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The k-tuple jumping champions among consecutive primes

by

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1. Introduction. The search for the most probable difference among consecutive primes has been conducted for a long time. The problem was proposed by H. Nelson [N1, N2] in the 1977-78 volume of the Journal of Recreational Mathematics. Assuming the prime pair conjecture of G. H. Hardy and J. E. Littlewood [HL], P. Erdős and E. G. Straus [ES] showed in 1980 that there is no most likely difference, since they found that the most likely difference grows as the number considered becomes larger.

J. H. Conway invented the term "jumping champion" to refer to the most common gap between consecutive primes not exceeding x. Let p_n denote the nth prime. The *jumping champions* are the integers d for which the counting function

$$N(x,d) = \sum_{\substack{p_n \le x \\ p_n - p_{n-1} = d}} 1$$

attains its maximum

$$N^*(x) = \max_d N(x,d).$$

In 1999 Odlyzko, Rubinstein and Wolf [ORW] formulated the following two hypotheses:

CONJECTURE 1.1. *The jumping champions greater than* 1 *are* 4 *and the primorials* $2, 6, 30, 210, 2310, \ldots$.

CONJECTURE 1.2. *The jumping champions tend to infinity. Furthermore, any fixed prime p divides all sufficiently large jumping champions.*

Conjecture 1.1 is now known as the Jumping Champion Conjecture. It is obvious that Conjecture 1.2 is a weaker consequence of Conjecture 1.1, and as already mentioned, the first assertion of Conjecture 1.2 was proved

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by Erdős and Straus [ES], under the assumption of the Hardy–Littlewood prime pair conjecture. Recently, Goldston and Ledoan [GL1] extended successfully Erdős and Straus's method to give a complete proof of Conjecture 1.2 under the same assumption. Soon after, in [GL2], they also gave a proof of Conjecture 1.1 for sufficiently large jumping champions by assuming a sufficiently strong form of the Hardy–Littlewood prime pair conjecture.

Motivated by the work of Goldston and Ledoan, we have been working on the problem what are the most probable differences among $k+1$ consecutive primes with $k \geq 1$.

Let $\mathcal{D}_k = \{d_1, \ldots, d_k\}$ be a set of k distinct integers with $d_1 < \cdots < d_k$. We define the *k-tuple jumping champions* to be the sets $\mathcal{D}_k$ for which the sum

$$N_k(x, \mathcal{D}_k) = \sum_{\substack{p_{n+k} \leq x \\ p_{n+i} - p_n = d_i,\, 1 \leq i \leq k}} 1$$

attains its maximum

$$N_k^*(x) = \max_{\mathcal{D}_k} N_k(x, \mathcal{D}_k).$$

Our main result in the present paper can be summarized as follows.

THEOREM 1.3. *Let k be any given positive integer. Assume Conjecture* 2.1. *The gcd* (*greatest common divisor*) *of all elements in the k-tuple jumping champions tends to infinity. Furthermore, any fixed prime p divides every element of all sufficiently large k-tuple jumping champions.*

THEOREM 1.4. *Assuming Conjecture* 2.2, *the gcd of any sufficiently large k-tuple jumping champion is square-free.*

In the following, we will denote $\mathcal{D}_k = d * \mathcal{D}'_k$, where $d = (d_1, \ldots, d_k)$ is the gcd of the elements in $\mathcal{D}_k$ and $\mathcal{D}'_k = \{d'_1, \ldots, d'_k\}$ with $d_i = dd'_i$ for any $i \leq k$. We let ϵ always denote an arbitrarily small positive constant which may have different values according to the context. Throughout the paper the implied constants in O, $\gg$, $\ll$ and o can depend on k.

2. The Hardy–Littlewood prime n-tuple conjecture. Let $\pi_n(x, \mathcal{D}_n)$ denote the number of positive integers $m \leq x$ such that $m + d_1, \ldots, m + d_n$ are all primes and $\nu_{\mathcal{D}_n}(p)$ represents the number of distinct residue classes modulo p occupied by elements of $\mathcal{D}_n$. The n-tuple conjecture says

$$\pi_n(x, D_n) \sim \mathfrak{S}(\mathcal{D}_n) \int_2^x \frac{dt}{\log^n t}$$

as $x \to \infty$, where

$$\mathfrak{S}(\mathcal{D}_n) = \prod_p \left(1 - \frac{1}{p}\right)^{-n} \left(1 - \frac{\nu_{\mathcal{D}_n}(p)}{p}\right)$$

with p running through all primes.

In the proof of Theorem 1.3, we need the following conjecture.

CONJECTURE 2.1. *If* $\mathfrak{S}(\{0\} \cup \mathcal{D}_k) \neq 0$, *then as* $x \to \infty$,

$$\pi_{k+1}(x, \{0\} \cup \mathcal{D}_k) = \mathfrak{S}(\{0\} \cup \mathcal{D}_k) \frac{x}{\log^{k+1} x}(1 + o(1))$$

uniformly for $\mathcal{D}_k \subset [2, \log^{k+1} x]$.

It is reasonable to suppose that the Hardy–Littlewood conjecture will hold uniformly for any $\mathcal{D}_k \subset [2, x]$, but the range $[2, \log^{k+1} x]$ is enough for our proof.

To prove Theorem 1.4, we need the following stronger form of the Hardy–Littlewood conjecture.

CONJECTURE 2.2. *For* $n = k+1, k+2$, *if* $\mathfrak{S}(\{0\} \cup \mathcal{D}_k) \neq 0$, *then as* $x \to \infty$,

$$\pi_n(x, \{0\} \cup \mathcal{D}_{n-1}) = \mathfrak{S}(\{0\} \cup \mathcal{D}_{n-1}) \frac{x}{\log^n x}(1 + E_n)$$

uniformly for $\mathcal{D}_{n-1} \subset [2, \log^{k+1} x]$, *where*

$$E_n = \begin{cases} o\left(\dfrac{1}{(\log\log x)^2}\right) & n = k+1, \\ o(1) & n = k+2. \end{cases}$$

We also need the following well-known sieve bound: for x sufficiently large,

$$\pi_n(x, \mathcal{D}_n) \le (2^n n! + \epsilon)\mathfrak{S}(\mathcal{D}_n) \frac{x}{\log^n x}, \tag{2.1}$$

for $\mathfrak{S}(\mathcal{D}_n) \neq 0$, which was given in Halberstam and Richert's excellent monograph [HR].

3. Lemma.

To prove Theorem 1.4, we need the following lemmas.

LEMMA 3.1. *For any set* $\mathcal{D}_k \subset [0, h]$, $H \le h$, *we have*

$$\sum_{\substack{1 \le d_0 \le H \\ d_0 \notin \mathcal{D}_k}} \mathfrak{S}(\mathcal{D}_k \cup \{d_0\}) = \mathfrak{S}(\mathcal{D}_k) H \left(1 + O_k\left(\frac{h^\epsilon}{H^{1/2}}\right)\right).$$

This lemma is about the average of the singular series, and the study of this is interesting in itself. We will give the proof in the last section.

LEMMA 3.2. *For any integer* $k \ge 1$, *assume Conjecture* 2.2. *Let* $\mathcal{D}_k$ *be a set of* k *distinct integers with* $\mathfrak{S}(\{0\} \cup \mathcal{D}_k) \neq 0$.

(i) *If* $2 \le d_k = o(\log x)$, *then*

$$N_k(x, \mathcal{D}_k) = \mathfrak{S}(\{0\} \cup \mathcal{D}_k) \frac{x}{\log^{k+1} x}\left\{1 - \frac{d_k}{\log x} + o\left(\frac{d_k}{\log x} + \frac{1}{(\log\log x)^2}\right)\right\}.$$

(ii) *If* $H \le d_k \le \log^{k+1} x$ *for some* H *with* $\log x/\log\log x \le H = o(\log x)$*, then*

$$N_k(x,\mathcal{D}_k) \le \mathfrak{S}(\{0\}\cup\mathcal{D}_k)\frac{x}{\log^{k+1} x}\left\{1-\frac{H}{\log x}+o\left(\frac{H}{\log x}\right)\right\}.$$

Proof. By inclusion-exclusion we have, for any integer $I \ge 0$ and any $1 \le H \le d_k$,

$$N_k(x,\mathcal{D}_k) \ge \sum_{i=0}^{2I+1}(-1)^i \sum_{\substack{0<m_1<\cdots<m_i<d_k\\ m_1,\dots,m_i\notin\mathcal{D}_k}} \pi_{k+1+i}(x,\{0,m_1,\dots,m_i\}\cup\mathcal{D}_k), \tag{3.1}$$

$$N_k(x,\mathcal{D}_k) \le \sum_{i=0}^{2I}(-1)^i \sum_{\substack{0<m_1<\cdots<m_i<H\\ m_1,\dots,m_i\notin\mathcal{D}_k}} \pi_{k+1+i}(x,\{0,m_1,\dots,m_i\}\cup\mathcal{D}_k). \tag{3.2}$$

By Conjecture 2.2 and Lemma 3.1,

$$\begin{aligned}\sum_{\substack{0<m_1<H\\ m_1\notin\mathcal{D}_k}} &\pi_{k+2}(x,\{0,m_1\}\cup\mathcal{D}_k)\\ &= \sum_{\substack{0<m_1<H\\ m_1\notin\mathcal{D}_k}} \mathfrak{S}(\{0,m_1\}\cup\mathcal{D}_k)\frac{x}{\log^{k+2} x}(1+o(1))\\ &= \mathfrak{S}(\{0\}\cup\mathcal{D}_k)\frac{x}{\log^{k+1} x}\frac{H}{\log x}\left(1+O\left(\frac{d_k^{\epsilon}}{H^{1/2}}\right)+o(1)\right).\end{aligned} \tag{3.3}$$

From (2.1) and Lemma 3.1, we also have, for any $1 \le H \le d_k$,

$$\begin{aligned}\sum_{\substack{0<m_1<m_2<H\\ m_1,m_2\notin\mathcal{D}_k}} &\pi_{k+3}(x,\{0,m_1,m_2\}\cup\mathcal{D}_k)\\ &\ll \sum_{\substack{0<m_1<m_2<H\\ m_1,m_2\notin\mathcal{D}_k}} \mathfrak{S}(\{0,m_1,m_2\}\cup\mathcal{D}_k)\frac{x}{\log^{k+3} x}\\ &\ll \mathfrak{S}(\{0\}\cup\mathcal{D}_k)\frac{x}{\log^{k+1} x}\left(\frac{H}{\log x}\right)^2\left(1+O\left(\frac{d_k^{\epsilon}}{H^{1/2}}\right)\right)^2.\end{aligned} \tag{3.4}$$

In the process of obtaining (3.3) and (3.4), we ignore all the terms with $\mathfrak{S}(\{0,m_1,\dots,m_i\})=0$, since they have $\pi_{k+i}(x,\{0,m_1,\dots,m_i\}\cup\mathcal{D})=0$ or 1 and contribute $\ll H^i$, which is absorbed in the error term.

Substituting (3.3) and (3.4) into (3.2) with $I = 1$, we have

$$\begin{aligned} N_k(x, \mathcal{D}_k) &\le \mathfrak{S}(\{0\} \cup \mathcal{D}_k) \frac{x}{\log^{k+1} x} \biggl\{ 1 - \frac{H}{\log x} \biggl(1 + O\biggl(\frac{d_k^{\epsilon}}{H^{1/2}} \biggr) + o(1) \biggr) \\ &\qquad + O\biggl(\frac{H}{\log x} \biggr)^2 \biggl(1 + O\biggl(\frac{d_k^{\epsilon}}{H^{1/2}} \biggr) \biggr)^2 \biggr\} \\ &\le \mathfrak{S}(\{0\} \cup \mathcal{D}_k) \frac{x}{\log^{k+1} x} \biggl\{ 1 - \frac{H}{\log x} + o\biggl(\frac{H}{\log x} \biggr) \biggr\} \end{aligned}$$

for any H with $\log x/\log\log x \le H = o(\log x)$ and $H \le d_k \le \log^k x$ since ϵ can be chosen arbitrarily small. Hence, we have proved part (ii) of the lemma.

To prove (i), we set $H = d_k$ in (3.3) and (3.4). Since $2 \le d_k = o(\log x)$, (i) follows by substituting (3.3) and (3.4) into (3.1) and (3.2) with $I = 1$. ∎

4. Proof of Theorem 1.3. We will only give the proof for $k \ge 2$, since the case $k = 1$ has been proved by Goldston and Ledoan [GL1].

It is not difficult to see that

$$\begin{aligned} \pi_{k+1}(x, \{0\} \cup \mathcal{D}_k) - \sum_{\substack{d' < d_k \\ d' \notin \mathcal{D}_k}} \pi_{k+2}(x, \{0, d'\} \cup \mathcal{D}_k) &\le N_k(x, \mathcal{D}_k) \\ &\le \pi_{k+1}(x, \{0\} \cup \mathcal{D}_k). \end{aligned}$$

Therefore, by (2.1), since $\mathfrak{S}(\mathcal{D}_n) \le d_n^{\epsilon}$ (the proof is as in Section 4 of [GL1]) and $\pi_n(x, \mathcal{D}_n) = 0$ or 1 for $\mathfrak{S}(\mathcal{D}_n) = 0$, it follows that

$$\sum_{\substack{d' < d_k \\ d' \notin \mathcal{D}_k}} \pi_{k+2}(x, \{0, d'\} \cup \mathcal{D}_k) \ll d_k^{1+\epsilon} \frac{x}{\log^{k+2} x}.$$

Hence, under the assumptions of Theorem 1.3, we have

$$N_k(x, \mathcal{D}_k) = \mathfrak{S}(\{0\} \cup \mathcal{D}_k) \frac{x}{\log^{k+1} x} (1 + o(1)) \tag{4.1}$$

uniformly for $2 \le d \le (\log x)^{1-\epsilon}$, and

$$N_k(x, \mathcal{D}_k) \le \mathfrak{S}(\{0\} \cup \mathcal{D}_k) \frac{x}{\log^{k+1} x} (1 + o(1)) \tag{4.2}$$

uniformly for $2 \le d \le \log^{k+1} x$.

Let

$$\mathcal{P}_n := 2 \cdot 3 \cdot 5 \cdots p_n$$

denote the nth primorial and write $\lfloor y \rfloor$ for the largest primorial not greater than y. Let $\mathcal{K} = \{1, 2, \ldots, k\}$; from (4.1), it follows that

$$\mathfrak{S}(\{0\} \cup \lfloor \log^{1/2} x \rfloor * \mathcal{K}) \frac{x}{\log^{k+1} x}(1 - o(1)) \leq \max_{2 \leq d_k \leq (\log x)^{1-\epsilon}} N_k(x, \mathcal{D}_k) \leq N_k^*(x). \tag{4.3}$$

Here the choice of $\mathcal{K}$ is insignificant. In fact, it can be replaced by any bounded set of k positive integers with gcd $= 1$. On the other hand,

$$N_k(x, \mathcal{D}_k) \leq \sum_{\substack{p_n \leq x \\ p_n - p_{n-k} \geq d_k}} 1 \leq \sum_{\substack{p_n \leq x \\ p_n - p_{n-k} \geq d_k}} \frac{p_n - p_{n-k}}{d_k} \leq \frac{kx}{d_k},$$

and so

$$N_k(x, \mathcal{D}_k) \leq \frac{kx}{\log^{k+1} x} \quad \text{for } d_k \geq \log^{k+1} x.$$

Now

$$\begin{aligned}
\mathfrak{S}(\{0\} \cup \lfloor \log^{1/2} x \rfloor * \mathcal{K}) &\geq \prod_{p \leq (1/2-\epsilon) \log\log x} \left(1 - \frac{1}{p}\right)^{-k} \\
&\quad \cdot \prod_{p \geq (1/2-\epsilon) \log\log x} \left(1 - \frac{1}{p}\right)^{-(k+1)} \left(1 - \frac{k+1}{p}\right) \\
&\gg \prod_{p \leq (1/2-\epsilon) \log\log x} \left(1 - \frac{1}{p}\right)^{-k} \\
&\gg \exp\left(k \sum_{p \leq (1/2-\epsilon) \log\log x} \frac{1}{p} + o(1)\right) \\
&\gg (\log\log\log x)^k
\end{aligned}$$

by an application of Mertens' formula (see Ingham's tract [I, Theorem 7, formula (23), p. 22])

$$\sum_{p \leq x} \frac{1}{p} = \log\log x + B + O\left(\frac{1}{\log x}\right) \quad \text{as } x \to \infty,$$

where B is a constant. By (4.3) for sufficiently large x, if $\mathcal{D}_k$ is a k-tuple jumping champion, then $d_k \leq \log^{k+1} x$.

Let $A > 1$ be any given constant. For $m \geq 3$, let $\mathcal{D}_m = \{d_1, \ldots, d_m\}$ be a set of m distinct integers with $d_1 < \cdots < d_m$ and $d_m \leq \log^A x$. Since

$$\mathcal{D}_m = d * \mathcal{D}'_m$$

with d being the gcd of all elements in $\mathcal{D}_m$, it is obvious that $d'_m < \log^A x$ and

$$2 \leq \nu_{\mathcal{D}'_m}(p) = \nu_{\mathcal{D}_m}(p) \leq k \tag{4.4}$$

for $p \nmid d$. Let

$$\Delta_{\mathcal{D}'_m} = \prod_{j<i} (d'_i - d'_j)$$

and $\Omega(n)$ be the total number of prime divisors of the positive integer n (with multiplicities). Then from the well known fact that for sufficiently large integer n,

$$\Omega(n) \le (1+\epsilon) \log n / \log\log n,$$

we see that, for sufficiently large x,

$$\Omega(\Delta_{\mathcal{D}'_m}) \le \frac{Am(m-1)}{2}(1+\epsilon) \log\log x / \log\log\log x. \tag{4.5}$$

Furthermore, if $\nu_{\mathcal{D}'_m}(p) < m$, this means that $p \mid \Delta_{\mathcal{D}'_m}$. We see that the number of such p with $\nu_{\mathcal{D}'_m}(p) < m$ is not more than $\frac{Am(m-1)}{2}(1+\epsilon)\frac{\log\log x}{\log\log\log x}$ for sufficiently large x. Then, from the definition of $\mathfrak{S}(\mathcal{D}_m)$ and (4.4),

(4.6)

$$\begin{aligned} \mathfrak{S}(\mathcal{D}_m) &= \prod_p \Big(1-\frac{1}{p}\Big)^{-m} \prod_{p \mid d}\Big(1-\frac{1}{p}\Big) \prod_{\substack{p \nmid d \\ p \mid \Delta_{\mathcal{D}'_m}}} \Big(1-\frac{\nu_{\mathcal{D}'_m}(p)}{p}\Big) \prod_{\substack{p \nmid d \\ p \nmid \Delta_{\mathcal{D}'_m}}} \Big(1-\frac{m}{p}\Big) \\ &\le \prod_p \Big(1-\frac{1}{p}\Big)^{-m} \prod_{p \mid d}\Big(1-\frac{1}{p}\Big) \prod_{\substack{p \nmid d \\ p \mid \Delta_{\mathcal{D}'_m}}} \Big(1-\frac{2}{p}\Big) \prod_{\substack{p \nmid d \\ p \nmid \Delta_{\mathcal{D}'_m}}} \Big(1-\frac{m}{p}\Big). \end{aligned}$$

Let d' be the greatest square-free factor of d. It is obvious that $\Omega(\lfloor d \rfloor) \ge \Omega(d')$. Since the combination of the last three products in the last expression of (4.6) is over all primes, we have

(4.7)

$$\begin{aligned} \mathfrak{S}(\mathcal{D}_m) &\le \prod_p \Big(1-\frac{1}{p}\Big)^{-m} \prod_{p \mid d'}\Big(1-\frac{1}{p}\Big) \prod_{\substack{p \nmid d' \\ p \mid \Delta_{\mathcal{D}'_m}}} \Big(1-\frac{2}{p}\Big) \prod_{\substack{p \nmid d' \\ p \nmid \Delta_{\mathcal{D}'_m}}} \Big(1-\frac{m}{p}\Big) \\ &\le \prod_p \Big(1-\frac{1}{p}\Big)^{-m} \prod_{p \le p_{\Omega(d')}} \Big(1-\frac{1}{p}\Big) \prod_{\substack{p > p_{\Omega(d')} \\ p \mid \Delta_{\mathcal{D}'_m}}} \Big(1-\frac{2}{p}\Big) \prod_{\substack{p > p_{\Omega(d')} \\ p \nmid \Delta_{\mathcal{D}'_m}}} \Big(1-\frac{m}{p}\Big) \\ &\le \prod_p \Big(1-\frac{1}{p}\Big)^{-m} \prod_{p \mid \lfloor d \rfloor}\Big(1-\frac{1}{p}\Big) \prod_{\substack{p \nmid \lfloor d \rfloor \\ p \mid \Delta_{\mathcal{D}'_m}}} \Big(1-\frac{2}{p}\Big) \prod_{\substack{p \nmid \lfloor d \rfloor \\ p \nmid \Delta_{\mathcal{D}'_m}}} \Big(1-\frac{m}{p}\Big). \end{aligned}$$

Here, the second inequality in (4.7) holds because we may interchange every prime greater than $p_{\Omega(d')}$ in the second product with a prime less than

$p_{\Omega(d')+1}$ in the last two products with an increase of the value to the formula. These interchanges can be made because the combination of the last three products in the formula is over all primes. The last inequality in (4.7) uses the fact $\Omega(d') \le \Omega(\lfloor d \rfloor)$. Let $\mathcal{M} = \{1, \ldots, m-1\}$; from (4.7) and the inequality $\lfloor d \rfloor \le \lfloor \log^A x \rfloor$, we may have

(4.8)
$$\frac{\mathfrak{S}(\mathcal{D}_m)}{\mathfrak{S}(\{0\} \cup \lfloor \log^A x \rfloor * \mathcal{M})} \le \frac{\prod_{p \mid \lfloor d \rfloor} \left(1 - \frac{1}{p}\right) \prod_{\substack{p \nmid \lfloor d \rfloor \\ p \mid \Delta_{\mathcal{D}'_m}}} \left(1 - \frac{2}{p}\right) \prod_{\substack{p \nmid \lfloor d \rfloor \\ p \nmid \Delta_{\mathcal{D}'_m}}} \left(1 - \frac{m}{p}\right)}{\prod_{p \mid \lfloor \log^A x \rfloor} \left(1 - \frac{1}{p}\right) \prod_{p \nmid \lfloor \log^A x \rfloor} \left(1 - \frac{m}{p}\right)}$$
$$\le \frac{\prod_{\substack{p \nmid \lfloor \log^A x \rfloor \\ p \mid \Delta_{\mathcal{D}'_m}}} \left(1 - \frac{2}{p}\right)}{\prod_{\substack{p \nmid \lfloor \log^A x \rfloor \\ p \mid \Delta_{\mathcal{D}'_m}}} \left(1 - \frac{m}{p}\right)} \le \prod_{\substack{p \nmid \lfloor \log^A x \rfloor \\ p \mid \Delta_{\mathcal{D}'_m}}} \frac{p-2}{p-m}.$$

Then, by the prime number theorem and (4.5), we have

$$\text{(4.9)} \qquad \frac{\mathfrak{S}(\mathcal{D}_m)}{\mathfrak{S}(\{0\} \cup \lfloor \log^A x \rfloor * \mathcal{M})} \le \prod_{\substack{p \nmid \lfloor \log^A x \rfloor \\ p \mid \Delta_{\mathcal{D}'_m}}} \frac{p-2}{p-m}$$
$$\le \prod_{p_{\Omega(\lfloor \log^A x \rfloor)} < p \le p_{\Omega(\lfloor \log^A x \rfloor) + \Omega(\Delta_{\mathcal{D}'_m})}} \frac{p-2}{p-m}$$
$$\le \prod_{(A-\epsilon) \log\log x \le p \le \left(\frac{A + Am(m-1)}{2} + \epsilon\right) \log\log x} \frac{p-2}{p-m}.$$

By the Mertens formula, we can continue this estimate as

$$\le \exp\Bigg(\sum_{(A-1) \log\log x \le p \le \left(\frac{Am(m-1)}{2} + A + 1\right) \log\log x} \log\left(1 + \frac{m-2}{p-m}\right)\Bigg)$$
$$\le \exp\Bigg(\sum_{(A-1) \log\log x \le p \le \left(\frac{Am(m-1)}{2} + A + 1\right) \log\log x} \frac{m-2}{p} + O\bigg(\sum_{n \ge (A-1) \log\log x} \frac{1}{n^2} \bigg)\Bigg)$$
$$\le \exp\left(\log^{m-2}\left(\frac{\log\log\log x + \log\left(\frac{Am(m-1)}{2} + A + 1\right)}{\log\log\log x + \log(A-1)} \right) + O\left(\frac{1}{\log\log\log x} \right)\right)$$
$$\le 1 + O\left(\frac{1}{\log\log\log x} \right).$$

Thus we have

$$\text{(4.10)} \qquad \frac{\mathfrak{S}(\mathcal{D}_m)}{\mathfrak{S}(\{0\} \cup \lfloor \log^A x \rfloor * \mathcal{M})} \le 1 + O\left(\frac{1}{\log\log\log x} \right).$$

From now on, we use $\mathcal{D}_k^*$ to denote a k-tuple jumping champion. Let $p^* < \log x$ be a given prime with $p^* \mid \lfloor \log^{k+1} x \rfloor$ but $p^* \nmid d^*$; it is obvious that $p^* d_k \le \log^{k+2} x$. Then using (4.10) with

$$\mathcal{D}_m = \{0\} \cup p^* * \mathcal{D}_k^* \quad \text{and} \quad A = k+2,$$

we see that

$$\begin{aligned} \text{(4.11)} \qquad \mathfrak{S}(\{0\} \cup \mathcal{D}_k^*)\left(1 + \frac{\nu_{p^*}(\{0\} \cup \mathcal{D}_k'^*) - 1}{p^* - \nu_{p^*}(\{0\} \cup \mathcal{D}_k'^*)}\right) &= \mathfrak{S}(\{0\} \cup p^* * \mathcal{D}_k^*) \\ &\le \mathfrak{S}(\{0\} \cup \lfloor \log^{k+2} x \rfloor * \mathcal{K})\left(1 + O\left(\frac{1}{\log\log\log x}\right)\right). \end{aligned}$$

Here, $\nu_{p^*}(\{0\} \cup \mathcal{D}_k'^*) < p^*$. This is because $\pi_{k+1}(x, \{0\} \cup \mathcal{D}_k^*) = 0$ or 1 if there exists $p \nmid d^*$ with $\nu_p(\{0\} \cup \mathcal{D}_k'^*) = p$, which cannot happen to the k-tuple jumping champion. On the other hand, from (4.2) and (4.3),

$$\begin{aligned} \mathfrak{S}(\{0\} \cup \lfloor \log^{1/2} x \rfloor * \mathcal{K}) \frac{x}{\log^{k+1} x}(1 - o(1)) &\le N_k(x, \mathcal{D}_k^*) \\ &\le \mathfrak{S}(\{0\} \cup \mathcal{D}_k^*) \frac{x}{\log^{k+1} x}(1 + o(1)). \end{aligned}$$

Hence

$$\text{(4.12)} \qquad \frac{\mathfrak{S}(\{0\} \cup \mathcal{D}_k^*)}{\mathfrak{S}(\{0\} \cup \lfloor \log^{1/2} x \rfloor * \mathcal{K})} \ge 1 - o(1).$$

From (4.11) and (4.12) we obtain

$$\begin{aligned} \text{(4.13)} \qquad 1 + &\frac{\nu_{p^*}(\{0\} \cup \mathcal{D}_k'^*) - 1}{p^* - \nu_{p^*}(\{0\} \cup \mathcal{D}_k'^*)} \\ &\le \frac{\mathfrak{S}(\{0\} \cup \lfloor \log^{k+2} x \rfloor * \mathcal{K})}{\mathfrak{S}(\{0\} \cup \lfloor \log^{1/2} x \rfloor * \mathcal{K})} \frac{\mathfrak{S}(\{0\} \cup \lfloor \log^{1/2} x \rfloor * \mathcal{K})}{\mathfrak{S}(\{0\} \cup \mathcal{D}_k^*)}(1 + o(1)) \\ &\le \frac{\mathfrak{S}(\{0\} \cup \lfloor \log^{k+2} x \rfloor * \mathcal{K})}{\mathfrak{S}(\{0\} \cup \lfloor \log^{1/2} x \rfloor * \mathcal{K})}(1 + o(1)), \end{aligned}$$

while

$$\frac{\mathfrak{S}(\{0\} \cup \lfloor \log^{k+2} x \rfloor * \mathcal{K})}{\mathfrak{S}(\{0\} \cup \lfloor \log^{1/2} x \rfloor * \mathcal{K})} \le \sum_{\frac{1}{2}(1-\epsilon)\log\log x \le p \le (k+2)(1+\epsilon)\log\log x} \frac{p-2}{p-(k+1)}.$$

Then an argument similar to the deduction of (4.10) from (4.9) gives

$$\text{(4.14)} \qquad \frac{\mathfrak{S}(\{0\} \cup \lfloor \log^{k+2} x \rfloor * \mathcal{K})}{\mathfrak{S}(\{0\} \cup \lfloor \log^{1/2} x \rfloor * \mathcal{K})} \le 1 + O\left(\frac{1}{\log\log\log x}\right).$$

Therefore, from (4.13) and (4.14) we have

$$1 + \frac{\nu_{p^*}(\{0\} \cup \mathcal{D}_k'^*) - 1}{p^* - \nu_{p^*}(\{0\} \cup \mathcal{D}_k'^*)} \le 1 + o(1)$$

with $2 \le \nu_{p^*}(\{0\} \cup \mathcal{D}_k'^*) \le \min(k+1, p-1)$. This means that $p^* \to \infty$ as $x \to \infty$. Hence any fixed prime p^* must divide every element of all sufficiently large k-tuple jumping champions. Thus we have proved Theorem 1.3.

5. Proof of Theorem 1.4. From Section 4, if $\mathcal{D}_k^*$ is a k-tuple jumping champion, then $d_k^* \le \log^{k+1} x$. Under the assumption of Theorem 1.4, we have $d_k^* = o(\log x)$. To see this, suppose that d_k^* does not satisfy this condition; taking $H = \log x/(\log\log\log x)^{1/2}$ in Lemma 3.2(ii), we have

$$\begin{aligned} &N_k(x, \mathcal{D}_k^*) \\ &\quad \le \mathfrak{S}(\{0\} \cup \mathcal{D}_k^*) \frac{x}{\log^{k+1} x} \left\{ 1 - \frac{1}{(\log\log\log x)^{1/2}} + o\left(\frac{1}{(\log\log\log x)^{1/2}} \right) \right\}. \end{aligned}$$

Then using (4.10) with $A = k+2$ and (4.14), we have

$$\begin{aligned} \mathfrak{S}(\{0\} \cup \mathcal{D}_k^*) &\le \mathfrak{S}(\{0\} \cup \lfloor \log^{1/2} x \rfloor * \mathcal{K}) \frac{\mathfrak{S}(\{0\} \cup \mathcal{D}_k^*)}{\mathfrak{S}(\{0\} \cup \lfloor \log^{k+2} x \rfloor * \mathcal{K})} \\ &\qquad \cdot \frac{\mathfrak{S}(\{0\} \cup \lfloor \log^{k+2} x \rfloor * \mathcal{K})}{\mathfrak{S}(\{0\} \cup \lfloor \log^{1/2} x \rfloor * \mathcal{K})} \\ &\le \mathfrak{S}(\{0\} \cup \lfloor \log^{1/2} x \rfloor * \mathcal{K}) \left\{ 1 + O\left(\frac{1}{\log\log\log x} \right) \right\}. \end{aligned}$$

It is easy to see that $\lfloor \log^{1/2} x \rfloor * k \le k \log^{1/2} x$. Then, from Lemma 3.2(i), we have

$$\begin{aligned} N_k(x, \mathcal{D}_k^*) &\le \mathfrak{S}(\{0\} \cup \lfloor \log^{1/2} x \rfloor * \mathcal{K}) \left(1 - \frac{1}{(\log\log\log x)^{1/2}} (1 + o(1)) \right) \\ &< N_k(x, \lfloor \log^{1/2} x \rfloor * \mathcal{K}), \end{aligned}$$

which contradicts the definition of $\mathcal{D}_k^*$. Hence if $\mathcal{D}_k^*$ is a k-tuple jumping champion, it must satisfy $d_k^* < H = o(\log x)$.

We also have $d_k^* \ge (1-\delta)\log x/(\log\log x)^2$ for any given $\delta > 0$. Indeed, if d_k^* does not satisfy this inequality, from the prime number theorem, we can find a prime $p' \le \log\log x$ with $p' \nmid d^*$. It is obvious that $p' d_k^* \le (1-\delta)\left(\frac{\log x}{\log\log x}\right)$. Since $\nu_{\{0\}\cup\mathcal{D}_k^*}(p') \ge 2$, it is easy to see that

$$\begin{aligned} \frac{\mathfrak{S}(\{0\} \cup \mathcal{D}_k^*)}{\mathfrak{S}(\{0\} \cup p' * \mathcal{D}_k^*)} &= \left(1 - \frac{\nu_{\{0\}\cup\mathcal{D}_k^*}(p')}{p'} \right) \left(1 - \frac{1}{p'} \right)^{-1} \\ &\le 1 - \frac{1}{\log\log x}. \end{aligned}$$

Then, from Lemma 3.2(i), we have

$$\begin{aligned}
N_k(x,\mathcal{D}_k^*) &\le \mathfrak{S}(\{0\}\cup\mathcal{D}_k^*)\frac{x}{\log^{k+1}x}\bigg(1+o\bigg(\frac{1}{(\log\log x)^2}\bigg)\bigg)\\
&\le \mathfrak{S}(\{0\}\cup p' * \mathcal{D}_k^*)\frac{x}{\log^{k+1}x}\bigg(1+o\bigg(\frac{1}{(\log\log x)^2}\bigg)\bigg)\bigg(1-\frac{1}{\log\log x}\bigg)\\
&\le N_k(x,p' * \mathcal{D}_k^*)\bigg(1-\frac{1-\epsilon}{\log\log x}\bigg)\bigg(1-(1+\epsilon)\frac{p' * d_k^*}{\log x}\bigg)^{-1}\\
&\le N_k(x,p' * \mathcal{D}_k^*)\bigg(1-\frac{\delta}{2}\,\frac{1}{\log\log x}\bigg) < N_k(x,p' * \mathcal{D}_k^*),
\end{aligned}$$

which contradicts the definition of $\mathcal{D}_k^*$. Hence $d_k^* \ge (1-\delta)\frac{\log x}{(\log\log x)^2}$ for any given $\delta>0$.

We now prove that d^* is square-free for $(1-\delta)\frac{\log x}{(\log\log x)^2} \le d_k^* = o(\log x)$ with $0<\delta<1$ given. Let p'' be a prime with $p''^2 \mid d^*$ and $\mathcal{D}_k^0 = (1/p'') * \mathcal{D}_k^*$. From Lemma 3.2(i), we have

$$\begin{aligned}
N_k(x,\mathcal{D}_k^*) &= \mathfrak{S}(\{0\}\cup\mathcal{D}_k^*)\frac{x}{\log^{k+1}x}\bigg\{1-\frac{d_k^*}{\log x}+o\bigg(\frac{d_k^*}{\log x}\bigg)+o\bigg(\frac{1}{(\log\log x)^2}\bigg)\bigg\}\\
&= \mathfrak{S}(\{0\}\cup\mathcal{D}_k^0)\frac{x}{\log^{k+1}x}\bigg(1-\frac{d_k^*}{\log x}+o\bigg(\frac{d_k^*}{\log x}\bigg)\bigg)\\
&= N_k(x,\mathcal{D}_k^0)\bigg(1-\frac{d_k^*}{\log x}+o\bigg(\frac{d_k^*}{\log x}\bigg)\bigg)\bigg(1-\frac{d_k^*}{p''\log x}+o\bigg(\frac{d_k^*}{\log x}\bigg)\bigg)^{-1}\\
&\le N_k(x,\mathcal{D}_k^0)\bigg(1-\frac{d_k^*}{3\log x}\bigg) < N_k(x,\mathcal{D}_k^0),
\end{aligned}$$

contrary to the definition of the k-tuple jumping champion. Therefore, we have proved that d^* is square-free and obtained Theorem 1.4.

6. Proof of Lemma 3.1. The original asymptotic formula of the average of the singular series was given by Gallagher [G] who proved that

$$\sum_{\substack{1\le d_1,\dots,d_k\le D\\ \text{distinct}}} \mathfrak{S}(\mathcal{D}_k) \sim D^k.$$

In 2004 Montgomery and Soundararajan [MS] strengthened this by proving that, for a fixed $k\ge 2$,

$$\sum_{\substack{1\le d_1,\dots,d_k\le D\\ \text{distinct}}} \mathfrak{S}(\mathcal{D}_k) = D^k - \binom{k}{2}D^{k-1}\log D + \binom{k}{2}(1-\gamma-\log 2\pi)D^{k-1} + O(D^{k-3/2+\epsilon}),$$

where γ is Euler's constant.

Compared to these formulas which concerned the average of the singular series over all the components of $\mathcal{D}_k$, in order to determine the precise point of transition between jumping champions, Odlyzko, Rubinstein and Wolf [ORW] proved asymptotic formulas for the special type of singular series average,

$$\sum_{1\le d_1<\cdots<d_{k-2}<D} \mathfrak{S}(0, d_1, \ldots, d_{k-2}, D) = \mathfrak{S}(D)\frac{D^{k-2}}{(k-2)!} + R_k(D)$$

with $R_k(D) \ll_k D^{k-2}/\log\log D$. They also presented numerical evidence that suggests that $R_k(D) \ll_k \mathfrak{S}(D)D^{k-3}\log D$. In [GL2], Goldston and Ledoan announced the estimate $R_k(D) \ll_k D^{k-3+\epsilon}$ for any $\epsilon > 0$, but did not give the proof.

In order to prove the jumping champion conjecture, Goldston and Ledoan [GL2] proved the following special type of singular series average, different from the asymptotic formulas given above:

$$\sum_{1\le d_1<\cdots<d_{k-2}<H} \mathfrak{S}(0, d_1, \ldots, d_{k-2}, D) = \mathfrak{S}(D)\frac{H^{k-2}}{(k-2)!}(1 + o(1))$$

for $k \ge 3$ and $D^\epsilon \le H \le D$. In this paper, we have improved this asymptotic formula and actually proved that

$$\sum_{1\le d_1<\cdots<d_{k-2}<H} \mathfrak{S}(0, d_1, \ldots, d_{k-2}, D) = \mathfrak{S}(D)\frac{H^{k-2}}{(k-2)!}\left(1 + O_k\left(\frac{D^\epsilon}{H^{1/2}}\right)\right)$$

for any $H \le D$. This can be deduced easily from Lemma 3.1.

We now come to the proof of Lemma 3.1, which follows Gallagher's method [G].

Proof of Lemma 3.1. First observe that if $\mathfrak{S}(\mathcal{D}_k) = 0$ then $\mathfrak{S}(\mathcal{D}_k \cup d_0) = 0$ and the assertion holds trivially. Therefore, we assume $\mathfrak{S}(\mathcal{D}_k) \ne 0$. Let

$$\mathscr{S}_{d_0} = \frac{\mathfrak{S}(\mathcal{D}_k \cup d_0)}{\mathfrak{S}(\mathcal{D}_k)} = \prod_p (1 + a(p, v_{\mathcal{D}_k\cup\{d_0\}}(p))),$$

where

$$a(p, v_{\mathcal{D}_k\cup\{d_0\}}(p)) = \frac{(v_{\mathcal{D}_k}(p) - v_{\mathcal{D}_k\cup\{d_0\}}(p) + 1)p - v_{\mathcal{D}_k}(p)}{(p - v_{\mathcal{D}_k}(p))(p-1)}.$$

We now let

$$\Delta_{d_0} = \prod_{1\le i\le k} |d_i - d_0|$$

and note that

$$v_{\mathcal{D}_k\cup\{d_0\}}(p) = \begin{cases} v_{\mathcal{D}_k}(p) + 1, & p \nmid \Delta_{d_0}, \\ v_{\mathcal{D}_k}(p), & p \mid \Delta_{d_0}. \end{cases}$$

It follows that

$$(6.1)\qquad a(p, v_{\mathcal{D}_k\cup\{d_0\}}(p)) \ll_k \begin{cases} p^{-2}, & p \nmid \Delta_{d_0}, \\ p^{-1}, & p \mid \Delta_{d_0}, \end{cases}$$

since $v_{\mathcal{D}_k}(p) \le k$ for any p. Hence the product for $\mathscr{S}_{d_0}$ converges. Defining $a_{d_0}(q)$ for square-free q by

$$a_{d_0}(1) = 1, \qquad a_{d_0}(q) = \prod_{p\mid q} a(p, v_{\mathcal{D}_k\cup\{d_0\}}(p)),$$

we get

$$\mathscr{S}_{d_0} = \sum_q \mu^2(q) a_{d_0}(q).$$

It is obvious that the series is convergent.

Let C be a large enough positive constant depending only on k. For large q, putting $q = q_1 q_2$ with $q_1 \mid \Delta_{d_0}$ and $(q_2, \Delta_{d_0}) = 1$, we have

$$\sum_{q>x} \mu^2(q)|a_{d_0}(q)| \le \sum_{q_1\mid \Delta_{d_0}} \frac{\mu^2(q_1) C^{\Omega(q_1)}}{q_1} \sum_{\substack{q_2 > x/q_1 \\ (q_2,\Delta_{d_0})=1}} \frac{\mu^2(q_2) C^{\Omega(q_2)}}{q_2^2}$$

$$\ll \sum_{q_1\mid \Delta_{d_0}} \frac{1}{q_1^{1-\epsilon}} \sum_{\substack{q_2 > x/q_1 \\ (q_2,\Delta_{d_0})=1}} \frac{1}{q_2^{2-\epsilon}} \ll \sum_{q_1\mid \Delta_{d_0}} \frac{1}{q_1^{1-\epsilon}} \frac{q_1^{1-\epsilon}}{x^{1-\epsilon}} \ll (xh)^{\epsilon}/x,$$

with the constant depending only on k and ϵ. It follows that

$$(6.2)\qquad \sum_{\substack{1\le d_0\le H \\ d_0\notin \mathcal{D}}} \mathscr{S}_{d_0} = \sum_{q\le x} \mu^2(q) \sum_{\substack{1\le d_0\le H \\ d_0\notin \mathcal{D}}} a_{d_0}(q) + O(H(xh)^{\epsilon}/x),$$

with the constant depending only on k and ϵ.

The inner sum in (6.2) is

$$\sum_v \prod_{p\mid q} a(p, v(p)) \Big(\sum{}' 1 + O(1)\Big),$$

where $\sum{}' 1$ stands for the number of integers d_0 with $1 \le d_0 \le H$ which, for each prime $p \mid q$, makes $\mathcal{D}_k \cup \{d_0\}$ occupy exactly $v(p)$ residue classes mod p; the outer sum is over all "vectors" $= (\ldots, v(p), \ldots)_{p\mid q}$ with components satisfying $v(p) = v_{\mathcal{D}_k}(p)$ or $v_{\mathcal{D}_k}(p) + 1$. Here the error term $O(1)$ comes from ignoring the condition $d_0 \notin \mathcal{D}_k$. The Chinese remainder theorem gives, for $q \le H$ (we choose $x = H^{1/2} \le H$ at the end, so this condition is satisfied),

$$\sum{}' 1 = \left(\frac{H}{q} + O(1)\right) \prod_{p\mid q} f(p, v(p)),$$

where $f(p,v(p))$ is the number of residue classes of d_0 such that $v_{\mathcal{D}_k\cup\{d_0\}}(p) = v(p)$. It follows that

$$f(p,v(p)) = \begin{cases} v_{\mathcal{D}_k}(p), & v(p) = v_{\mathcal{D}_k}(p), \\ p - v_{\mathcal{D}_k}(p), & v(p) = v_{\mathcal{D}_k}(p) + 1. \end{cases}$$

Thus the inner sum in (6.2) is

$$\left(\frac{H}{q}\right)A(q) + B(q),$$

with

$$A(q) = \sum_v \prod_{p|q} a(p,v(p))f(p,v(p)),$$
$$B(q) = \sum_v \prod_{p|q} |a(p,v(p))|f(p,v(p)) + \sum_v \prod_{p|q} |a(p,v(p))|.$$

We have

$$A(q) = \prod_{p|q}\Big(\sum_{v(p)} a(p,v(p))f(p,v(p))\Big),$$
$$B(q) = \prod_{p|q}\Big(\sum_{v(p)} |a(p,v(p))|f(p,v(p))\Big) + \prod_{p|q}\Big(\sum_{v(p)} |a(p,v(p))|\Big).$$

From the definition of $a(p,v(p))$ and $f(p,v(p))$,

$$\sum_{v(p)} a(p,v(p))f(p,v(p)) = \frac{p - v_{\mathcal{D}_k}(p)}{(p - v_{\mathcal{D}_k}(p))(p-1)}v_{\mathcal{D}_k}(p) + \frac{-v_{\mathcal{D}_k}(p)}{(p - v_{\mathcal{D}_k}(p))(p-1)}(p - v_{\mathcal{D}_k}(p)) = 0.$$

Hence, $A(q) = 0$ for $q > 1$.

Using the bounds (6.1) for $a(p, v_{\mathcal{D}_k\cup\{d_0\}}(p))$ and the definition of $f(p,v(p))$, we have

$$B(q) \le C^{\Omega(q)}.$$

Employing this in (6.2) shows that

$$\sum_{\substack{1\le d_0\le H \\ d_0\notin\mathcal{D}_k}} \mathscr{S}_{d_0} = H + O\Big(\sum_{q\le x} C^{\Omega(q)}\Big) + O(H(xh)^\epsilon/x)$$
$$= H + O(x^{1+\epsilon}) + O(H(hx)^\epsilon/x) = H + O(H^{1/2}h^\epsilon)$$

on choosing $x = H^{1/2}$, and the lemma follows. ∎

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