ACTA ARITHMETICA
156.4 (2012)

Odd parts of tame kernels of dihedral extensions

by

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1. Introduction. Let F be a number field and $\mathcal{O}_F$ its ring of integers. The problem of computing the higher K-groups of F and of $\mathcal{O}_F$ has a rich history (see [1], [5], [6] and [18]). Quillen [18] proved that for all $n > 0$ the K-groups $K_n(\mathcal{O}_F)$ are finitely generated. There are various conjectures about their torsion subgroups. One of them, due to Lichtenbaum, is:

LICHTENBAUM CONJECTURE. For all $n \geq 2$,

$$\zeta_F^*(1-n) = \pm \frac{|K_{2n-2}(\mathcal{O}_F)|}{|K_{2n-1}(\mathcal{O}_F)_{\mathrm{tors}}|} R_n^B(F)$$

up to powers of 2, where $R_n^B(F)$ is the Borel regulator and $\zeta_F^*(1-n)$ is the first non-vanishing coefficient in the Taylor expansion of the zeta function $\zeta_F(s)$ at $s = 1-n$.

For $n = 1$, there is an exact sequence (see [19])

$$0 \to K_2(\mathcal{O}_F) \to K_2(F) \xrightarrow{\oplus\tau_{\mathfrak{p}}} \bigoplus_{\mathfrak{p}\ \mathrm{finite}} k_{\mathfrak{p}}^* \to 0.$$

The map $\oplus\tau_{\mathfrak{p}}$ is explicitly given by the so-called tame symbol, namely, for each prime ideal $\mathfrak{p}$ of $\mathcal{O}_F$, the map

$$\tau_{\mathfrak{p}} : K_2(F) \to k_{\mathfrak{p}}^*$$

defined by

$$\tau_{\mathfrak{p}}(\{a,\ b\}) = (-1)^{v_{\mathfrak{p}}(a)v_{\mathfrak{p}}(b)} a^{v_{\mathfrak{p}}(b)} b^{-v_{\mathfrak{p}}(a)} \bmod \mathfrak{p},$$

where $v_{\mathfrak{p}}$ denotes the $\mathfrak{p}$-adic valuation. Therefore $K_2(\mathcal{O}_F) = \ker(\oplus\tau_{\mathfrak{p}})$, and hence $K_2(\mathcal{O}_F)$ is also called the *tame kernel*.

For a quadratic number field F, the 2-primary part of $K_2(\mathcal{O}_F)$ has been intensively studied (see [4, 12–16]). For an odd prime p, some results on p-primary parts of tame kernels of number fields can be found in [2, 3, 5,

2010 *Mathematics Subject Classification*: 11R70, 12F10.
Key words and phrases: tame kernels, Klein's four group, dihedral extension of number fields.

DOI: 10.4064/aa156-4-3

9–10, 17, 20–22]. Browkin [2] studied tame kernels of cubic cyclic fields with exactly one ramified prime. Li, Qin, Wu and the author obtained some results on tame kernels of cubic and quintic number fields in [10, 17, 20–22].

Throughout the paper we use the following notation:

- C_n is a cyclic group of order n.
- A_n is the alternating group of order $n!/2$.
- D_n is the dihedral group of order $2n$.
- $V_4 = C_2 \times C_2$ is Klein's four group.
- If A is a finite group, then we denote by $|A|$ the number of elements of A.
- Let p be a prime and $a,\ b$ two positive integers. If $a = p^n a_0$, $b = p^n b_0$, a_0 and b_0 are prime to p, then we write $a =_p b$.

Recently, J. Browkin and H. Gangl [3] discussed the following conjecture:

Conjecture. For every number field F, one has

$$|\zeta_F^*(-1)| = \frac{\tilde{R}_2(F)|K_2(\mathcal{O}_F)|}{\omega_2(F)}, \tag{1}$$

where $\tilde{R}_2(F)$ is the second dilogarithmic regulator of F, and $\omega_2(F)$ is the maximal order of the root of unity belonging to the compositum of all quadratic extensions of F.

Let E be a Galois extension of $\mathbb{Q}$ with dihedral Galois group D_p, where p is an odd prime, E_0 the unique quadratic subfield of E, and E_1 a subfield of degree p. Assuming the conjectural formula (1) and applying the Brauer–Kuroda relations between the Dedekind zeta functions of a number field E and of some of its subfields, J. Browkin and H. Gangl [3] proved that if E is not totally real and $\omega_2(E) = \omega_2(E_0) = \omega_2(E_1) = 24$, then

$$|K_2(\mathcal{O}_E)| = \frac{Q_2(E)}{4p}|K_2(\mathcal{O}_{E_1})|^2|K_2(\mathcal{O}_{E_0})| \tag{2}$$

for some $Q_2(E) \in \mathbb{N}$. Moreover, in numerical examples in [3], it is always the case that $Q_2(E) = 1$ or p.

Let E/F be a Galois extension of number fields. In this paper, using some basic properties of the transfer mapping in K-theory, we mainly prove some relations among the orders of odd parts of tame kernels of some subfields of E/F. In Section 2, we obtain two main results, Theorems 1 and 2. For $F = \mathbb{Q}$ and $\mathrm{Gal}(E/F) = V_4$, the following formula can be obtained from Zhou [21]:

$$|K_2(\mathcal{O}_E)| =_p |K_2(\mathcal{O}_{F_0})|\,|K_2(\mathcal{O}_{F_1})|\,|K_2(\mathcal{O}_{F_2})|,$$

where F_i, $i = 0, 1, 2$, are three quadratic subfields of E. Our result in Theorem 1 generalizes the above formula. Let E/F be a dihedral extension with Galois group D_l, where l is an odd prime. We will prove (see Theorem 2) that for every odd prime $p \neq l$,

$$|K_2(\mathcal{O}_E)|\,|K_2(\mathcal{O}_F)|^2 =_p |K_2(\mathcal{O}_K)|^2 |K_2(\mathcal{O}_k)|, \tag{3}$$

where K/F is a subextension of degree l in E/F and k is the unique quadratic subfield. For $F = \mathbb{Q}$, the formula (3) implies that $Q_2(E)$ in (2) has prime factors 2 and p only.

As applications, in Section 3 we give some formulae for the odd parts of the tame kernels of E, F and of some subfields of E over F, when E/F is a Galois extension of number fields with Galois group D_4 or A_4.

2. Main theory. Let E/F be a finite extension of number fields. In [11], the transfer $\mathrm{tr}_{E/F}$ was defined, which is a group homomorphism $\mathrm{tr}_{E/F} : K_2(E) \to K_2(F)$. Now, we recall its basic properties (see [8]) and some well known facts about K_2-groups of a number field and of its ring of integers.

(i) The composite

$$K_2(F) \xrightarrow{j} K_2(E) \xrightarrow{\mathrm{tr}_{E/F}} K_2(F),$$

where j is induced by the inclusion $F \subset E$, is multiplication by the degree of E/F.

(ii) If E/F is a Galois extension with Galois group G, then $j\,\mathrm{tr}_{E/F} = \mathrm{N}_{E/F}$, where $\mathrm{N}_{E/F}$ is the group norm, $\mathrm{N}_{E/F}(x) = \prod_{\sigma\in G} \sigma(x)$.

(iii) If $j : K_2(F) \to K_2(E)$ and $\mathrm{tr} : K_2(E) \to K_2(F)$ are restricted to the groups $K_2(\mathcal{O}_E)$, $K_2(\mathcal{O}_F)$, then the analogues of (i) and (ii) hold for these groups as well.

We denote by $K_2(E/F)$ the kernel of the map $\mathrm{tr}_{E/F} : K_2(\mathcal{O}_E) \to K_2(\mathcal{O}_F)$. The Sylow p-subgroup $K_2(E/F)(p)$ of $K_2(E/F)$ is the kernel of the restriction of $\mathrm{tr}_{E/F}$ to the Sylow p-subgroup $K_2(\mathcal{O}_E)(p)$ of $K_2(\mathcal{O}_E)$. For every prime $p \nmid (E : F)$, the exact sequence

$$1 \to K_2(E/F)(p) \to K_2(\mathcal{O}_E)(p) \xrightarrow{\mathrm{tr}} K_2(\mathcal{O}_F)(p) \to 1$$

splits, and we conclude that $K_2(\mathcal{O}_E)(p) \cong K_2(E/F)(p) \times K_2(\mathcal{O}_F)(p)$.

LEMMA 1. *Let E/F be a Galois extension of number fields and p a prime not dividing $(E : F)$. Then the map $j : K_2(\mathcal{O}_F)(p) \to K_2(\mathcal{O}_E)(p)$ is injective, the transfer $\mathrm{tr}_{E/F} : K_2(\mathcal{O}_E)(p) \to K_2(\mathcal{O}_F)(p)$ is surjective, and $K_2(\mathcal{O}_E)(p) \cong K_2(E/F)(p) \times K_2(\mathcal{O}_F)(p)$.*

Theorem 1. *Let E/F be a Galois extension of number fields with the Galois group $V_4 = \{1, \sigma_0, \sigma_1, \sigma_2\}$ and let $F_i := E^{\sigma_i}$, $i = 0, 1, 2$, be subfields quadratic over F. Then for every odd prime p,*

$$K_2(E/F)(p) \cong K_2(F_0/F)(p) \times K_2(F_1/F)(p) \times K_2(F_2/F)(p), \tag{4}$$

and

$$|K_2(\mathcal{O}_E)|\,|K_2(\mathcal{O}_F)|^2 =_p |K_2(\mathcal{O}_{F_0})|\,|K_2(\mathcal{O}_{F_1})|\,|K_2(\mathcal{O}_{F_2})|. \tag{5}$$

Proof. We shall define a mapping

$$\lambda : K_2(E/F)(p) \to K_2(F_0/F)(p) \times K_2(F_1/F)(p) \times K_2(F_2/F)(p)$$

and prove that it is an isomorphism.

For an odd prime p, squaring is an automorphism of the p-part of any finite abelian group. So for every element $c \in K_2(E/F)(p)$ there is a unique element $d \in K_2(E/F)(p)$ such that $c = d^2$.

From $\mathrm{tr}_{E/F} = \mathrm{tr}_{F_i/F}\,\mathrm{tr}_{E/F_i}$, it follows that $\mathrm{tr}_{E/F_i}(\ker \mathrm{tr}_{E/F}) \subseteq \ker \mathrm{tr}_{F_i/F}$. Hence

$$\mathrm{tr}_{E/F_i}(K_2(E/F)(p)) \subseteq K_2(F_i/F)(p). \tag{6}$$

By assumption, we have $1 = \mathrm{tr}_{E/F}(d) = d^{1+\sigma_0+\sigma_1+\sigma_2}$, and, by (6), $\mathrm{tr}_{E/F_i}(d) = d^{1+\sigma_i} \in K_2(F_i/F)(p)$.

We define

$$\lambda(c) := (d^{1+\sigma_0}, d^{1+\sigma_1}, d^{1+\sigma_2}).$$

Obviously, λ is a homomorphism.

If $\lambda(c) = 1$, then $d^{1+\sigma_0} = d^{1+\sigma_1} = d^{1+\sigma_2} = 1$. Hence

$$c = d^2 = d^{2+1+\sigma_0+\sigma_1+\sigma_2} = d^{(1+\sigma_0)+(1+\sigma_1)+(1+\sigma_2)} = 1.$$

So λ is injective.

For every $b_i \in K_2(F_i/F)(p)$ there exists $d_i \in K_2(F_i/F)(p)$ such that $b_i = d_i^2$, $i = 0, 1, 2$. We have $d^{1+\sigma_i} = d_i^2 = b_i$, since d_i is fixed by σ_i. Moreover, $d_i^{1+\sigma_j} = \mathrm{tr}_{F_i/F}(d_i) = 1$ for $j \neq i$.

Hence taking $d := d_0 d_1 d_2$ and $c := d^2$ we get

$$\lambda(c) = (d^{1+\sigma_0}, d^{1+\sigma_1}, d^{1+\sigma_2}) = (b_0, b_1, b_2),$$

so λ is surjective.

Thus we have proved (4). By (4), we have

$$|K_2(E/F)| =_p |K_2(F_0/F)|\,|K_2(F_1/F)|\,|K_2(F_2/F)|. \tag{7}$$

By Lemma 1, we have $|K_2(\mathcal{O}_E)| =_p |K_2(E/F)|\,|K_2(\mathcal{O}_F)|$ and $|K_2(\mathcal{O}_{F_i})| =_p |K_2(F_i/F)|\,|K_2(\mathcal{O}_F)|$, $i = 0, 1, 2$. Substituting this in (7) proves the theorem.

COROLLARY 1. *Let E/F be a Galois extension of number fields with Galois group $V_4 = \{1, \sigma_0, \sigma_1, \sigma_2\}$ and let $F_i := E^{\sigma_i}$, $i = 0, 1, 2$, be the quadratic subextensions of E/F. Then, for every odd prime p,*

$$K_2(E/F_0)(p) \cong K_2(F_1/F)(p) \times K_2(F_2/F)(p), \tag{8}$$

$$K_2(E/F_1)(p) \cong K_2(F_0/F)(p) \times K_2(F_2/F)(p), \tag{9}$$

$$K_2(E/F_2)(p) \cong K_2(F_0/F)(p) \times K_2(F_1/F)(p). \tag{10}$$

Proof. By Lemma 1, we get $K_2(\mathcal{O}_E)(p) \cong K_2(E/F_0)(p) \times K_2(\mathcal{O}_{F_0})(p)$ and $K_2(\mathcal{O}_{F_0})(p) \cong K_2(F_0/F)(p) \times K_2(\mathcal{O}_F)(p)$. Putting them together gives $K_2(\mathcal{O}_E)(p) \cong K_2(E/F_0)(p) \times K_2(F_0/F)(p) \times K_2(\mathcal{O}_F)(p)$. Comparing this with $K_2(\mathcal{O}_E)(p) \cong K_2(E/F)(p) \times K_2(\mathcal{O}_F)(p)$ we deduce

$$K_2(E/F)(p) \cong K_2(E/F_0)(p) \times K_2(F_0/F)(p).$$

Together with (4) this gives (8) as claimed. The proofs of (9) and (10) are similar.

Let l be an odd prime and let D_l be the dihedral group of order $2l$. Then D_l has a unique subgroup of order l, and l subgroups of order 2.

THEOREM 2. *Let E/F be a dihedral extension with Galois group D_l, k its quadratic subfield fixed by $\langle\sigma\rangle$, K the fixed field of τ, and K' the fixed field of $\sigma\tau$. Then for every odd prime $p \neq l$, we have*

$$K_2(E/k)(p) \cong K_2(K/F)(p) \times K_2(K'/F)(p),$$
$$|K_2(\mathcal{O}_E)|\,|K_2(\mathcal{O}_F)|^2 =_p |K_2(\mathcal{O}_K)|^2 |K_2(\mathcal{O}_k)|.$$

Proof. For $a \in K_2(K/F)(p)$, there exists $b \in K_2(\mathcal{O}_E)(p)$ such that $a = \mathrm{tr}_{E/K}(b) = b\tau(b)$ by Lemma 1. Then

$$\mathrm{N}_{E/F}(b) = \mathrm{tr}_{E/F}(b) = (\mathrm{tr}_{K/F}\,\mathrm{tr}_{E/K})(b) = \mathrm{tr}_{K/F}(a) = 1.$$

So we have $\mathrm{tr}_{E/k}(a) = \mathrm{tr}_{E/k}(b\tau(b)) = \mathrm{N}_{E/F}(b) = 1$. This implies that $a \in K_2(E/k)(p)$. Similarly, $K_2(K'/F)(p)$ can also be considered as a subgroup of $K_2(E/k)(p)$.

Since $\mathrm{tr}_{K/F}\,\mathrm{tr}_{E/K} = \mathrm{tr}_{k/F}\,\mathrm{tr}_{E/k}$ and $\mathrm{tr}_{K'/F}\,\mathrm{tr}_{E/K'} = \mathrm{tr}_{k/F}\,\mathrm{tr}_{E/k}$, we can define the mapping

$$\phi : K_2(E/k)(p) \to K_2(K/F)(p) \times K_2(K'/F)(p)$$

by $\phi(a) = (\mathrm{tr}_{E/K}(a), \mathrm{tr}_{E/K'}(a))$. Clearly, it is a homomorphism. We shall prove that ϕ is injective: If $\mathrm{tr}_{E/K}(a) = \mathrm{tr}_{E/K'}(a) = 1$, then $a\tau(a) = a((\sigma\tau)(a)) = 1$, so $\sigma(a) = a$. Hence, $a^l = \mathrm{tr}_{E/k}(a) = 1$. This implies $a = 1$ since $p \neq l$.

For $c \in K_2(K/F)(p) \cap K_2(K'/F)(p)$, it is clear that c is fixed by τ and by $\sigma\tau$, hence by σ. Since $K_2(K/F)(p)$ and $K_2(K'/F)(p)$ are subgroups of $K_2(E/k)(p)$, we have $c^l = \mathrm{tr}_{E/k}(c) = 1$. This implies

$$K_2(K/F)(p) \cap K_2(K'/F)(p) = 1$$

since $p \neq l$. Therefore

$$|K_2(E/k)(p)| = |K_2(K/F)(p)|\,|K_2(K'/F)(p)|. \tag{11}$$

Since K and K' are isomorphic, we conclude that $|K_2(\mathcal{O}_{K'})(p)| = |K_2(\mathcal{O}_K)(p)|$. This proves the theorem by Lemma 1 and (11).

3. Applications. Let E/F be a number field extension with Galois group D_4 or A_4. By Galois theory, we get the following information about subextensions of E/F.

(I) Let $D_4 = \langle a, x \mid a^4 = x^2 = 1, xax^{-1} = a^{-1}\rangle$. Then there are five subgroups of order 2: $\{e, a^2x\}$, $\{e, x\}$, $\{e, a^2\}$, $\{e, a^3x\}$ and $\{e, ax\}$. The corresponding fixed subfields are respectively K_1', K_1, K, K_2 and K_2'. Since there is only one normal subgroup $\{e, a^2\}$, we have only one Galois subextension K/F of degree 4. Moreover $\{e, a^2x\}$ and $\{e, x\}$ are conjugate subgroups, so K_1 and K_1' are isomorphic subfields over F. Similarly, K_2 and K_2' are isomorphic subfields over F.

There are three subgroups of order 4: $\{e, x, a^2, a^2x\}$, $\{e, ax, a^2, a^3x\}$, and $\{e, a, a^2, a^3\}$. The corresponding fixed subfields are respectively k_1, k_2 and k. Since every quadratic extension is normal, there are three Galois subextensions of degree 2 in E/F.

(II) The alternating group A_4 has the following subgroups except for trivial subgroups:

(a) Three subgroups of order two, each generated by the product of two transpositions. These subgroups are conjugate.
(b) A subgroup of order four, i.e., $\{(1), (12)(34), (13)(24), (14)(23)\}$. It is the Klein four-group V_4. It is a normal subgroup of A_4.
(c) Four subgroups of order three, each generated by a 3-cycle. These subgroups are conjugate.

Hence, the corresponding subfields K_i, $i = 1, 2, 3$, of E in (a) are isomorphic over F, and the corresponding subextension K/F of E/F in (b) is a Galois extension.

PROPOSITION 1. *Let E/F be a dihedral extension with Galois group D_4 described as in* (I). *Then, for any odd prime p,*

$$|K_2(\mathcal{O}_E)|\,|K_2(\mathcal{O}_{k_1})|^2 =_p |K_2(\mathcal{O}_{K_1})|^2 |K_2(\mathcal{O}_K)|, \tag{12}$$

$$|K_2(\mathcal{O}_E)|\,|K_2(\mathcal{O}_{k_2})|^2 =_p |K_2(\mathcal{O}_{K_2})|^2 |K_2(\mathcal{O}_K)|, \tag{13}$$

$$|K_2(\mathcal{O}_K)|\,|K_2(\mathcal{O}_F)|^2 =_p |K_2(\mathcal{O}_{k_1})|\,|K_2(\mathcal{O}_{k_2})|\,|K_2(\mathcal{O}_k)|, \tag{14}$$

$$|K_2(\mathcal{O}_E)|\,|K_2(\mathcal{O}_F)|^2 =_p |K_2(\mathcal{O}_{K_1})|\,|K_2(\mathcal{O}_{K_2})|\,|K_2(\mathcal{O}_k)|. \tag{15}$$

Proof. By (I), we have three Galois subextensions with Galois group V_4 in E/F. They are respectively E/k_1 with subextensions K_1/k_1, K_1'/k_1 and K/k_1; E/k_2 with subextensions K_2/k_2, K_2'/k_2; and K/k_2, and K/F with subextensions k_1/F, k_1/F and k/F. Since K_i and K_i', $i = 1, 2$, are isomorphic fields over F, it is easy to see that $K_2(\mathcal{O}_{K_i})$, $i = 1, 2, 3$, are isomorphic. The first three formulae follow from Theorem 1. Finally, we get the last formula from (12)–(14).

COROLLARY 2. *Let E/F be a dihedral extension with Galois group D_4 denoted as in* (I). *Then for any odd prime p and any integer $i > 0$, we have*

$$p^i\text{-rank}(K_2(\mathcal{O}_{K_1})) - p^i\text{-rank}(K_2(\mathcal{O}_{k_1})) = p^i\text{-rank}(K_2(\mathcal{O}_{K_2})) - p^i\text{-rank}(K_2(\mathcal{O}_{k_2})).$$

Proof. We know that the tame kernels of K_j and K_j' are isomorphic since K_j and K_j' are isomorphic fields over F, where $j = 1, 2$. So for every odd prime p and every integer $i > 0$,

$$p^i\text{-rank}(K_2(\mathcal{O}_{K_j})) = p^i\text{-rank}(K_2(\mathcal{O}_{K_j'})).$$

By Corollary 1, we know that for every odd prime p,

$$\begin{aligned} K_2(K_1/k_1)(p) \times K_2(K_1'/k_1)(p) &\cong K_2(E/K)(p) \\ &\cong K_2(K_2/k_2)(p) \times K_2(K_2'/k_2)(p). \end{aligned}$$

This proves the corollary, by Lemma 1.

EXAMPLE. Let $E = \mathbb{Q}(\sqrt[4]{2}, i)$. It is easy to see that $\mathrm{Gal}(E/\mathbb{Q}) = D_4$, $k_1 = \mathbb{Q}(\sqrt{2})$, $k_2 = \mathbb{Q}(\sqrt{-2})$, $k = \mathbb{Q}(i)$, $K = \mathbb{Q}(\sqrt{2}, i)$, $K_1 = \mathbb{Q}(\sqrt[4]{2})$, $K_1' = \mathbb{Q}(i\sqrt[4]{2})$, $K_2 = \mathbb{Q}((1-i)\sqrt[4]{2})$ and $K_2' = \mathbb{Q}((1+i)\sqrt[4]{2})$. For every odd prime p, we see that $K_2(\mathcal{O}_k)(p) = K_2(\mathcal{O}_{k_1})(p) = K_2(\mathcal{O}_{k_2})(p) = 1$. So $K_2(\mathcal{O}_K)(p)$ is trivial by Theorem 1. By Proposition 1, we have $|K_2(\mathcal{O}_{K_1})| =_p |K_2(\mathcal{O}_{K_2})|$ and

$$|K_2(\mathcal{O}_E)| =_p |K_2(\mathcal{O}_{K_1})|^2 =_p |K_2(\mathcal{O}_{K_2})|^2.$$

PROPOSITION 2. *Let E/F be a Galois extension with Galois group A_4 denoted as in* (II). *Then, for any odd prime p,*

$$|K_2(\mathcal{O}_E)|\,|K_2(\mathcal{O}_K)|^2 =_p |K_2(\mathcal{O}_{K_1})|^3.$$

Proof. This follows from (II) and Theorem 1.

COROLLARY 3. *Let E/F be a Galois extension with Galois group A_4 denoted as in* (II). *Then, for any odd prime p and any integer $i > 0$, we have*

$$p^i\text{-rank}(K_2(\mathcal{O}_E)) = 3p^i\text{-rank}(K_2(\mathcal{O}_{K_1})) - 2p^i\text{-rank}(K_2(\mathcal{O}_K)).$$

Proof. This follows at once from Corollary 1 by observing that the fields K_i, $i = 1, 2, 3$, are isomorphic over F.

Acknowledgements. The author thanks Professors Jerzy Browkin and Herbert Gangl for sending her their paper [3] and the anonymous referees for their very careful reading of the paper and useful comments.

Supported by NSFC 10801076, 10971098, 11071110, Post-Doctor Funds of Jiangsu 1201065C and BK 2010362.

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Received on 5.10.2011
and in revised form on 19.6.2012 (6852)