. The thin line is $|L_{0}(1/2 + it, \Delta \times \Delta)|$ for $0 \leq t \leq 30$, the line of medium thickness is $|L_{10}(1/2 + it, \Delta \times \Delta)|$ for $0 \leq t \leq 30$, and the thick line is $|L_{100}(1/2 + it, \Delta \times \Delta)|$ for $0 \leq t \leq 30$.

In Figures 1 and 2, dot points • are zeros of $L(s, \Delta \times \Delta) = \zeta(s)L(s, \text{sym}^2 \Delta)$ on the critical line ([34, Table 3]). Interestingly, we observe that the lower zeros of $L(s, \Delta \times \Delta)$ on the critical line are approximated by zeros of the sum of the Riemann zeta-function $L_0(s, \Delta \times \Delta)$. Needless to say, this is not true for zeros of $L(s, \Delta \times \Delta)$ whose imaginary part becomes large. Figure 3 is the comparison of the absolute values $|L_{0}(s, \Delta \times \Delta)|$, $|L_{10}(s, \Delta \times \Delta)|$ and $|L_{100}(s, \Delta \times \Delta)|$ on the critical line. It shows that to know the value of $L(s, \Delta \times \Delta)$ for large $|t|$, we need many terms in $W_{m,ij}^{\pm}(s)$ as large as $|t|$. 
3.2. The case \(k = 24\). This is the first case in which \(d > 1\). We have \(\dim S_{24} = 2\). Two functions \(f\) and \(g\) given by

\[
f(z) = E_{12}(z)\Delta(z) + 12 \left( \frac{27017}{691} + \sqrt{144169} \right) \Delta^2(z)
\]

\[
= \sum_{n=1}^{\infty} A_f(n)e(nz),
\]

\[
g(z) = E_{12}(z)\Delta(z) + 12 \left( \frac{27017}{691} - \sqrt{144169} \right) \Delta^2(z)
\]

\[
= \sum_{n=1}^{\infty} A_g(n)e(nz)
\]

are distinct normalized Hecke eigen cusp forms of \(S_{24}\), where

\[
E_{12}(z) = 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n)e(nz).
\]

Put \(F = \{f/(f,f)^{1/2}, g/(g,g)^{1/2}\}\). Then \(F\) is an orthonormal basis of \(S_{24}\).

Applying Corollary 2 to \(m = (1, 2)\), we obtain

\[
L(1/2, f \times f)/(f,f) = \frac{1}{D} \frac{(4\pi)^{23}}{\Gamma(23)} \left\{ \frac{\Gamma'(47/2)}{\Gamma(47/2)} \left( A_g(2) - \frac{A_f(2)}{\sqrt{2}} \right) \right. + \left( A_g(2) \log \frac{e^\gamma}{16\pi^2m} - \frac{A_f(2)}{\sqrt{2}} \log \frac{e^\gamma}{32\pi^2m} \right) \}
\]

\[
+ \frac{1}{D} \frac{4\pi^{24} \Gamma(46)}{\Gamma(23)^2 \Gamma(24)} \left\{ \sum_{n=1}^{\infty} A_g(2) \frac{A_f(n+1)}{(n+1)^{23}} \frac{\sigma_0(n)}{\sqrt{n}} F \left( \frac{1}{2}, \frac{1}{2}, 24; -\frac{1}{n} \right) \right. \]

\[
- 2^{23} \sum_{n=1}^{\infty} A_f(n+2) \frac{\sigma_0(n)}{\sqrt{n}} F \left( \frac{1}{2}, \frac{1}{2}, 24; -\frac{2}{n} \right) - 2^{23} F \left( \frac{1}{2}, \frac{1}{2}, 24; -1 \right) \}
\]

\[
L(1/2, f \times g)/(f,f)(g,g) = \frac{A_f(2)}{D} \frac{\sqrt{(g,g)}}{\sqrt{(f,f)}} \frac{(4\pi)^{23}}{\Gamma(23)} \left\{ \frac{\Gamma'(47/2)}{\Gamma(47/2)} \left( \frac{1 - \sqrt{2}}{\sqrt{2}} \right) \right. \]

\[
+ \left( \frac{1}{\sqrt{2}} \log \frac{e^\gamma}{32\pi^2m} - \log \frac{e^\gamma}{16\pi^2m} \right) \}
\]

\[
- \frac{1}{D} \frac{4\pi^{24} \Gamma(46)}{\Gamma(23)^2 \Gamma(24)} \frac{\sqrt{(g,g)}}{\sqrt{(f,f)}} \left\{ \sum_{n=1}^{\infty} A_f(2) \frac{A_f(n+1)}{(n+1)^{23}} \frac{\sigma_0(n)}{\sqrt{n}} F \left( \frac{1}{2}, \frac{1}{2}, 24; -\frac{1}{n} \right) \right. \]

\[
- 2^{23} \sum_{n=1}^{\infty} A_f(n+2) \frac{\sigma_0(n)}{\sqrt{n}} F \left( \frac{1}{2}, \frac{1}{2}, 24; -\frac{2}{n} \right) - 2^{23} F \left( \frac{1}{2}, \frac{1}{2}, 24; -1 \right) \}
\]
and
\[
\frac{L(1/2, g \times g)}{(g, g)} = \frac{1}{D} \frac{(4\pi)^{23}}{\Gamma(23)} \left\{ \frac{\Gamma'}{\Gamma} \left( \frac{47}{2} \right) \left( \frac{A_g(2)}{\sqrt{2}} - A_f(2) \right) + \left( A_g(2) \log \frac{e^{\gamma}}{32\pi^2 m} - A_f(2) \log \frac{e^{\gamma}}{16\pi^2 m} \right) \right\} \\
- \frac{1}{D} \frac{4\pi^{24}\Gamma(46)}{\Gamma(23)^2\Gamma(24)} \left\{ \sum_{n=1}^{\infty} A_f(2) \frac{A_g(n+1)}{(n+1)^{23}} \frac{\sigma_0(n)}{\sqrt{n}} F\left( \frac{1}{2}, \frac{1}{2}, 24; -\frac{1}{n} \right) \right\} \\
- 2^{23} \sum_{n=1}^{\infty} \frac{A_g(n+2)}{(n+2)^{23}} \frac{\sigma_0(n)}{\sqrt{n}} F\left( \frac{1}{2}, \frac{1}{2}, 24; -\frac{2}{n} \right) - 2^{23} F\left( \frac{1}{2}, \frac{1}{2}, 24; -1 \right),
\]
where \( D = A_g(2) - A_f(2) \). As \((f, f) = 1.28993 \times 10^{-4}\) and \((g, g) = 1.07837 \times 10^{-4}\), we obtain the central values
\[
L(1/2, f \times f) = -3.07917 \ldots ,
\]
\[
L(1/2, f \times g) = +9.79843 \ldots \times 10^{-3},
\]
\[
L(1/2, g \times g) = -2.55952 \ldots .
\]
Further, if \( \det A_{\mathcal{F}, m} \neq 0 \), we have
\[
D_{m,11}(s) = \frac{1}{D_m} \left\{ \frac{A_f(m_1)A_g(m_2)}{m^s_1} - \frac{A_f(m_2)A_g(m_1)}{m^s_2} \right\},
\]
\[
D_{m,12}(s) = \frac{A_f(m_1)A_f(m_2)}{D_m} \frac{\sqrt{(g, g)}}{\sqrt{(f, f)}} \left\{ \frac{1}{m^s_2} - \frac{1}{m^s_1} \right\},
\]
\[
D_{m,21}(s) = \frac{A_g(m_1)A_g(m_2)}{D_m} \frac{\sqrt{(f, f)}}{\sqrt{(g, g)}} \left\{ \frac{1}{m^s_1} - \frac{1}{m^s_2} \right\},
\]
\[
D_{m,22}(s) = \frac{1}{D_m} \left\{ \frac{A_f(m_1)A_g(m_2)}{m^s_2} - \frac{A_f(m_2)A_g(m_1)}{m^s_1} \right\},
\]
where \( D_m = A_f(m_1)A_g(m_2) - A_f(m_2)A_g(m_1) \). We find that \( A_{\mathcal{F},(1,2)}, A_{\mathcal{F},(2,3)} \) and \( A_{\mathcal{F},(3,5)} \) are invertible by calculating their determinants directly. Using (3.1), we can determine the location of zeros of \( D_{m,11}(s) \) for a given vector \( m \). For example, all zeros of \( D_{m,11}(s) \) lie on the line \( \sigma = 0.343579 \ldots \) for \( m = (1, 2), \sigma = -5.69519 \ldots \) for \( m = (2, 3) \) and \( \sigma = 1.72665 \ldots \) for \( m = (3, 5) \). These examples show that the location of zeros of \( D_{m,ij}(s) \) strongly depends on the choice of the vector \( m \). It is not clear whether we can always choose a vector \( m \) such that \( D_{m,ij}(s) \) satisfies the assumption of Theorem 2 in the case of large dimension of \( S_k \).

4. Preliminaries

4.1. Poincaré series. Let \( m \) be a nonnegative integer. The \( m \)th Poincaré series \( P_m(z) \) of weight \( k \) on \( \Gamma \) is defined by
\[ P_m(z) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} j(\gamma, z)^{-k}e(m\gamma z), \]

where \( \Gamma_\infty = \{ \pm(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}) : n \in \mathbb{Z} \} \subset \Gamma \) and \( j(\gamma, z) = cz + d \) for \( \gamma = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \). If \( k > 2 \), the series on the right-hand side converges absolutely and uniformly on every compact subset of \( \mathfrak{h} \). If \( m \geq 1 \), \( P_m(z) \) is a cusp form, or may vanish identically. In particular, \( P_m(z) \) vanishes identically for \( k \leq 10 \) and \( k = 14 \), since a cusp form of weight \( k \) on \( \Gamma \) exists only for \( k = 12 \) and \( k \geq 16 \). Petersson \([15, 16]\) showed that a basis of \( \mathcal{S} \) can be chosen from the Poincaré series \( P_m(z) \), and the set \( \{ P_1(z), \ldots, P_d(z) \} \) \( (d = \dim \mathcal{S}) \) is a basis of \( \mathcal{S} \).

### 4.2. Nonholomorphic Eisenstein series

The nonholomorphic Eisenstein series \( E(z, s) \) is defined by

\[
E(z, s) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} (\operatorname{Im} \gamma z)^s = \frac{1}{2} \sum_{(c,d) \in \mathbb{Z}^2} \frac{y^s}{|cz + d|^{2s}}.
\]

The right-hand side converges absolutely for \( \sigma > 1 \). The modified function

\[ E^*(z, s) = \zeta^*(2s)E(z, s) \]

is often called the *completed nonholomorphic Eisenstein series*. The function \( E^*(z, s) \) is continued meromorphically to the whole \( s \)-plane, and is holomorphic except for simple poles at \( s = 0 \) and \( 1 \). It satisfies the functional equation \( E^*(z, s) = E^*(z, 1 - s) \). On the other hand, \( E((az + b)/(cz + d), s) = E(z, s) \) for every \( (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma \). Hence, in particular, \( E^*(z, s) \) has the Fourier expansion

\[ E^*(z, s) = \sum_{n=0}^{\infty} a_n(y, s) \cos(2\pi nx), \]

where

\[
a_0(y, s) = \begin{cases} 
\zeta^*(2s)y^s + \zeta^*(2s - 1)y^{1-s}, & s \neq 0, 1/2, 1, \\
y^{1/2}\log y + (\gamma - \log 4\pi)y^{1/2}, & s = 1/2,
\end{cases}
\]

and

\[
a_n(y, s) = 4\sqrt{y} \sum_{n=1}^{\infty} \tau_{s-1/2}(n) K_{s-1/2}(2\pi ny)
\]

for \( n \neq 0 \). Here \( \gamma = 0.57721\ldots \) is the Euler constant, \( \tau_\nu(n) = n^\nu \sigma_{-2\nu}(n) \), \( \sigma_\nu(n) = \sum_{d|n} d^\nu \) and \( K_\nu(t) \) is the \( K \)-Bessel function.

### 4.3. \( C^\infty \)-modular forms

A smooth function \( f \) on \( \mathfrak{h} \) satisfying \( f(\gamma z) = j(\gamma, z)^kf(z) \) for every \( \gamma \in \Gamma \) is called a \( C^\infty \)-modular form of weight \( k \). The *Petersson inner product* \( (f, g) \) of \( C^\infty \)-modular forms \( f \) and \( g \) is defined by

\[
(f, g) := \int_{\Gamma \setminus \mathfrak{h}} f(z) \overline{g(z)} y^{k-2} \, dx \, dy,
\]
if the right-hand side converges. In particular, \((f, g)\) is defined if one of \(f\) and \(g\) belongs to \(M_k\), and the other to \(S_k\), where \(M_k\) is the space of all holomorphic modular forms of weight \(k\) on \(\Gamma\). A \(C^\infty\)-modular form \(f\) of weight \(k\) is called a \(C^\infty\)-modular form of bounded growth if

\[
\int_0^1 \int_0^1 |f(z)|y^{k-2}e^{-\varepsilon y} \, dx \, dy < \infty \quad \text{for every } \varepsilon > 0.
\]

(4.3)

4.4. Inner product with Poincaré series. Let \(f(z) = \sum_{n \in \mathbb{Z}} a_n(y)e(nx)\) be a \(C^\infty\)-modular form of bounded growth. By the unfolding method we derive

\[
(f, P_m) = \int_0^1 \int_0^1 f(z)e(-m\overline{z})y^{k-2} \, dx \, dy
\]

for all \(m \geq 0\). Substituting the Fourier expansion of \(f\) for the right-hand side, we obtain

\[
(f, P_m) = \int_0^\infty a_m(y)e^{-2\pi my}y^{k-2} \, dy \quad (m \geq 0).
\]

(4.4)

Interchanging integration and summation is justified by the growth condition (4.3) ([21, Proposition 1]). Hence, equality (4.4) holds for all \(C^\infty\)-modular forms of bounded growth. Thus we have

\[
(f, P_m) = a_f(m)m^{(k-1)/2} \int_0^\infty e^{-4\pi my}y^{k-2} \, dy
\]

\[
= (4\pi)^{-k+1} \Gamma(k-1)a_f(m)m^{-(k-1)/2}
\]

for every nonnegative integer \(m\), since the holomorphic cusp form \(f(z) = \sum_{n=1}^\infty a_f(n)n^{(k-1)/2}e(nz)\) satisfies the condition (4.3).

4.5. Rankin–Selberg \(L\)-functions. Let \(f(z) = \sum_{n=1}^\infty a_f(n)n^{(k-1)/2}e(nz)\) and \(g(z) = \sum_{n=0}^\infty a_g(n)n^{(k-1)/2}e(nz)\) be modular forms in \(S_k\) and \(M_k\), respectively. The Rankin–Selberg \(L\)-function \(L(s, f \otimes g)\) is defined by (2.1) if the real part of \(s\) is sufficiently large. The function \(F(z) = y^kf(z)\overline{g(z)}\) is a bounded \(\Gamma\)-invariant function on \(\mathfrak{h}\) with rapid decay as \(y \to +\infty\). Its Fourier expansion is

\[
F(x + iy) = y^kf(z)\overline{g(z)}
\]

\[
= y^k \sum_{n \in \mathbb{Z}} \left( \sum_{m=1-n}^\infty a_f(m+n)\overline{a_g(m)}(m+n)^{(k-1)/2}m^{(k-1)/2}e^{-2\pi(2m+n)y} \right) e(nx).
\]
Therefore we obtain
\[
\int_{\Gamma \setminus \mathfrak{h}} y^k f(z) \overline{g(z)} E(z, s) \, d\mu(z)
\]
\[
= \int_0^\infty \left( \sum_{n=1}^{\infty} a_f(n) a_g(n) n^{k-1} e^{-4\pi ny} \right) y^{s+k-1} \frac{dy}{y}
\]
for \( \sigma > 1 \) by the unfolding method. The right-hand side is equal to
\[
(4\pi)^{-s-k+1} \Gamma(s + k - 1) \sum_{m=1}^{\infty} a_f(n) a_g(n) n^{-s}
\]
for \( \sigma > k/2 + 1 \), since the series converges absolutely there by the estimates
\( a_f(n) = O(n^{1/2}) \) and \( a_g(n) = O(n^{(k-1)/2}) \). Hence we obtain
\[
(4.5) \quad (f E_s^*, g) = \int_{\Gamma \setminus \mathfrak{h}} y^k f(z) \overline{g(z)} E^*(z, s) \, d\mu(z)
\]
\[
= \pi^{-s}(4\pi)^{-s-k+1} \Gamma(s) \Gamma(s + k - 1) L(s, f \times \overline{g})
\]
for \( \sigma > k/2 + 1 \), where \( E_s^*(z) = E^*(z, s) \). The left-hand side is defined for all \( s \in \mathbb{C} \) except for the poles of \( E^*(z, s) \), since \( f \) is a cusp form. Therefore (4.5) gives the meromorphic continuation of \( L(s, f \times \overline{g}) \) to \( \mathbb{C} \).

5. **Proof of Theorem 1.** Theorem 1 is a consequence of the following proposition.

**Proposition 1.** Let \( \mathcal{F} = \{f_1, \ldots, f_d\} \) be an orthonormal basis of \( S_k \), and let \( f_j(z) = \sum_{n=1}^{\infty} a_j(n) n^{(k-1)/2} e(nz) \) be the Fourier expansion of \( f_j \) at \( i\infty \). For every \( f \in S_k \),

\[
(5.1) \quad (4\pi)^{-k+1} \Gamma(k-1) \sum_{j=1}^{d} a_j(m) L^*(s, f \times \overline{f}_j)
\]
\[
= a_f(m) [(4\pi)^{-s-k+1} \Gamma(s + k - 1) \zeta^*(2s)m^{-s} \\
+ (4\pi)^{s-k} \Gamma(k-s) \zeta^*(2s-1)m^{s-1}] \\
+ (4\pi)^{-k+1} \Gamma(s + k - 1) \Gamma(k-s) \\
\times \sum_{n=1}^{m-1} a_f(m-n) \frac{\tau_{s-1/2}(n)}{\sqrt{n}} P_{s-1}^{1-k} \left( \frac{2m-n}{n} \right) \\
+ (4\pi)^{-k+1} \Gamma(s + k - 1) \Gamma(k-s) \\
\times \sum_{n=1}^{\infty} a_f(m+n) \frac{\tau_{s-1/2}(n)}{\sqrt{n}} P_{s-1}^{1-k} \left( \frac{2m+n}{n} \right)
\]
in the strip (2.5) if the first term \( a_f(m)[\cdots] \) in (5.1) is replaced by

\[
a_f(m)(4\pi)^{-k+1/2} \Gamma\left( k - \frac{1}{2} \right) \left\{ \frac{\Gamma'}{\Gamma} \left( k - \frac{1}{2} \right) + \log \frac{e^\gamma}{16\pi^2m} \right\} \frac{1}{\sqrt{m}}
\]

at the point \( s = 1/2 \). The series on the right-hand side of (5.1) converges absolutely and uniformly on every compact subset of the vertical strip (2.5).

**Proof.** We denote \( E^*(z, s) \) by \( E^*_s(z) \). Calculating the Petersson inner product \( (f E^*_s, P_m) \) in two ways, we will obtain Proposition 1.

Let \( m \) be a positive integer, and let \( \mathcal{F} = \{f_1, \ldots, f_d\} \) be an orthonormal basis of \( S_k \). Expanding \( P_m(z) \) with respect to the basis \( \mathcal{F} \), we have

\[
P_m(z) = (4\pi)^{-k+1} \Gamma(k-1)m^{-(k-1)/2} \sum_{j=1}^{d} a_j(m) f_j(z),
\]

where \( a_j(m) \) is the \( m \)th Fourier coefficient of \( f_j \). Using this expansion, we obtain the first formula

\[
(5.2) \quad (f E^*_s, P_m) = (4\pi)^{-k+1} \Gamma(k-1)m^{-(k-1)/2} \sum_{j=1}^{d} a_j(m) L^* (s, f \times \bar{f}_j)
\]

for \( \sigma > 1 \). Further (5.2) holds for all \( s \in \mathbb{C} \), since \( f \) is a cusp form. By Lemma 1 of [14], the product \( f(z)E(z, s) \) is a \( C^\infty \)-modular form of bounded growth for \( 0 < \sigma < 1 \). Hence, by (4.4), we have

\[
(5.3) \quad (f E^*_s, P_m) = \int_0^{\infty} \left( \sum_{n=1}^{\infty} a_f(n)a_{m-n}(y, s) n^{(k-1)/2} e^{-2\pi ny} \right) e^{-2\pi my} y^{k-2} dy
\]

for \( 0 < \sigma < 1 \), where \( a_n(y, s) \) is the \( n \)th Fourier coefficient of \( E^*(z, s) \) given in (4.1) and (4.2). Formally, the right-hand side of (5.3) is equal to

\[
\sum_{n=0}^{m-1} a_f(m - n)(m - n)^{(k-1)/2} \int_0^{\infty} a_n(y, s) e^{-2\pi(2m-n)y} y^{k-2} dy
\]

\[+
\sum_{n=1}^{\infty} a_f(m + n)(m + n)^{(k-1)/2} \int_0^{\infty} a_n(y, s) e^{-2\pi(2m+n)y} y^{k-2} dy.
\]

This formal calculation is justified, since interchanging summation and integration is allowed by the estimates
\[ |a_0(y, s)| \ll y^{\sigma} + y^{1-\sigma}, \]
\[ |a_n(y, s)| \ll y^{\sigma} |\sigma_{1-2s}(n)| e^{-\pi ny/2} \quad (n \neq 0), \]

and Fubini’s theorem. For \( n = 0 \) and \( s \neq 0, 1/2, 1 \), we have

\[ (5.4) \quad \int_0^\infty a_0(y, s) e^{-4\pi my} y^{k-2} dy \]
\[ = \zeta^*(2s) \int_0^\infty e^{-4\pi my} y^{k+s-2} dy + \zeta^*(2s-1) \int_0^\infty e^{-4\pi my} y^{k-s-1} dy \]
\[ = (4\pi m)^{-s-k+1} \Gamma(s+k-1) \zeta^*(2s) + (4\pi m)^{s-k} \Gamma(k-s) \zeta^*(2s-1). \]

For \( n = 0 \) and \( s = 1/2 \), we have

\[ (5.5) \quad \int_0^\infty a_0(y, 1/2) e^{-4\pi my} y^{k-2} dy \]
\[ = \int_0^\infty e^{-4\pi my} y^{k-3/2} \log y dy + (\gamma - \log 4\pi) \int_0^\infty e^{-4\pi my} y^{k-3/2} dy \]
\[ = (4\pi m)^{-k+1/2} \Gamma \left( k - \frac{1}{2} \right) \left\{ \frac{\Gamma'}{\Gamma} \left( k - \frac{1}{2} \right) + \log \frac{e^\gamma}{16\pi^2 m} \right\}. \]

For \( n \geq 1 \), we have

\[ (5.6) \quad \int_0^\infty a_n(y, s) e^{-2\pi(2m \pm n)y} y^{k-2} dy \]
\[ = 2\tau_{s-1/2}(n) \int_0^\infty K_{s-1/2}(2\pi ny) e^{-2\pi(2m \pm n)y} y^{k-3/2} dy \]
\[ = (4\pi)^{-k+1} m^{-k+1} \Gamma(s+k-1) \Gamma(k-s) \]
\[ \times \frac{\tau_{s-1/2}(n)}{\sqrt{n}} \left( \frac{m}{m \pm n} \right)^{(k-1)/2} P_{1-k}^1 \left( \frac{2m \pm n}{n} \right) \]

by using the formula

\[ \int_0^\infty K_{\nu}(x) e^{-ax} x^{\mu-1} dx = \sqrt{\frac{\pi}{2}} \frac{\Gamma(\mu + \nu) \Gamma(\mu - \nu)}{(a^2 - 1)^{\mu/2 - 1/4}} P_{\nu-1/2}^{-\mu + 1/2}(a) \]

for \( \text{Re}(a) > -1 \) and \( \text{Re}(\mu) > |\text{Re}(\nu)| \) ([30, p. 388]). By (5.3), (5.4) and (5.6), we obtain the second formula.
(5.7) \( (fE^*_s, P_m) = m^{-(k-1)/2}a_f(m)[(4\pi)^{-s-k+1}m^{-s}\Gamma(s+k-1)\zeta^*(2s) + (4\pi)^{s-k}m^{-1}\Gamma(k-s)\zeta^*(2s-1)] \\
+ m^{-(k-1)/2}(4\pi)^{-k+1}\Gamma(s+k-1)\Gamma(k-s) \\
\times \sum_{n=1}^{m-1} a_f(m-n) \frac{\tau_{s-1/2}(n)}{\sqrt{n}} P_{s-1}^{1-k}(\frac{2m-n}{n}) \\
+ m^{-(k-1)/2}(4\pi)^{-k+1}\Gamma(s+k-1)\Gamma(k-s) \\
\times \sum_{n=1}^{\infty} a_f(m+n) \frac{\tau_{s-1/2}(n)}{\sqrt{n}} P_{s-1}^{1-k}(\frac{2m+n}{n}). \)

Combining (5.2) and (5.7), we obtain (5.1) for \( 0 < \sigma < 1 \) except for \( s = 1/2 \). For \( s = 1/2 \), we use (5.5) instead of (5.4).

To complete the proof of Proposition 1, it suffices to show that the series on the right-hand side of (5.1) converges absolutely in the vertical strip (2.5), since the left-hand side of (5.1) is defined for all \( s \in \mathbb{C} \) except for the possible poles at \( s = 1 \) and 0. Moreover, it suffices to show that the series

\[
(5.8) \sum_{n=1}^{\infty} a_f(m+n) \frac{\tau_{s-1/2}(n)}{\sqrt{n}} P_{s-1}^{1-k}(\frac{2m+n}{n})
\]

converges absolutely in the strip (2.5), since

\[ P_{s-1}^{1-k}(z) = \frac{\Gamma(s-k+1)}{\Gamma(s+k-1)} P_{s-1}^{1-k}(z) \]

for every positive integer \( k \geq 2 \). Suppose that \( |a_f(n)| \ll n^{1/2-\alpha+\varepsilon} \) for some real number \( 0 \leq \alpha \leq 1/2 \). Then

\[
\sum_{n=1}^{\infty} |a_f(m+n)| \frac{\tau_{s-1/2}(n)}{\sqrt{n}} \left| P_{s-1}^{1-k}\left(\frac{2m+n}{n}\right)\right| \ll m \sum_{n=1}^{\infty} n^{s-1/2-\alpha+\varepsilon} \left| P_{s-1}^{1-k}\left(\frac{2m+n}{n}\right)\right|,
\]

since \( |\tau_{s-1/2}(n)| = |n^{s-1/2}\sigma_{1-2s}(n)| \ll n^{s-1/2+\varepsilon} \). Using the formula

\[
P_{s-1}^{1-k}(z) = \frac{\Gamma(s+k-1)(z^2-1)^{(k-1)/2}}{2^{k-1}\sqrt{\pi} \Gamma(k-1/2)\Gamma(s+k-1)} \\
\times \int_0^{\pi} (z+\sqrt{z^2-1}\cos \theta)^{s-k} \sin^{2k-2}\theta d\theta
\]

for \( \text{Re}(z) > 0 \) and \( k \geq 1 \) ([11, p. 199]), we have

\[
\left| P_{s-1}^{1-k}\left(\frac{2m+n}{n}\right)\right| \ll m n^{-(k-1)/2}.
\]
Hence we obtain

$$|\text{series (5.8)}| \ll_m \sum_{n=1}^{\infty} n^{\sigma-1/2}-(k-1)/2-\alpha+\epsilon.$$  

The right-hand side converges absolutely for $2-k/2-\alpha < \Re(s) < k/2+\alpha-1$. Hence the Ramanujan–Deligne estimate $|a_f(n)| \ll \epsilon n^\epsilon$ implies that the series on the right-hand side of (5.1) converges absolutely in the vertical strip (2.5). ■

**Proof of Theorem 1.** Let $\mathcal{F} = \{f_1, \ldots, f_d\}$, $m = (m_1, \ldots, m_d) \in \mathbb{Z}_d^d$ with $0 < m_1 < \cdots < m_d$ and $\{\alpha_{ij}\}$ be as in the statement of the theorem. By Proposition 1, 

$$\begin{align*}
(4\pi)^{-k+1}\Gamma(k-1)A_{\mathcal{F},m} \mathcal{L}_{\mathcal{F},f}(s) &= N_{m,f}(s),
\end{align*}$$

where

$$\begin{align*}
\mathcal{L}_{\mathcal{F},f}(s) &= \begin{pmatrix}
L^*(s, f \times \tilde{f}_1) \\
\vdots \\
L^*(s, f \times \tilde{f}_d)
\end{pmatrix}, \\
N_{m,f}(s) &= \begin{pmatrix}
N_f(s, m_1) \\
\vdots \\
N_f(s, m_d)
\end{pmatrix},
\end{align*}$$

and

$$\begin{align*}
N_f(s, m_h) &= a_f(m_h)[(4\pi)^{-s-k+1}\Gamma(s+k-1)\zeta^*(2s)m_h^{-s} \\
&\quad + (4\pi)^{s-k}\Gamma(k-s)\zeta^*(2s-1)m_h^{s-1}]
\end{align*}$$

$$\begin{align*}
&\times \sum_{n=1}^{m_h-1} a_f(m_h-n) \frac{\tau_{s-1/2}(n)}{\sqrt{n}} P_{s-1}^{-k} \left( \frac{2m_h-n}{n} \right) \\
&\times \sum_{n=1}^{\infty} a_f(m_h+n) \frac{\tau_{s-1/2}(n)}{\sqrt{n}} P_{s-1}^{-k} \left( \frac{2m_h+n}{n} \right).
\end{align*}$$

Multiplying (5.9) by the inverse matrix $A^{-1}_{\mathcal{F},m}$, we have

$$(4\pi)^{-k+1}\Gamma(k-1)\mathcal{L}_{\mathcal{F},f}(s) = A^{-1}_{\mathcal{F},m}N_{m,f}(s).$$

Comparing the $j$th components of both sides, we obtain

$$(4\pi)^{-k+1}\Gamma(k-1)L^*(s, f \times \tilde{f}_j) = \sum_{h=1}^{d} \alpha_{jh}N_f(s, m_h).$$

Taking $f = f_i$, we obtain equality (2.4) of Theorem 1. ■

**6. Proof of Theorem 2.** It suffices to investigate the zeros of $L^N_{m,ij}(s)$ in $\sigma \geq 1/2$, because of the functional equation (2.12) of $L^N_{m,ij}(s)$. By definition
(2.11) of $L_{m,ij}^N(s)$, we have
\begin{equation}
L_{m,ij}^N(s) = (4\pi)^{-s-k+1} \Gamma(s+k-1) \zeta^*(2s) D_{m,ij}(s) \{1 + R_{m,ij}^N(s)\},
\end{equation}
where
\begin{equation*}
R_{m,ij}^N(s) = (4\pi)^{2s-1} \frac{\Gamma(k-s)\zeta^*(2s-1)}{\Gamma(s+k-1)\zeta^*(2s)} \frac{D_{m,ij}(1-s)}{D_{m,ij}(s)} + (4\pi)^s \frac{\Gamma(k-s)\{W_{m,ij}^+(N) + W_{m,ij}^-(s)\}}{\zeta^*(2s)D_{m,ij}(s)}.
\end{equation*}

By the assumption on the location of zeros of $D_{m,ij}(s)$ in Theorem 2, the factor $\zeta^*(2s)D_{m,ij}(s)$ in (6.1) has only finitely many zeros in $\sigma \geq 1/2$. Hence, if the inequality
\begin{equation}
|R_{m,ij}^N(s)| < 1
\end{equation}
is valid for $1/2 < \sigma \leq a$ and sufficiently large $|t|$, then $L_{m,ij}^N(s) \neq 0$ in that region. Now we show that there exists $T_{N,a,\varepsilon} > 1$ such that
\begin{equation}
|R_{m,ij}^N(\sigma + it)| \ll |t|^{1-2\sigma} \log |t|
\end{equation}
for $1/2 \leq \sigma \leq a$ and $|t| \geq T_{N,a,\varepsilon}$. We define
\begin{equation}
I_{m,ij}(s) = (4\pi)^{2s-1} \frac{\Gamma(k-s)\zeta^*(2s-1)}{\Gamma(s+k-1)\zeta^*(2s)} \frac{D_{m,ij}(1-s)}{D_{m,ij}(s)},
\end{equation}
and
\begin{equation}
J_{m,ij}^N(s) = (4\pi)^s \frac{\Gamma(k-s)\{W_{m,ij}^+(N) + W_{m,ij}^-(s)\}}{\zeta^*(2s)D_{m,ij}(s)}
\end{equation}
so that
\begin{equation}
R_{m,ij}^N(s) = I_{m,ij}(s) + J_{m,ij}^N(s).
\end{equation}

For $I_{m,ij}(s)$ and $J_{m,ij}^N(s)$, we obtain the following estimates.

**Lemma 1.** There exists $T_1 > 0$ such that
\begin{equation*}
|I_{m,ij}(s)| = O(|t|^{1-2\sigma})
\end{equation*}
for $1/2 \leq \sigma \leq a$ and $|t| \geq T_1$, where the implied constant depends on $m$, $i$ and $j$.

**Lemma 2.** There exists $T_2 > 0$ such that
\begin{equation*}
|J_{m,ij}^N(s)| = O(|t|^{1-2\sigma} \log |t|)
\end{equation*}
for $1/2 \leq \sigma \leq a$ and $|t| \geq T_2$, where the implied constant depends on $N$, $m$, $i$ and $j$.

Lemma 1, Lemma 2 and (6.5) imply (6.2). Hence the proof of Theorem 2 will be completed if we prove Lemmas 1 and 2. To do that, we use the following lemma.
Lemma 3. Let \( g(s) \) be an exponential polynomial having the form

\[
g(s) = \sum_{j=1}^{n} p_j e^{\beta_j s}, \quad 0 = \beta_0 < \beta_1 < \cdots < \beta_n,
\]

where \( 0 \neq p_j \in \mathbb{C} \ (0 \leq j \leq n) \). Then \( |g(s)| \) is uniformly bounded away from zero if \( s \) is uniformly separated from the zeros of \( g(s) \).

Proof. See Theorem 12.6 of [2].

Proof of Lemma 1. Let \( \xi(s) = s(s - 1)\zeta^*(s) \). We have

\[
\left| (4\pi)^{2s-1} \frac{\Gamma(k-s)\zeta^*(2s-1)}{\Gamma(s+k-1)\zeta^*(2s)} \right| = \left| \frac{t}{4\pi} \right|^{1-2\sigma} 1 + O(|t|^{-1}) \left| \frac{s}{s-1} \right| \frac{\xi(2s-1)}{\xi(2s)}
\]

for \( 1/2 \leq \sigma \leq a \) and \(|t| \geq 1\) by using Stirling’s formula

\[
|\Gamma(\sigma + it)| = \sqrt{2\pi} |t|^{|\sigma-1/2|} e^{-\pi/4} (1 + O(|t|^{-1}))
\]

for \( \sigma_1 \leq \sigma \leq \sigma_2 \) and \(|t| \geq 1\). By the proof of Theorem 2 in [10], we have

\[
\left| \frac{\xi(2s-1)}{\xi(2s)} \right| \leq 1
\]

for \( \sigma \geq 1/2 \). Hence, we obtain

\[
(6.6) \quad \left| (4\pi)^{2s-1} \frac{\Gamma(k-s)\zeta^*(2s-1)}{\Gamma(s+k-1)\zeta^*(2s)} \right| = O(|t|^{1-2\sigma})
\]

for \( 1/2 \leq \sigma \leq a \) and \(|t| \geq t_1 (> 1)\). By Lemma 3 and the assumption on the location of zeros of \( D_{m,ij}(s) \), we have

\[
(6.7) \quad \left| \frac{D_{m,ij}(1-s)}{D_{m,ij}(s)} \right| = O(1)
\]

for \( 1/2 \leq \sigma \leq a \) and \(|t| \geq t_2\). By (6.3), (6.6) and (6.7), we obtain the estimate in Lemma 1.

Proof of Lemma 2. The asymptotic formula (A.1) of the Appendix yields

\[
(4\pi)^s \frac{\Gamma(k-s)}{\zeta^*(2s)} P_{s-1}^{1-k}(\cosh \zeta)
\]

where

\[
= \frac{(2\pi)^{2s}}{\sqrt{\pi}} \frac{\Gamma(k-s)}{\Gamma(s+k-1)} \frac{1}{\zeta(2s)} \frac{1}{(s-1)^{1/2}} \frac{e^{-\zeta/2}}{\sqrt{1-e^{-2\zeta}}}
\]

\[
\times [e^{(s-1/2)\zeta} + e^{\pm i(k-1/2)} e^{(-s+1/2)\zeta} + O(|s-1|^{-1})]
\]
where the implied constant depends on $\zeta > 0$. Therefore,
\[
\left| (4\pi)^s \frac{\Gamma(k-s)}{\zeta^*(2s)} P_{s-1}^{1-k}(\cosh \zeta) \right|
\]
\[
= \frac{(2\pi)^{2\sigma}}{\sqrt{\pi}} \left| \frac{\Gamma(k-s)}{\Gamma(s+k-1)} \right| \frac{1}{|\zeta(2s)|} \frac{1}{\sqrt{|s-1|}}
\]
\[
\times \frac{e^{-\zeta/2}}{\sqrt{1-e^{-2\zeta}}} \left[ e^{(\sigma-1/2)\zeta} + e^{(-\sigma+1/2)\zeta} + O(|s-1|^{-1}) \right].
\]

Using Stirling’s formula, we have
\[
\left| \frac{\Gamma(k-s)}{\Gamma(s+k-1)} \right| = |t|^{1-2\sigma} \frac{1 + O(|t|^{-1})}{1 + O(|t|^{-1})} \ll |t|^{1-2\sigma}
\]
for $1/2 < \sigma < a$ and $|t| \geq t_3$. On the other hand,
\[
\frac{1}{|\zeta(s)|} = O(\log(|t| + 2))
\]
for $\sigma \geq 1 - A/\log(|t| + 2)$ ([27, p. 60]). Hence we have
\[
(6.8) \quad \left| (4\pi)^s \frac{\Gamma(k-s)}{\zeta^*(2s)} P_{s-1}^{1-k}(\cosh \zeta) \right| = O(|t|^{1-2\sigma} \log |t|)
\]
for $1/2 - A'/\log |t| \leq \sigma \leq a$ and $|t| \geq t_4$. By Lemma 3 and the assumption on the location of zeros of $D_{m,ij}(s)$, we have
\[
(6.9) \quad \left| \frac{1}{D_{m,ij}(s)} \right| = O(1)
\]
for $1/2 \leq \sigma \leq a$ and $|t| \geq t_5$. Here we note that
\[
1 + \frac{2}{m_h - 1} < \frac{2m_h - n}{n} < 2m_h - 1 \quad (1 \leq n \leq m_h - 1, 1 \leq h \leq d),
\]
\[
1 + \frac{2m_h}{N} < \frac{2m_h + n}{n} < 2m_h + 1 \quad (1 \leq n \leq N, 1 \leq h \leq d)
\]
for fixed $m = (m_1, \ldots, m_d)$. Combining (2.10), (6.4), (6.8), (6.9) and (6.10), we obtain Lemma 2.

7. Relation with the holomorphic projection. In this section, we reconsider the argument of Section 5 from the viewpoint of the holomorphic projection of Sturm [21]. Let $\mathcal{F} = \{f_1, \ldots, f_d\}$ be an orthonormal basis of $S_k$. Define
\[
(7.1) \quad K(z, w) = \sum_{i=1}^{d} f_i(z) \overline{f_i(w)}.
\]
Then \( K(z, w) \) belongs to \( S_k \) as a function of \( z \in \mathfrak{h} \) for every fixed \( w \in \mathfrak{h} \), and has the reproducing property:

\[
(g(z), K(z, w)) = g(w) \quad \text{for any } g \in S_k.
\]

For a \( C^\infty \)-modular form \( F \) of bounded growth, we define

\[
\pi(F)(w) := (F(z), K(z, w)).
\]

Then \( \pi(F)(w) \) belongs to \( S_k \), and is called the holomorphic projection of \( F \).

Using the formula

\[
K(z, w) = \sum_{m=1}^{\infty} \frac{(4\pi m)^{k-1}}{\Gamma(k-1)} P_m(z) e(-m\overline{w})
\]

([21, p. 333]), we obtain

\[
\pi(F)(w) = \sum_{m=1}^{\infty} \frac{(4\pi m)^{k-1}}{\Gamma(k-1)} (F, P_m) e(mw),
\]

where the inner product \((F, P_m)\) is given by (4.4). Using (7.2), we have

\[
((F(z), K(z, w)), g(w)) = (F(z), (g(w), K(w, z))) = (F(z), g(z)).
\]

Hence, we obtain

\[
(F, g) = (\pi(F), g).
\]

Applying (7.5) to \( F(z) = (fE_s^*)(z) := f(z)E^*(z, s) \), we have

\[
L^*(s, f \times \overline{g}) = (\pi(fE_s^*), g)
\]

by (4.5) (compare (7.6) with (2.10) of [12]). By (7.3) and (7.5), we have

\[
(F, g) = (\pi(F), g) = \sum_{m=1}^{\infty} \frac{(4\pi m)^{k-1}}{\Gamma(k-1)} \phi_m(g)(F, P_m),
\]

where

\[
\phi_m(g) = \int_{\Gamma \backslash \mathfrak{h}} g(w) e(mw) d\mu(w).
\]

Applying (7.7) to \( F = fE_s^* \), we obtain

\[
L^*(s, f \times \overline{g}) = \sum_{m=1}^{\infty} \frac{(4\pi m)^{k-1}}{\Gamma(k-1)} \phi_m(g)(fE_s^*, P_m)
\]

by (7.6). However, this formula for \( L(s, f \times \overline{g}) \) is not useful for application, because each \( \phi_m(g) \) depends on a choice of a fundamental domain of \( \Gamma \).

To improve formula (7.8) of \( L(s, f \times \overline{g}) \), we consider the Fourier coefficients of \( \pi(fE_s^*) \). Let \( \mathcal{F} = \{f_1, \ldots, f_d\} \) be an orthogonal basis of \( S_k \).

Applying (7.4) to \( F = fE_s^* \), we have

\[
\pi(fE_s^*)(z) = \sum_{m=1}^{\infty} \frac{(4\pi m)^{k-1}}{\Gamma(k-1)} (fE_s^*, P_m) e(mz).
\]
Because \( \pi(fE^*_s) \in S_k \), there exist functions \( C_j(s) \) of \( s \) such that

\[
\pi(fE^*_s)(z) = \sum_{j=1}^{d} C_j(s) f_j(z).
\]

By (7.1) and (7.6), we have

\[
C_j(s) = \frac{1}{(f_j, f_j)} \frac{1}{(f, f)} L^*(s, f \times f_j).
\]

Here we have used the Fourier expansion \( f_j(z) = \sum_{n=1}^{\infty} a_j(n) n^{(k-1)/2} e(nz) \).

Combining (7.9)–(7.11), and comparing the \( m \)th Fourier coefficients of both sides, we obtain

\[
\sum_{j=1}^{d} C_j(s) a_j(m) = \sum_{j=1}^{d} \frac{a_j(m)}{(f_j, f_j)} L^*(s, f \times f_j) = \frac{(4\pi m)^{k-1}}{\Gamma(k-1)} (fE^*_s, P_m)
\]

\[
= \frac{a_f(m)}{(f, f)} \left\{ (4\pi m)^{-s} \frac{\Gamma(s+k-1)}{\Gamma(k-1)} \zeta^*(2s) + (4\pi m)^{s-1} \frac{\Gamma(k-s)}{\Gamma(k-1)} \zeta^*(2s-1) \right. 
\]

\[
+ \frac{\Gamma(s+k-1)\Gamma(k-s)}{\Gamma(k-1)} \sum_{n=1}^{m-1} \frac{a_f(m-n)}{a_f(m)} \frac{\tau_s(n)}{\sqrt{n}} P_{s-1}^{1-k} \left( \frac{2m-n}{n} \right) 
\]

\[
+ \frac{\Gamma(s+k-1)\Gamma(k-s)}{\Gamma(k-1)} \sum_{n=1}^{\infty} \frac{a_f(m+n)}{a_f(m)} \frac{\tau_s(n)}{\sqrt{n}} P_{s-1}^{1-k} \left( \frac{2m+n}{n} \right) \right\}.
\]

This is nothing other than equality (5.1).

**Appendix. Asymptotic expansion of \( P^\mu_\nu(z) \).** In this section, we give an asymptotic expansion of the associated Legendre functions \( P^\mu_\nu(z) \) for large \( |\nu| \) according to Watson [29], where \( \nu \) and \( \mu \) do not have to be integers. The associated Legendre function \( P^\mu_\nu(z) \) of the first kind is defined by

\[
P^\mu_\nu(z) = \frac{1}{\Gamma(1-\mu)} \left( \frac{z+1}{z-1} \right)^{\mu/2} F\left(-\nu, \nu+1, 1-\mu; \frac{1-z}{2} \right)
\]

for \( z-1 \in \mathbb{C} \setminus (-\infty, 0] \). We write \( z = \cosh \zeta, \ \zeta = \xi + i\eta \ (\xi, \eta \in \mathbb{R}) \) for \( z-1 \in \mathbb{C} \setminus (-\infty, 0] \), and define the values \( \omega_i = \omega_i(z) \ (i = 1, 2) \) by

\[
\omega_1 = -\arctan\left( \frac{\eta - \pi}{\xi} \right), \quad \omega_2 = \arctan\left( \frac{\eta}{\xi} \right)
\]

if \( \eta \geq 0 \), and by

\[
\omega_1 = -\arctan\left( \frac{\eta}{\xi} \right), \quad \omega_2 = -\arctan\left( \frac{\eta + \pi}{\xi} \right)
\]
if $\eta \leq 0$. In each case arctan denotes an acute angle, positive or negative. Define

$$\tau = \log \left( \frac{t - z}{t^2 - 1} \right) + \log(2e^\zeta).$$

We define the numbers $c_n$ and $d_n$ by using the expansion

$$(1 - t)^\mu (1 + t)^{-\mu} (z - t)^{-1} \frac{dt}{d\tau} = \pm C \sum_{n=0}^{\infty} c_n \tau^{n-1/2} + \sum_{n=0}^{\infty} d_n \tau^n,$$

where $C = 2^{-1}(1 - e^\zeta)^{\mu+1/2}(1 + e^\zeta)^{1/2-\mu}(z - e^\zeta)^{-1}$ and multiple-valued functions are specified by the conventions

$$|\arg(1 - e^\zeta)| < \pi, \quad |\arg(1 + e^\zeta)| < \pi.$$

In particular,

$$c_0 = 1, \quad c_1 = \frac{8\mu^2 - 3 + 3e^{2\zeta}}{4(1 - e^{2\zeta})}.$$  

We define the numbers $c'_n$ from $c_n$ by changing the sign of $\zeta$. In particular,

$$c'_0 = 1, \quad c'_1 = \frac{8\mu^2 - 3 + 3e^{-2\zeta}}{4(1 - e^{-2\zeta})}.$$

**Proposition 2 (Watson).** Let $z$ be a complex number such that $z - 1 \in \mathbb{C} \backslash (-\infty, 0]$. In the range of $\arg \nu$ depending on $z$ and given by

$$-\frac{\pi}{2} - \omega_2 + \delta \leq \arg \nu \leq \frac{\pi}{2} + \omega_1 + \delta,$$

the associated Legendre function $P_\nu^\mu(z)$ has the asymptotic expansion

$$(A.1) \quad P_\nu^\mu(z) = \frac{\Gamma(\nu + 1)}{\Gamma(\nu - \mu + 1)} \frac{e^{-\zeta/2}}{(\nu \pi)^{1/2}(1 - e^{-2\zeta})^{1/2}}
\times \left[ e^{(\nu+1/2)\zeta} \sum_{n=0}^{N-1} \frac{\Gamma(n + 1/2)}{\Gamma(1/2)} c_n \nu^{-n}
+ e^{\mp i\pi(\mu-1/2)} e^{-(\nu+1/2)\zeta} \sum_{n=0}^{N-1} \frac{\Gamma(n + 1/2)}{\Gamma(1/2)} c'_n \nu^{-n} + O(|\nu|^{-N}) \right]$$

as $|\nu| \to +\infty$, where the implied constant depends on $z$ and $\mu$.

**References**


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