ACTA ARITHMETICA
136.1 (2009)

Arithmetic functions on Beatty sequences

by

ALEX G. ABERCROMBIE (Pembroke),
WILLIAM D. BANKS (Columbia, MO) and
IGOR E. SHPARLINSKI (Sydney)

1. Introduction

1.1. *Background.* For a real number $\alpha > 1$, the *homogeneous Beatty sequence* corresponding to α is the sequence of natural numbers given by

$$\mathcal{B}_\alpha = (\lfloor \alpha k \rfloor)_{k\in\mathbb{N}},$$

where $\lfloor t \rfloor$ denotes the greatest integer $\le t$. Beatty sequences appear in a variety of contexts and have been extensively explored in the literature. In particular, summatory functions of the form

$$S_\alpha(f,x) = \sum_{n\le x,\, n\in\mathcal{B}_\alpha} f(n) \tag{1}$$

have been studied when the arithmetic function f is

- a multiplicative or an additive function (see [1, 2, 8, 9, 10, 11]);
- a Dirichlet character (see [2, 3, 5]);
- the characteristic function of primes or smooth numbers (see [4, 6, 7]).

For an arbitrary arithmetic function f we define

$$S(f,x) = S_1(f,x) = \sum_{n\le x} f(n). \tag{2}$$

Abercrombie [1] has shown that for the divisor function τ the asymptotic formula

$$S_\alpha(\tau,x) = \alpha^{-1} S(\tau,x) + O(x^{5/7+\varepsilon}) \tag{3}$$

holds for any $\varepsilon > 0$ and almost all $\alpha > 1$ (with respect to Lebesgue measure), where the implied constant depends only on α and ε. This result has been

2000 *Mathematics Subject Classification*: 11B83, 11J83, 11K65.
Key words and phrases: Beatty sequence, arithmetic function.

DOI: 10.4064/aa136-1-6

improved and extended by Zhai [14] as follows. For a fixed integer $r \geq 1$, let $\tau_r(n)$ be the number of ways to express n as a product of r natural numbers, expressions with the same factors in a different order being counted as different (in particular, $\tau_2 = \tau$ is the usual divisor function). In [14] it is shown that the asymptotic formula

$$S_\alpha(\tau_r, x) = \alpha^{-1} S(\tau_r, x) + O(x^{(r-1)/r+\varepsilon}) \tag{4}$$

holds for any $\varepsilon > 0$ and almost all $\alpha > 1$ (in the special case $r = 2$ a similar result has also been obtained by Begunts [8]). The estimate (4) has been further improved by Lü and Zhai [11] as follows:

$$S_\alpha(\tau_r, x) = \alpha^{-1} S(\tau_r, x) + \begin{cases} O(x^{(r-1)/r+\varepsilon}) & \text{if } 2 \leq r \leq 4, \\ O(x^{4/5+\varepsilon}) & \text{if } r \geq 5. \end{cases} \tag{5}$$

1.2. *Our result.* In this paper, we use the methods of [1] to derive an asymptotic formula for $S_\alpha(f, x)$ which holds for almost all $\alpha > 1$ whenever f satisfies a rather mild growth condition. In particular, we do not stipulate any conditions on the multiplicative or additive properties of f (or on any other properties of f except for the rate of growth). Our general result, when applied to the divisor functions, yields a statement stronger than (3) and an improvement of (5) for all $r \geq 4$, and it can be applied to many other number-theoretic functions (and to powers and products of such functions), including:

- the Möbius function $\mu(n)$,
- the Euler function $\varphi(n)$,
- the number of prime divisors $\omega(n)$,
- the sum $\sigma_g(n)$ of the digits of n in a given base $g \geq 2$.

On the other hand, we note that although the results of [1, 11, 14] are formulated as bounds which hold for almost all α, the methods of those papers are somewhat more explicit than ours, and the results can be applied to any "individual" numbers α whose rational approximations satisfy certain hypotheses; thus, one can derive variants of (3), (4) and (5) for specific values of α (or over some interesting classes of α, such as the class of algebraic numbers).

1.3. *Notation.* Throughout the paper, implied constants in the symbols O, $\ll$ and $\gg$ may depend (where obvious) on the parameters α, ε but are absolute otherwise. We recall that the notations $U = O(V)$, $U \ll V$, and $V \gg U$ are all equivalent to the assertion that the inequality $|U| \leq cV$ holds with some constant $c > 0$.

We also use $\|t\|$ to denote the distance from $t \in \mathbb{R}$ to the nearest integer.

2. Main result

2.1. *Formulation.* We define

$$\Delta_\alpha(f,x) = |S_\alpha(f,x) - \alpha^{-1}S(f,x)| \tag{6}$$

and

$$M(f,x) = 1 + \max\{|f(n)| : n \le x\}.$$

THEOREM 1. *For fixed $\varepsilon > 0$ and almost all real numbers $\alpha > 1$, the following bound holds:*

$$\Delta_\alpha(f,x) \ll x^{2/3+\varepsilon} M(f,x).$$

2.2. *Preparations.* We follow the arguments of [1]. For any real number $x \ge 1$, let ψ_x be the trigonometric polynomial of Vaaler [13] given by

$$\psi_x(t) = \sum_{1 \le |m| \le x^{1/2}} a_x(m) e^{2\pi i m t} \qquad (t \in \mathbb{R}),$$

where for each integer m in the sum we put

$$a_x(m) = -\frac{\pi m_x(1-|m_x|)\cot(\pi m_x) + |m_x|}{2\pi i m} \quad \text{with} \quad m_x = \frac{m}{x^{1/2}+1}. \tag{7}$$

As in [1, Section 3] we note that the inequality

$$|u(1-u)\cot(\pi u)| \le 1 \qquad (0 \le u \le 1)$$

immediately implies the uniform bound

$$a_x(m) \ll \frac{1}{|m|} \qquad (1 \le |m| \le x^{1/2}). \tag{8}$$

The function ψ_x is an exceptionally good approximation to the "sawtooth" function $\psi(t) = \{t\} - 1/2$, where $\{t\}$ denotes the fractional part of $t \in \mathbb{R}$. Indeed, by [1, Corollary 2.9] we have

$$|\psi(t) - \psi_x(t)| \le \frac{\csc^2(\pi t)}{2(x^{1/2}+1)^2} \ll \frac{\csc^2(\pi t)}{x}. \tag{9}$$

To prove the theorem, we can clearly assume that $\alpha > 1$ is irrational. In this case, one sees that a natural number n is a term in the Beatty sequence $\mathcal{B}_\alpha$ (that is, $n = \lfloor \alpha k \rfloor$ for some $k \in \mathbb{N}$) if and only if $\alpha^{-1}n$ lies in the set

$$\{t \in \mathbb{R} : 1 - \alpha^{-1} \le \{t\} < 1\}.$$

As the characteristic function ξ_α of that set satisfies the relation

$$\xi_\alpha(t) = \alpha^{-1} + \psi(t) - \psi(t + \alpha^{-1})$$

for every $t \in \mathbb{R}$, it follows that

$$\sum_{n\le x,\, n\in\mathcal{B}_\alpha} f(n) = \sum_{n\le x} f(n)\xi_\alpha(\alpha^{-1}n)$$
$$= \sum_{n\le x} f(n)(\alpha^{-1} + \psi(\alpha^{-1}n) - \psi(\alpha^{-1}(n+1))).$$

Taking into account the definitions (1), (2) and (6), we see that

$$\Delta_\alpha(f,x) \le |Q_\alpha(f,x)| + \sum_{n\le x} |f(n)| R_\alpha(n,x), \tag{10}$$

where

$$Q_\alpha(f,x) = \sum_{n\le x} f(n)(\psi_x(\alpha^{-1}n) - \psi_x(\alpha^{-1}(n+1))),$$
$$R_\alpha(n,x) = |\psi(\alpha^{-1}n) - \psi_x(\alpha^{-1}n)| + |\psi(\alpha^{-1}(n+1)) - \psi_x(\alpha^{-1}(n+1))|.$$

2.3. *Growth of the function $Q_\alpha(f,x)$.* We need the following estimate on the finite differences of the function $Q_\alpha(f,x)$, which could be of independent interest.

LEMMA 1. *For a fixed irrational $\alpha > 1$ we have*

$$Q_\alpha(f,y) - Q_\alpha(f,x) \ll (y-x)M(f,y) \qquad (1\le x\le y\le 2x).$$

Proof. For any $t \in \mathbb{R}$ we have

$$\psi_y(t) - \psi_x(t) = S_1 + S_2,$$

where

$$S_1 = \sum_{1\le |m|\le x^{1/2}} (a_y(m) - a_x(m))e^{2\pi i m t} \quad \text{and} \quad S_2 = \sum_{x^{1/2}<|m|\le y^{1/2}} a_y(m)e^{2\pi i m t}.$$

In view of (8) the latter sum is bounded by

$$S_2 \ll \sum_{x^{1/2}<|m|\le y^{1/2}} \frac{1}{|m|} \ll \frac{y^{1/2}-x^{1/2}}{x^{1/2}} \ll \frac{y-x}{x}.$$

To bound S_1, we put

$$F(u) = \pi u(1-|u|)\cot(\pi u) + |u|,$$

so that $a_x(m) = -F(m_x)/(2\pi i m)$ in the notation of (7). If $1 \le |m| \le x^{1/2}$ then

$$m_y - m_x = \frac{m(x^{1/2}-y^{1/2})}{(x^{1/2}+1)(y^{1/2}+1)} \ll \frac{|m|(y-x)}{x^{3/2}},$$

and since F is continuous and piecewise-differentiable on the interval $(-1,1)$ it follows that

$$a_y(m) - a_x(m) = -\frac{F(m_y)-F(m_x)}{2\pi i m} \ll \frac{y-x}{x^{3/2}}.$$

Therefore,

$$|S_1| \le \sum_{1\le |m|\le x^{1/2}} |a_y(m) - a_x(m)| \ll \frac{y-x}{x}.$$

Thus, we have established the uniform bound

$$\psi_y(t) - \psi_x(t) \ll \frac{y-x}{x} \qquad (t\in\mathbb{R},\ 1\le x\le y\le 2x). \tag{11}$$

Now write

$$Q_\alpha(f,y) - Q_\alpha(f,x) = \widetilde{S}_1 + \widetilde{S}_2 + \widetilde{S}_3,$$

where

$$\begin{aligned}
\widetilde{S}_1 &= \sum_{n\le x} f(n)(\psi_y(\alpha^{-1}n) - \psi_x(\alpha^{-1}n)),\\
\widetilde{S}_2 &= -\sum_{n\le x} f(n)(\psi_y(\alpha^{-1}(n+1)) - \psi_x(\alpha^{-1}(n+1))),\\
\widetilde{S}_3 &= \sum_{x<n\le y} f(n)(\psi_y(\alpha^{-1}n) - \psi_y(\alpha^{-1}(n+1))).
\end{aligned}$$

Using (11) we see that

$$\widetilde{S}_j \ll (y-x)M(f,x) \qquad (j=1,2),$$

and clearly,

$$\widetilde{S}_3 \ll (y-x)M(f,y).$$

This completes the proof. ∎

2.4. *Concluding the proof of Theorem 1.* Now put $\lambda = \alpha^{-1}$ and expand $Q_\alpha(f,x)$ as a Fourier series in λ:

$$\begin{aligned}
Q_{\lambda^{-1}}(f,x) &= \sum_{n\le x} f(n) \sum_{1\le |m|\le x^{1/2}} a_x(m)(e^{2\pi i m n\lambda} - e^{2\pi i m(n+1)\lambda})\\
&= \sum_{n\le x+1} g(n) \sum_{1\le |m|\le x^{1/2}} a_x(m) e^{2\pi i m n\lambda}\\
&= \sum_{1\le |k|\le (x+1)x^{1/2}} e^{2\pi i k\lambda} \sum_{\substack{n\le x+1\\ |m|\le x^{1/2}\\ nm=k}} g(n)a_x(m),
\end{aligned}$$

where

$$g(n) = \begin{cases} f(n) & \text{if } n=1,\\ f(n)-f(n-1) & \text{if } 2\le n\le x,\\ -f(n-1) & \text{if } x<n\le x+1.\end{cases}$$

By the Parseval identity we have

$$\int_0^1 |Q_{\lambda^{-1}}(f,x)|^2\,d\lambda = \sum_{1\le|k|\le(x+1)x^{1/2}} \Big| \sum_{\substack{n\le x+1\\ |m|\le x^{1/2}\\ nm=k}} g(n)a_x(m)\Big|^2. \tag{12}$$

The inner sum on the right of (12) is bounded above by

$$\begin{aligned}\sum_{\substack{n\le x+1\\ |m|\le x^{1/2}\\ nm=k}} g(n)a_x(m) &\ll M(f,x) \sum_{\substack{|k|/(x+1)\le|m|\le x^{1/2}\\ m\mid k}} \frac{1}{|m|}\\ &\ll M(f,x)\tau(|k|)\min\{1,x/|k|\}.\end{aligned}$$

Thus, the integral on the left of (12) is bounded by

$$\begin{aligned}\int_0^1 |Q_{\lambda^{-1}}(f,x)|^2\,d\lambda &\ll \Big(\sum_{k\le x}\tau(k)^2 + x^2 \sum_{x<k\le(x+1)x^{1/2}} \frac{\tau(k)^2}{k^2}\Big) M(f,x)^2\\ &\ll x(\log x)^3 M(f,x)^2,\end{aligned}$$

where we have used the bound (see [12, Chapter 1, Theorem 5.4])

$$\sum_{k\le x}\tau(k)^2 \ll x(\log x)^3$$

together with partial summation (for the second sum).

Now put

$$\Theta = \frac{3}{1+3\varepsilon},$$

and observe that the preceding bound implies

$$\int_0^1 |Q_{\lambda^{-1}}(f,N^\Theta)|^2\,d\lambda \ll N^\Theta(\log N)^3 M(f,N^\Theta)^2 \quad (N\ge 1).$$

Then, since

$$\sum_{N=1}^{\infty}\int_0^1 \frac{|Q_{\lambda^{-1}}(f,N^\Theta)|^2}{N^{\Theta+1}(\log N)^6 M(f,N^\Theta)^2}\,d\lambda \ll \sum_{N=1}^{\infty}\frac{1}{N(\log N)^3} < \infty,$$

it follows that the integral

$$\int_0^1 \Big(\sum_{N=1}^{\infty} \frac{|Q_{\lambda^{-1}}(f,N^\Theta)|^2}{N^{\Theta+1}(\log N)^6 M(f,N^\Theta)^2}\Big)\,d\lambda$$

converges. This implies that the series

$$\sum_{N=1}^{\infty} \frac{|Q_\alpha(f,N^\Theta)|^2}{N^{\Theta+1}(\log N)^6 M(f,N^\Theta)^2}$$

converges for almost all $\alpha > 1$. Let α be fixed with that property, and note that

$$Q_\alpha(f, N^\Theta) \ll N^{(\Theta+1)/2}(\log N)^3 M(f, N^\Theta) \quad (N \ge 1).$$

For any given real number $x \ge 1$, let N be the unique integer for which $N^\Theta \le x < (N+1)^\Theta$. Then

$$\begin{aligned} Q_\alpha(f, N^\Theta) &\ll x^{(\Theta+1)/(2\Theta)}(\log x)^3 M(f,x) = x^{2/3+\varepsilon/2}(\log x)^3 M(f,x) \\ &\ll x^{2/3+\varepsilon} M(f,x). \end{aligned}$$

By Lemma 1 we also see that

$$\begin{aligned} Q_\alpha(f,x) - Q_\alpha(f, N^\Theta) &\ll ((N+1)^\Theta - N^\Theta) M(f,x) \ll N^{\Theta-1} M(f,x) \\ &\ll x^{(\Theta-1)/\Theta} M(f,x) = x^{2/3-\varepsilon} M(f,x). \end{aligned}$$

Therefore,

$$Q_\alpha(f,x) \ll x^{2/3+\varepsilon} M(f,x) \tag{13}$$

for almost all α.

To bound the sum in (10) we put

$$L = \left\lfloor \frac{\log x}{2\log 2} \right\rfloor,$$

and for each $j = 1, \ldots, L$ we denote by $\mathcal{N}_j$ the set of natural numbers $n \le x$ for which

$$2^{-j-1} < \min\{\|\alpha^{-1}n\|, \|\alpha^{-1}(n+1)\|\} \le 2^{-j}.$$

We also denote by $\mathcal{N}_*$ the set of natural numbers $n \le x$ such that

$$\min\{\|\alpha^{-1}n\|, \|\alpha^{-1}(n+1)\|\} \le 2^{-(L+1)}.$$

If $n \in \mathcal{N}_j$, then (9) implies that

$$\begin{aligned} R_\alpha(n,x) &\ll (\csc^2(\pi\alpha^{-1}n) + \csc^2(\pi\alpha^{-1}(n+1)))x^{-1} \\ &\ll (\|\alpha^{-1}n\|^{-2} + \|\alpha^{-1}(n+1)\|^{-2})x^{-1} \ll 2^{2j}x^{-1}, \end{aligned}$$

and the bound $|\psi(t) - \psi_x(t)| \le 1$, which follows from [1, Lemma 2.8] (which in turn follows from [13]), implies that $R_\alpha(n,x) \ll 1$ holds for all $n \in \mathcal{N}_*$; therefore,

$$\begin{aligned} \sum_{n \le x} |f(n)| R_\alpha(n,x) &\ll x^{-1} \sum_{j=1}^{L} 2^{2j} \sum_{n \in \mathcal{N}_j} |f(n)| + \sum_{n \in \mathcal{N}_*} |f(n)| \\ &\le \Big(x^{-1} \sum_{j=1}^{L} 2^{2j} |\mathcal{N}_j| + |\mathcal{N}_*| \Big) M(f,x). \end{aligned}$$

Using [1, Lemma 2.4 and Corollary 2.7] one sees that for almost all $\alpha > 1$ and uniformly for $x \ge 1$, the upper bounds

$$|\mathcal{N}_j| \ll 2^{-j}x + (\log x)^3 \qquad (j = 1, \dots, L)$$

and

$$|\mathcal{N}_*| \ll 2^{-L}x + (\log x)^3$$

hold. Since $2^L \asymp x^{1/2}$, it follows that

$$\sum_{n \le x} |f(n)| R_\alpha(n, x) \ll x^{1/2} M(f, x) \tag{14}$$

for almost all $\alpha > 1$.

Combining (10), (13) and (14), we obtain our main result. ∎

Acknowledgements. During the preparation of this paper, I. S. was supported in part by ARC grant DP0556431.

References

[1] A. G. Abercrombie, *Beatty sequences and multiplicative number theory*, Acta Arith. 70 (1995), 195–207.

[2] W. D. Banks and I. E. Shparlinski, *Non-residues and primitive roots in Beatty sequences*, Bull. Austral. Math. Soc. 73 (2006), 433–443.

[3] —, —, *Short character sums with Beatty sequences*, Math. Res. Lett. 13 (2006), 539–547.

[4] —, —, *Prime divisors in Beatty sequences*, J. Number Theory 123 (2007), 413–425.

[5] —, —, *Character sums with Beatty sequences on Burgess-type intervals*, in: Analytic Number Theory—Essays in Honour of Klaus Roth, Cambridge Univ. Press, Cambridge, to appear.

[6] —, —, *Prime numbers with Beatty sequences*, Colloq. Math., to appear.

[7] A. V. Begunts, *On prime numbers in an integer sequence*, Moscow Univ. Math. Bull. 59 (2004), no. 2, 60–63.

[8] —, *An analogue of the Dirichlet divisor problem*, ibid. no. 6, 37–41.

[9] —, *On the distribution of the values of sums of multiplicative functions on generalized arithmetic progressions*, Chebyshevskiĭ Sb. 6 (2005), no. 2, 52–74 (in Russian).

[10] A. M. Güloğlu and C. W. Nevans, *Sums with multiplicative functions over a Beatty sequence*, preprint, 2008, http://arxiv.org/abs/0801.2796.

[11] G. S. Lü and W. G. Zhai, *The divisor problem for the Beatty sequences*, Acta Math. Sinica (Chin. Ser.) 47 (2004), 1213–1216 (in Chinese, with English summary).

[12] K. Prachar, *Primzahlverteilung*, Springer, Berlin, 1957.

[13] J. D. Vaaler, *Some extremal functions in Fourier analysis*, Bull. Amer. Math. Soc. 12 (1985), 183–216.

[14] W. G. Zhai, *A note on a result of Abercrombie*, Chinese Sci. Bull. 42 (1997), 1151–1154.

3 Middle Row, Golden Hill
Pembroke SA71 4TD, UK
E-mail: alexabercrombie@hotmail.co.uk

Department of Mathematics
University of Missouri
Columbia, MO 65211, U.S.A.
E-mail: bbanks@math.missouri.edu

Department of Computing
Macquarie University
Sydney, NSW 2109, Australia
E-mail: igor@ics.mq.edu.au

Received on 19.5.2008 (5710)