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Kloosterman sums with prime variable

by

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1. Introduction. We are concerned with the exponential sum

$$S_q(a;x) = \sum_{\substack{x<p\le 2x\\(p,q)=1}} e\left(\frac{a\bar{p}}{q}\right)$$

where $x \ge 2$; $q \ge 2$ is an integer, $(a,q) = 1$ and $\bar{w}$ denotes inverse of w modulo q. As usual, $e(\theta) = e^{2\pi i\theta}$ and $e_q(\theta) = e(\theta/q)$. The sum is taken over primes p.

Using bounds for multidimensional exponential sums coming from algebraic geometry, Fouvry and Michel [3] showed that

$$\sum_{\substack{x<p\le 2x\\(p,q)=1}} e\left(\frac{f(p)}{q}\right) \ll_{f,\varepsilon} q^{3/16+\varepsilon}x^{25/32} \tag{1.1}$$

for q prime, $2 \le x \le q$, and $f(X)$ a rational function with integer coefficients, not of the form $cX + d$. (Values of p with the denominator of $f(p)$ divisible by q are excluded in (1.1).) Here and below, ε denotes an arbitrary positive number, which we may suppose is small. As for the particular case $f(X) = aX^{-1}$, Fouvry and Michel showed that for every $\delta > 0$, there exists $\eta = \eta(\delta) > 0$ such that

$$S_q(a;x) \ll_\delta x^{1-\eta} \tag{1.2}$$

for q prime, $(a,q) = 1$ and $q^{3/4+\delta} \le x \le q$. This was sharpened by Bourgain [2], using an ingenious elementary method that will be discussed below. It is shown in [2] that for every $\delta > 0$, (1.2) holds for q prime, $(a,q) = 1$, some $\eta = \eta(\delta) > 0$ and $q^{1/2+\delta} \le x \le q$.

An effective version of (1.2) has been given by Garaev [5] for prime q and extended to general modulus q by Fouvry and Shparlinski [4]. In [4] it

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is shown that for $q^{3/4} \le x \le q^{4/3}$,

$$S_q(a;x) \ll (x^{15/16} + q^{1/4}x^{2/3})q^{\varepsilon}. \tag{1.3}$$

Fouvry and Shparlinski also give the average bound

$$\sum_{q\sim Q} \max_{(a,q)=1} |S_q(a;x)| \ll (Q^{13/10}x^{3/5} + Q^{13/12}x^{5/6})Q^{\varepsilon} \tag{1.4}$$

for $Q^{3/2} \ge x \ge 1$. We use '$q \sim Q$' as an abbreviation for '$Q < q \le 2Q$'.

We extend Bourgain's result, but with a limitation on the multiplicative structure of q. We shall write, for an integer $q \ge 2$,

$$q = uv, \quad (u,v) = 1,\ u \text{ squarefree},\ v \text{ squarefull}.$$

THEOREM 1. *Let $x \ge 2$, $q \ge 2$ and $v \le x^{1/4}$. Let $0 < \delta \le 1/24$. Then*

$$S_q(a;x) \ll_\delta x^{1-\delta^4/2000} \quad \textit{for } (a,q) = 1 \textit{ and } vq^{1/2+\delta} \le x \le q^{3/4+\delta}.$$

Obviously it would be desirable to reduce the lower bound on x to $q^{1/2+\delta}$. We also give an improvement of (1.3) for part of the range of x, which is nontrivial for $x \ge Q^{1/2+\delta}$.

THEOREM 2. *We have*

$$\sum_{q\sim Q} \max_{(a,q)=1} |S_q(a;x)| \ll (Q^{11/10}x^{4/5} + Qx^{11/12})Q^{\varepsilon} \quad \textit{for } Q^{1/2} \le x \le 2Q.$$

A nice application of (1.4) given in [4] concerns the values of the quadratic form

$$A(X_1, X_2, X_3) = X_1X_2 + X_1X_3 + X_2X_3$$

at prime triplets. We write $P^+(N)$ for the largest prime factor of $N \ge 2$, and $P^+(1) = 1$. Let $\theta_0 = 1.1002\ldots$ be the unique root of the equation

$$13\theta - 16 + 12\log\left(\frac{13\theta - 12}{2}\right) = 0.$$

Then for $\theta < \theta_0$ and $x > x_0(\theta)$,

$$|\{(p_1,p_2,p_3) : p_i \sim x,\ P^+(A(p_1,p_2,p_3)) > x^{\theta}\}| \ge \frac{c(\theta)x^3}{(\log x)^3} \tag{1.5}$$

([4, Corollary 1.6]). Here and below, $|E|$ denotes the cardinality of a finite set E, or the number of elements (counted with multiplicity) of a multiset E.

In the present paper I improve this a little, by applying Theorem 2 and imposing a simple restriction on the set of triples (p_1, p_2, p_3) considered.

THEOREM 3. *Let $\theta_1 = 1.1673\ldots$ be the unique root of the equation*

$$24\theta - 37 + 22\log\left(\frac{12\theta - 11}{2}\right) = 0.$$

Then for any $\theta < \theta_1$, there exists $c(\theta) > 0$ and $x_1(\theta)$ such that (1.5) *holds for $x > x_1(\theta)$.*

2. Proof of Theorem 1. We recall some results about the Fourier transform on the additive group $G := \mathbb{Z}/q\mathbb{Z}$. For $f, g : G \to \mathbb{C}$, let

$$\hat{f}(y) = \sum_{x\in G} f(x)e_q(-xy), \qquad \check{f}(y) = \frac{1}{q}\sum_{x\in G} f(x)e_q(xy),$$
$$(f * g)(y) = \sum_{\substack{x,z\in G\\ x+z=y}} f(x)g(z).$$

It may readily be verified that $(\hat{f})^{\vee} = (\check{f})^{\wedge} = f$, $(f*g)^{\wedge} = \hat{f}\hat{g}$, and

$$\sum_{y\in G} |\hat{f}(y)|^2 = q\sum_{x\in G} |f(x)|^2. \tag{2.1}$$

Let δ_x be the point mass at x. For complex measures

$$\nu = \sum_{x\in G} a(x)\delta_x, \qquad \mu = \sum_{y\in G} b(y)\delta_y,$$

with respective density functions $a(\ldots)$, $b(\ldots)$, we define $\nu * \mu$ to be the measure with density function $a * b$, and define $\hat{\nu} = \hat{a}$, so that $(\nu * \mu)^{\wedge} = \hat{\nu}\hat{\mu}$. We write $\nu^{(k)}$ for the k-fold convolution $\nu * \cdots * \nu$, and $\|\nu\| = \sum_{x\in G} |a(x)|$. Clearly

$$\|\nu * \mu\| \le \|\nu\|\,\|\mu\|.$$

We write χ_E for the indicator function of E.

Lemma 1. *Let $S \subseteq G$. For a measure ν on G,*

$$\nu(S) = \frac{1}{q}\sum_{y\in G} \hat{\nu}(y)\hat{\chi}_{-S}(y). \tag{2.2}$$

Proof. It suffices to prove this for $\nu = \delta_x$. Here the left-hand side of (2.2) is $\chi_S(x)$. The right-hand side is

$$(\hat{\nu}\,\hat{\chi}_{-S})^{\vee}(0) = ((f * \chi_{-S})^{\wedge})^{\vee}(0) = (f * \chi_{-S})(0),$$

where $f(y) = 1$ for $y = x$ and $f(y) = 0$ otherwise. The last expression is

$$\sum_{z+w=0} f(z)\chi_{-S}(w) = \chi_{-S}(-x) = \chi_S(x). \ \blacksquare$$

Lemma 2. *Let p be prime and let $(b_1, \ldots, b_{2k})$ be a $2k$-tuple of integers such that $(b_{k+1}, \ldots, b_{2k})$ is not a permutation of $(b_1, \ldots, b_k)$ modulo p. Then the congruence*

$$(y + b_1)^{-} + \cdots + (y + b_k)^{-} - (y + b_{k+1})^{-} - \cdots - (y + b_{2k})^{-} \equiv 0 \pmod{p}$$

has at most $2k - 1$ solutions in the set $\{y \pmod{p} : (y + b_j, p) = 1\ (j = 1, \ldots, 2k)\}$.

Proof. After removing pairs with $j \le k < h$ for which $b_j \equiv b_h \pmod{p}$ until no such pairs remain, and combining like terms, we must solve

$$\sum_{j\in A} a_j(b_j+y)^- - \sum_{h\in B} c_h(b_h+y)^- \equiv 0 \pmod{p} \tag{2.3}$$

where $A \subseteq \{1,\dots,k\}$, $B \subseteq \{k+1,\dots,2k\}$ are nonempty sets, the integers b_j $(j \in A\cup B)$ are distinct modulo p, and $1 \le a_j, c_h \le k$.

Since the result is trivial for $p \le k$, suppose that $p > k$. We multiply (2.3) by $\prod_{j\in A\cup B}(y+b_j)$, obtaining a polynomial congruence

$$G(y) \equiv 0 \pmod{p}$$

of degree $\le 2k-1$. Since for $j \in A$,

$$G(-b_j) = a_j \prod_{\substack{l\in A\cup B\\ l\ne j}} (b_l - b_j) \not\equiv 0 \pmod{p},$$

G is not identically zero modulo p, and the result follows from Lagrange's theorem. ∎

LEMMA 3. *Let $\mathcal{B}$ be the set of $\boldsymbol{b} = (b_1,\dots,b_{2k})$ in $\mathbb{Z}^{2k}$ with $1 \le b_j \le B$ $(j = 1,\dots,2k)$, where $B \ge 1$. Let $N(\boldsymbol{b},q)$ be the number of solutions y $(\mathrm{mod}\ q)$ of*

$$(y+b_1)^- + \dots + (y+b_k)^- - (y+b_{k+1})^- - \dots - (y+b_{2k})^- \equiv 0 \pmod{q}$$

subject to $(y+b_j,q) = 1$ $(j = 1,\dots,q)$. Then

$$\sum_{\boldsymbol{b}\in\mathcal{B}} N(\boldsymbol{b},q) \ll_{k,\varepsilon} q^{\varepsilon}(B^{2k}v + B^k q).$$

Proof. For each factorization $u = u_1u_2$, let $\mathcal{B}(u_1,u_2)$ be the set of $\boldsymbol{b}$ in $\mathcal{B}$ for which:

- if $p \mid u_1$, $(b_{k+1},\dots,b_{2k})$ is not a permutation of $(b_1,\dots,b_k)$ modulo p;
- if $p \mid u_2$, $(b_{k+1},\dots,b_{2k})$ is a permutation of $(b_1,\dots,b_k)$ modulo p.

It suffices to show that

$$\sum_{\boldsymbol{b}\in\mathcal{B}(u_1,u_2)} N(\boldsymbol{b},q) \ll_{k,\varepsilon} q^{\varepsilon/2}(B^{2k}v + B^k q). \tag{2.4}$$

For $\boldsymbol{b} \in \mathcal{B}(u_1,u_2)$,

$$\begin{aligned} N(\boldsymbol{b},q) &= \Big(\prod_{p\mid u_1} N(\boldsymbol{b},p)\Big) N(\boldsymbol{b},u_2v) \\ &\le u_2v \prod_{p\mid u_1} N(\boldsymbol{b},p) \quad \text{(trivially)} \\ &\le u_2v(2k-1)^{\omega(u_1)} \ll u_2vq^{\varepsilon/4} \end{aligned} \tag{2.5}$$

by Lemma 2. Here $\omega(u_1)$ denotes the number of prime divisors of u_1.

Let $l > k$. Given the first k coordinates of $\boldsymbol{b}$, the number of possibilities for b_l is $\le k^{\omega(u_2)}(B/u_2+1)$, since there are $k^{\omega(u_2)}$ possibilities for $b_l \pmod{u_2}$. Hence,

$$(2.6)\qquad |\mathcal{B}(u_1,u_2)| \ll k^{\omega(u_2)} B^k\left(\frac{B}{u_2}+1\right)^k \ll q^{\varepsilon/4}\left(\frac{B^{2k}}{u_2}+B^k\right).$$

Now (2.4) follows on combining (2.5), (2.6). ∎

In the proofs in the remainder of this section, we sometimes suppose implicitly that q is 'sufficiently large'. An *interval* $I = (a,b]$ denotes $\{x \in \mathbb{Z} : a < x \le b\}$, rather than $\{x \in \mathbb{R} : a < x \le b\}$; similarly for $I = [a,b]$. We write

$$I^* = \{n \in I : (n,q) = 1\}, \qquad -I = \{q-n : n \in I\}$$

and, for $\alpha > 0$,

$$\Omega(I,\alpha) = \Big\{\zeta \in [1,q] : \Big|\sum_{x\in I^*} e_q(\zeta\bar{x})\Big| > |I^*|q^{-\alpha}\Big\}.$$

Lemma 4. *Let* $0 < \alpha \le 1/5$, $I = (c, c+M] \subseteq [1,q]$ *and suppose that* $M \le q^{1/2}$. *Then*

$$|\Omega(I,\alpha)| \ll_\alpha vq^{1+5\alpha^{1/2}}M^{-2}.$$

Proof. Let $\tau = \alpha^{1/2}$ and $k = [1/\tau]+1$. Let us write

$$\Omega = \Omega(I,\alpha), \qquad \mathcal{A} = [1, q^{-2\tau}M]^*.$$

Since the result is trivial for $M < q^{2\tau+\varepsilon}$, we suppose that $M \ge q^{2\tau+\varepsilon}$. It follows that

$$|\mathcal{A}| \gg q^{-2\tau-\varepsilon}M.$$

Let $a \in \mathcal{A}$ and $b \in [1,q^\tau]$. Then

$$\sum_{x\in I^*} e_q(\zeta\bar{x}) = \sum_{\substack{w+ab\in I\\ (w+ab,q)=1}} e_q(\zeta(w+ab)^-) = \sum_{\substack{w\in I\\ (w+ab,q)=1}} e_q(\zeta(w+ab)^-) + O(q^{-\tau}M).$$

Since $\tau > \alpha$,

$$(2.7)\qquad S(\Omega) := \sum_{\zeta\in\Omega}\Big|\sum_{a\in\mathcal{A}}\sum_{1\le b\le q^\tau}\sum_{\substack{w\in I\\ (w+ab,q)=1}} e_q(\zeta(w+ab)^-)\Big| \gg |\Omega| q^{-\tau-\alpha-2\varepsilon}M^2.$$

Now

$$S(\Omega) \le \sum_{\zeta\in\Omega}\sum_{a\in\mathcal{A}}\sum_{w\in I}\Big|\sum_{\substack{1\le b\le q^\tau\\ (w+ab,q)=1}} e_q(\zeta(w+ab)^-)\Big| = \sum_{1\le y,z\le q} \mu(y,z)\Big|\sum_{\substack{1\le b\le q^\tau\\ (y+b,q)=1}} e_q(z(y+b)^-)\Big|, \tag{2.8}$$

where

$$\mu(y,z) = |\{(w,a,\zeta)\in I\times\mathcal{A}\times\Omega : \bar a\zeta \equiv z \pmod q,\ \bar a w\equiv y \pmod q\}|.$$

By Hölder's inequality, the last expression in (2.8) is at most

$$\Big(\sum_{1\le y,z\le q}\mu(y,z)\Big)^{1-1/k}\Big(\sum_{1\le y,z\le q}\mu(y,z)^2\Big)^{1/2k} \times\Big(\sum_{1\le y,z\le q}\Big|\sum_{\substack{1\le b\le q^\tau\\ (y+b,q)=1}} e_q(z(y+b)^-)\Big|^{2k}\Big)^{1/2k}. \tag{2.9}$$

Clearly

$$\sum_{1\le y,z\le q}\mu(y,z) \ll M|\mathcal{A}|\,|\Omega| \ll q^{-2\tau}M^2|\Omega|. \tag{2.10}$$

Now

$$\sum_{1\le y,z\le q}\mu(y,z)^2 = |\{(w_1,a_1,\zeta_1,w_2,a_2,\zeta_2) : w_j\in I,\ a_j\in\mathcal{A},\ \zeta_j\in\Omega,\ \bar a_1\zeta_1\equiv \bar a_2\zeta_2 \pmod q,\ \bar a_1 w_1\equiv\bar a_2 w_2 \pmod q\}|. \tag{2.11}$$

The contribution to the right-hand side of (2.11) from tuples with $w_1 = w_2$ is

$$\ll q^{-2\tau+\varepsilon}M^2|\Omega|. \tag{2.12}$$

To see this, let d be a divisor of q. It suffices to give the bound

$$\ll q^{-2\tau}M^2|\Omega|$$

for the contribution from $w_1 = w_2$, $(w_1,q) = d$. There are $\le M/d+1$ possibilities for w_1. Once w_1 is fixed, the congruence

$$a_1w_1 \equiv a_2w_1 \pmod q$$

implies $a_1\equiv a_2 \pmod{q/d}$, and there are $\le q^{-2\tau}M(1+q^{-2\tau}Md/q)$ possible pairs a_1, a_2. Once a_1, a_2 are fixed, we have $a_1\zeta_2\equiv a_2\zeta_1 \pmod q$, and there are $|\Omega|$ possible pairs (ζ_1,ζ_2). Thus the number of tuples $(a_1,w_1,\zeta_1,a_2,w_1,\zeta_2)$ in question is

$$\le \Big(\frac{M}{d}+1\Big)\Big(1+\frac{q^{-2\tau}Md}{q}\Big)q^{-2\tau}M|\Omega|.$$

Since

$$\left(\frac{M}{d}+1\right)\left(1+\frac{q^{-2\tau}Md}{q}\right) \ll M + q^{-2\tau-1}M^2 \ll M,$$

we have verified the bound (2.12).

To estimate the contribution to the right-hand side of (2.11) from tuples with $w_1 \neq w_2$, we fix the values of $a = a_1 - a_2$, w_1 and ζ_1. We have $a_2 w_1 \equiv a_1 w_2 \pmod{q}$, hence

$$a_1(w_1 - w_2) \equiv a w_1 \pmod{q}.$$

Since $0 < |a_1(w_1 - w_2)| \le q^{-2\tau}M^2 < q$, this determines $a_1(w_1 - w_2)$, and in turn determines a_1 and w_2 to within $O(q^{\varepsilon})$ possibilities. Now a_2 is determined by $a_2 = a_1 - a$, and ζ_2 is determined by $a_1\zeta_2 \equiv a_2\zeta_1 \pmod{q}$. It follows that

$$\sum_{1\le y,z\le q} \mu(y,z)^2 \ll q^{-2\tau+\varepsilon} M^2 |\Omega|. \tag{2.13}$$

We rewrite the last factor F in (2.9) as

$$\begin{aligned} F^{2k} &= \sum_{1\le y\le q} \sum_{\substack{1\le b_1,\dots,b_{2k}\le q^{\tau}\\ (y+b_j,q)=1\ (j=1,\dots,q)}} \sum_{1\le z\le q} e_q(z((y+b_1)^- + \dots + (y+b_k)^- \\ &\qquad\qquad - (y+b_{k+1})^- - \dots - (y+b_{2k})^-)) \\ &= \sum_{1\le b_1,\dots,b_{2k}\le q^{\tau}} qN(\boldsymbol{b},q) \\ &\ll q^{1+\varepsilon}(q^{2k\tau}v + q^{1+k\tau}) \quad \text{(by Lemma 2)} \\ &\ll q^{1+2k\tau+\varepsilon}v \end{aligned}$$

by the choice of k. Combining this with (2.8), (2.9), (2.10), (2.12), we obtain

$$S(\Omega) \ll q^{\varepsilon}(q^{-2\tau}M^2\Omega)^{1-1/2k}(vq^{1+2k\tau})^{1/2k}.$$

In conjunction with (2.7), this gives

$$q^{-\tau-\alpha-2\varepsilon}M^2|\Omega| \ll q^{\varepsilon}(q^{-2\tau}M^2|\Omega|)^{1-1/2k}(vq^{1+2k\tau})^{1/2k},$$
$$M^2|\Omega| \ll vq^{2k\alpha+2\tau+1+6k\varepsilon}.$$

Since $2k\alpha + 2\tau < 4\alpha^{1/2} + 2\alpha < 5\alpha^{1/2} - 6k\varepsilon$, the lemma follows. ∎

LEMMA 5. *Let ν be the measure*

$$\nu = \frac{1}{|A|}\sum_{x\in A}\delta_x$$

where $A \subseteq G$. Let $0 < \alpha \le 1/3$, $q > 32$ and let l be an integer, $l > 1/\alpha$; let

$$B = B(\nu,\alpha) = \{\zeta \in G : |\hat{\nu}(\alpha)| > q^{-\alpha}\}.$$

Then for any set $S \subseteq G$ *with*

$$|B(\nu,\alpha)|\,|S| < \frac{1}{2}\,q^{1-\alpha}, \tag{2.14}$$

we have

$$\nu^{(l)}(S) < q^{-\alpha}. \tag{2.15}$$

Proof. Suppose that (2.15) is false; then

$$\frac{1}{q}\sum_{y\in G}\hat{\nu}(y)^l\hat{\chi}_{-S}(y) \ge q^{-\alpha} \tag{2.16}$$

by Lemma 1, while

$$\Big|\sum_{y\notin B}\hat{\nu}(y)^l\hat{\chi}_{-S}(y)\Big| \le q^{-\alpha l}\sum_{y\in G}|\hat{\chi}_{-S}(y)| \le q^{1-\alpha l}|S|^{1/2} \le q^{3/2-\alpha l} \tag{2.17}$$

by (2.1) and Cauchy's inequality. Since $q^{3/2-\alpha l} < q^{1/2} < \frac{1}{2}q^{1-\alpha}$, we deduce from (2.16) and (2.17) that

$$\Big|\sum_{y\in B}\hat{\nu}(y)^l\hat{\chi}_{-S}(y)\Big| > \frac{1}{2}q^{1-\alpha}.$$

Moreover,

$$\sum_{y\in B}\hat{\nu}(y)^l\hat{\chi}_{-S}(y) = \sum_{y\in G}\hat{\nu}(y)^l(\hat{\chi}_{-S}\chi_B)(y) = \sum_{y\in G}\hat{\nu}(y)^l(\chi_{-S}*\check{\chi}_B)^\wedge(y).$$

For convenience, write $\nu^{(l)} = \sum_{z\in G} b(z)\delta_z$; then $\sum_{z\in G}|b(z)| \le \|\nu\|^l = 1$. We have shown that

$$\Big|\sum_{y\in G}(b*\chi_{-S}*\check{\chi}_B)^\wedge(y)\Big| \ge \frac{1}{2}q^{1-\alpha}. \tag{2.18}$$

The left-hand side of (2.18) is

$$\begin{aligned} q|(b*\chi_{-S}*\check{\chi}_B)(0)| &= q\sum_{u+v+w=0} b(u)\chi_{-S}(v)\check{\chi}_B(w)\\ &\le q\|\check{\chi}_B\|_\infty\sum_{u\in G}|b(u)|\sum_{v\in G}|\chi_{-S}(v)| \le |B|\,|S|, \end{aligned}$$

and (2.14) is false. ∎

Lemma 6. *Let* $0 < \alpha \le 1/5$, $l > 1/\alpha$, $I = (c, c+M] \subseteq [1,q]$ *where* $M \le q^{1/2}$. *Let* $S \subseteq G$ *and*

$$|S| \ll_\alpha v^{-1}q^{-6\alpha^{1/2}}M^2. \tag{2.19}$$

Let

$$\nu_1 = \frac{1}{|I^*|}\sum_{x\in I^*}\delta_{\bar{x}}, \qquad \nu_2 = \frac{1}{|I^*\cup(-I)^*|}\sum_{x\in I^*\cup(-I)^*}\delta_{\bar{x}}.$$

Then $\nu_1^{(l)}(S) \ll_\alpha q^{-\alpha}$. *If* $I \cap (-I) = \emptyset$, *then*

$$\nu_2^{(l)}(S) \ll_\alpha q^{-\alpha}.$$

Proof. We take $A = \{\bar{x} : x \in I^*\}$, $\nu = \nu_1$ in Lemma 5. Then

$$B(\nu_1, \alpha) = \Omega(I, \alpha), \qquad |B(\nu_1, \alpha)| \ll vq^{1+5\alpha^{1/2}} M^{-2}$$

from Lemma 4. Since we may suppose that q is large,

$$|B(\nu_1, \alpha)|\,|S| < q^{1-\alpha^{1/2}+\varepsilon} < \frac{1}{2} q^{1-\alpha}.$$

Now $\nu_1^{(l)}(S) < q^{-\alpha}$ from Lemma 5.

Let $\nu_3 = |(-I)^*|^{-1} \sum_{x \in (-I)^*} \delta_{\bar{x}}$; then $\hat{\nu}_3 = \bar{\hat{\nu}}_1$. Assume now $I \cap (-I) = \emptyset$; then $\nu_2 = \frac{1}{2}(\nu_1 + \nu_3)$, $\hat{\nu}_2 = \operatorname{Re} \nu_1$, and $B(\nu_2, \alpha) \subseteq \Omega(I, \alpha)$. We can complete the proof for ν_2 as before, and the lemma follows. ∎

Lemma 7. *Let* $I = (c, c+M]$, $J = (d, d+N]$ *be intervals in* $[1, q]$, *with* $J \cap (-J) = \emptyset$. *Let*

$$\nu = \frac{1}{|J^* \cup (-J)^*|} \sum_{x \in J^* \cup (-J)^*} \delta_{\bar{x}},$$

$$S(I, J) = \sum_{m \in I^*} \sum_{n \in J^*} \alpha_m \beta_n e_q(a \bar{m} \bar{n}) \tag{2.20}$$

where $|\alpha_m| \le 1$, $|\beta_n| \le 1$. *Then for any even natural number* k, *and any* $\alpha > 0$,

$$|S(I, J)|^{4k} \le (MN)^{4k} \{q^{-\alpha} + 2^{2k} \nu^{(2k)}(\Omega(I, \alpha))\}.$$

Proof. Let $k = 2h$. By Cauchy's inequality,

$$\begin{aligned} |S(I, J)|^2 &\le M \sum_{m \in I^*} \Big| \sum_{n \in J^*} \beta_n e_q(a \bar{m} \bar{n}) \Big|^2 \\ &= M \sum_{n_1, n_2 \in J^*} \beta_{n_1} \bar{\beta}_{n_2} \sum_{m \in I^*} e_q(a \bar{m}(\bar{n}_1 - \bar{n}_2)) \\ &\le M \sum_{n_1, n_2 \in J^*} \Big| \sum_{m \in I^*} e_q(a \bar{m}(\bar{n}_1 - \bar{n}_2)) \Big|. \end{aligned}$$

Using Cauchy's inequality again leads to

$$|S(I, J)|^4 \le M^2 N^2 \sum_{m_1, m_2 \in I^*} \Big| \sum_{n_1, n_2 \in J^*} e_q(a(\bar{m}_1 - \bar{m}_2)(\bar{n}_1 - \bar{n}_2)) \Big|.$$

By Hölder's inequality,

$$|S(I, J)|^{8h} \le M^{8h-2} N^{4h} \sum_{m_1, m_1 \in I^*} \sum_{n_1, n_2 \in J^*} \cdots \sum_{n_{4h-1}, n_{4h} \in J^*} e_q(a(\bar{m}_1 - \bar{m}_2)(\bar{n}_1 - \bar{n}_2 + \cdots - (\bar{n}_{4h-1} - \bar{n}_{4h}))).$$

Renumbering the variables $n_1, \ldots, n_{4h}$, and treating the variable m_2 trivially, we obtain

$$(2.21)\qquad |S(I,J)|^{4k} \le M^{4k-1}N^{2k} \times \sum_{(n_1,\ldots,n_{2k})\in(J^*)^{2k}} \Big|\sum_{m_1\in I^*} e_q(a\bar{m}_1(\bar{n}_1+\cdots+\bar{n}_k-(\bar{n}_{k+1}+\cdots+\bar{n}_{2k}))\Big|.$$

For brevity, let

$$\Omega = \Omega(I,\alpha), \qquad T = \{\bar{x} : x \in J^* \cup (-J)^*\}.$$

We partition $(J^*)^{2k}$ into two sets $\mathcal{A}_1$, $\mathcal{A}_2$, where

$$\mathcal{A}_1 = \{(n_1,\ldots,n_{2k}) \in (J^*)^{2k} : \bar{n}_1+\cdots+\bar{n}_k-(\bar{n}_{k+1}+\cdots+\bar{n}_{2k}) \in \Omega\}.$$

The contribution to the right-hand side of (2.21) from $(n_1,\ldots,n_{2k})$ in $\mathcal{A}_2$ is

$$\le M^{4k-1}N^{4k}Mq^{-\alpha} = (MN)^{4k}q^{-\alpha}.$$

We also observe that

$$\nu^{(2k)}(\Omega) = \frac{1}{|T|^{2k}} \sum_{\substack{(z_1,\ldots,z_{2k})\in T^{2k}\\ z_1+\cdots+z_{2k}\in\Omega}} 1 \ge \frac{1}{|T|^{2k}}\,|\mathcal{A}_1|.$$

Accordingly, the contribution to the right-hand side of (2.21) from $(n_1,\ldots,n_{2k})$ in $\mathcal{A}_2$ is

$$\le M^{4k}N^{2k}|T|^{2k}\nu^{(2k)}(\Omega),$$

and the lemma follows. ∎

LEMMA 8. *Let $0<\delta\le 1/3$. Make the hypothesis of Lemma 7 and suppose in addition that $v \le q^{1/4}$ and*

$$(2.22)\qquad vq^\delta \ll |I| \ll |J|, \qquad |I|\,|J| \gg vq^{1/2+\delta}.$$

Then

$$(2.23)\qquad S(I,J) \ll |I|\,|J|q^{-\delta^4/2100}.$$

Proof. If $|J| > q^{1/2}$, we partition J into intervals of length between $\frac{1}{2}q^{1/2}$ and $q^{1/2}$, and similarly for I. A pair of intervals I', J' obtained in this way satisfies

$$vq^\delta \ll |I'| \ll |J'| \ll q^{1/2}, \qquad |I'|\,|J'| \gg vq^{1/2+\delta}.$$

It now suffices to prove (2.21) for I', J' in place of I, J. Thus we may add to (2.22) the hypothesis

$$|J| \le q^{1/2}.$$

Let $\alpha = \delta^2/32$, $k = [16/\delta^2]+j$, where $j=1$ or 2 is chosen to produce even k. Then

$$\frac{\alpha}{4k} \ge \frac{\delta^2}{128(16\delta^{-2}+2)} = \frac{\delta^4}{2024+256\delta^2} \ge \frac{\delta^4}{2100}.$$

In view of Lemma 7, it suffices to show that

$$\nu^{(2k)}(\Omega(I,\alpha)) \ll q^{-\alpha},$$

where ν is given by (2.20).

We are going to apply Lemma 6 with $2k$, J, $\Omega(I,\alpha)$ in place of l, I, S. The hypothesis (2.19) is satisfied, since

$$M^2N^2 \geq v^2 q^{1+2\delta} \geq v^2 q^{1+11\alpha^{1/2}}$$

and

$$|\Omega(I,\alpha)| \ll vq^{1+5\alpha^{1/2}} M^{-2} \ll v^{-1} q^{-6\alpha^{1/2}} N^2$$

by Lemma 4. We conclude that (2.23) holds. ∎

In [2], Lemma A.7 corresponds to Lemma 8 above. The author of [2] has inadvertently omitted to assume any lower bound on $|I|$ ($|I_1|$ in his notation), but it is implicit in his proof of Lemma A.7, being required to get a suitable lower bound for the quantity $|I'|\,|J'|$. The reader will easily see that Lemma 8 would not be true without a lower bound on $|I|$.

Proof of Theorem 1. We begin by recalling some facts from Heath-Brown's decomposition [6] of $\Lambda(n)$. A function $f(n)$ on $K = (x,(1+\beta)x]$ is given, where $0 < \beta \leq 1$. The decomposition enables us to express $\sum_{r\in K,\,(r,q)=1} \Lambda(r)f(r)$ as a sum of $O((\log x)^6)$ sums S_{I}, S'_{I}, S_{II}. Here

(2.24)
$$S_{\mathrm{I}} = S_{\mathrm{I}}(q,a) = \sum_{\substack{m\sim N\\ mn\in K\\ (mn,q)=1}}\sum_{n\sim N} a_m f(mn), \qquad S_{\mathrm{I}'} = \sum_{\substack{m\sim N\\ mn\in K\\ (mn,q)=1}}\sum_{n\sim N} (\log n) a_m f(mn),$$

with $a_m \ll x^{\varepsilon}$ for every $\varepsilon > 0$, and $MN \asymp x$, $N \gg x^{1-\lambda}$; while

$$S_{\mathrm{II}} = S_{\mathrm{II}}(q,a) = \sum_{\substack{m\sim N\\ mn\in K}} a_m \sum_{n\sim N} b_n f(mn) \tag{2.25}$$

with $a_m, b_n \ll x^{\varepsilon}$ for every $\varepsilon > 0$; $MN \asymp x$, $x^{\lambda} \ll N \ll x^{1/2}$. Here the parameter λ in $\big(0,1/3\big]$ is at our disposal. See [1] for a discussion of an almost identical situation. We can reduce S'_{I} to S_{I} (with a different β) by partial summation. Let $\delta_1 = 99\delta/100$. For the proof of Theorem 1 we take $f(r) = e_q(a\bar{r})$, $K = (x,x']$, $x' \leq 2x$, $x^{\lambda} = vq^{\delta_1} \leq x^{1/3}$ (since $v \leq x^{1/4}$, $q \leq x^2$, $\delta \leq 1/24$). We shall show that S_{I}, S_{II} are $O(x^{1-\delta^4/2000-\varepsilon})$, leading to a suitable bound for $\sum_{x<r\leq x',\,(r,q)=1} \Lambda(r)e_q(ar)$. The corresponding bound for $S_q(a;x)$ follows easily.

Lemma 8, with δ_1 in place of δ, gives

$$S_{\mathrm{II}} \ll xq^{-\delta_1^4/2100+\varepsilon} \ll x^{1-\delta^4/2000-\varepsilon}$$

for $vq^{\delta_1} \ll N \ll x(vq^{\delta_1})^{-1}$. This requires a short calculation: we have $\delta_1^4 \ge \frac{96}{100}\delta^4$, $x \le q^{19/24}$ and

$$x^{\delta^4/2000} \le q^{\frac{19}{24}\frac{100}{96}\frac{\delta_1^4}{2000}} \le q^{\delta_1^4/2100-2\varepsilon}.$$

It remains to show that

$$S_{\mathrm{I}} \ll x^{1-\delta^4/2000-2\varepsilon} \qquad \text{for } N \gg \frac{x}{vq^{\delta_1}}.$$

We note that

$$N \gg \frac{x}{vq^{\delta_1}} \gg q^{1/2+\delta/100}.$$

By a standard estimate (see e.g. [4, Lemma 2.1]), the inner sum in S_{I} is $\ll q^{1/2+\varepsilon}$. Hence

$$S_{\mathrm{I}} \ll Mq^{1/2+\varepsilon} \ll x\,\frac{q^{1/2+\varepsilon}}{N} \ll xq^{-\delta/200}.$$

This completes the proof of Theorem 1. ∎

3. Proof of Theorem 2. In the present section, we suppose that Q is large and $Q^{1/2} \le x \le 2Q$. It is convenient for use in Section 4 to work with the sum

$$S_q(a; x, \beta) = \sum_{x<p\le(1+\beta)x} e\left(\frac{a\bar{p}}{q}\right)$$

where β is a constant in $(0,1]$. Define S_{I} and S_{II} by (2.24), (2.25) with $f(r) = e_q(a\bar{r})$. We now take $\lambda = 1/3$ in our application of Heath-Brown's identity. Thus in order to show that

$$\sum_{q\sim Q} \max_{(a,q)=1} |S_q(a; x, \beta)| \ll (Q^{11/10}x^{4/5} + Qx^{11/12})Q^{\varepsilon}, \tag{3.1}$$

it is sufficient to show that

$$\sum_{q\sim Q} \max_{(a,q)=1} |S_{\mathrm{I}}(q, a)| \ll (Q^{11/10}x^{4/5} + Qx^{11/12})Q^{\varepsilon/2} \tag{3.2}$$

whenever $N \gg x^{2/3}$, and that

$$\sum_{q\sim Q} \max_{(a,q)=1} |S_{\mathrm{II}}(q, a)| \ll (Q^{11/10}x^{4/5} + Qx^{11/12})Q^{\varepsilon/2} \tag{3.3}$$

whenever $x^{1/2} \ll N \ll x^{2/3}$.

Let $J_K(q)$ denote the number of solutions of the congruence

$$\bar{n}_1 + \bar{n}_2 \equiv \bar{n}_3 + \bar{n}_4 \pmod{q} \quad \text{with } 1 \le n_i \le K.$$

LEMMA 9. *For $M \le q$, $N \le q$, $(a,q) = 1$ we have*

$$S_{\mathrm{II}}(q,a) \ll q^{1/8+\varepsilon/4}(MN)^{1/2} J_M(q)^{1/8} J_N(q)^{1/8}.$$

Proof. See Garaev [5]. The restriction to prime q in [5] plays no role in the argument. ∎

LEMMA 10. *We have, for $K \ge 1$,*

$$\sum_{q\sim Q} J_K(q) \ll (K^2 Q + K^4) K^{\varepsilon}.$$

Proof. This is Lemma 2.3 of [4]. ∎

LEMMA 11. *Let $M \le N \le Q$, $MN \asymp x$. We have*

$$\sum_{q\sim Q} \max_{(a,q)=1} |S_{\mathrm{II}}(q,a)| \ll Q^{\varepsilon/2}(Q^{9/8}x^{3/4} + Qx^{3/4}N^{1/4}).$$

Proof. By Hölder's inequality and Lemma 10,

$$\begin{aligned}\sum_{q\sim Q} J_M(q)^{1/8} J_N(q)^{1/8} &\le Q^{3/4}\Big(\sum_{q\sim Q} J_M(q)\Big)^{1/8}\Big(\sum_{q\sim Q} J_N(q)\Big)^{1/8}\\ &\ll Q^{3/4+\varepsilon/4}(M^{1/4}Q^{1/8} + M^{1/2})(N^{1/4}Q^{1/8} + N^{1/2})\\ &\ll Q^{3/4+\varepsilon/4}(x^{1/4}Q^{1/4} + x^{1/4}N^{1/4}Q^{1/8})\end{aligned}$$

since $x^{1/2} \le 2Q^{1/4}x^{1/4}$. Combining this with Lemma 9, we get

$$\begin{aligned}\sum_{q\sim Q} \max_{(a,q)=1} |S_{\mathrm{II}}(q,a)| &\ll Q^{1/8+\varepsilon/2}x^{1/2} \sum_{q\sim Q} J_M(q)^{1/8} J_N(q)^{1/8}\\ &\ll Q^{\varepsilon/2}(Q^{9/8}x^{3/4} + Qx^{3/4}N^{1/4}).\ ∎\end{aligned}$$

Proof of Theorem 2. We begin by showing that (3.2) holds for $N \gg x^{2/3}$. We distinguish two cases.

CASE 1: $N > Q^{2/5}x^{1/5}$. For each $m \sim M$, $(m,q) = 1$, we have the estimate

$$\sum_{\substack{n\sim N\\ x/m<n\le(1+\beta)x/m\\ (n,q)=1}} e_q(a\bar{m}\bar{n}) \ll q^{1/2+\varepsilon/2}$$

for $q \ge 1$, $(a,q) = 1$, as noted earlier. Thus for $q \sim Q$, $(a,q) = 1$,

$$S_{\mathrm{I}}(q,a) \ll N^{-1}xq^{1/2+\varepsilon/2} \ll Q^{1/10+\varepsilon/2}x^{4/5}$$

and

$$\sum_{q\sim Q} \max_{(a,q)=1} |S_{\mathrm{I}}(q,a)| \ll Q^{11/10+\varepsilon/2}x^{4/5}.$$

CASE 2: $x^{2/3} \ll N \le Q^{2/5}x^{1/5}$. (This case occurs only if $Q \gg x^{7/6}$.) We observe that $N \le Q$. By Lemma 11,

$$\sum_{q\sim Q} \max_{(a,q)=1} |S_{\mathrm{I}}(q,a)| \ll Q^{\varepsilon/2}(Q^{9/8}x^{3/4} + Q^{11/10}x^{4/5}) \ll Q^{\varepsilon/2}(Q^{11/10}x^{4/5} + Qx^{11/12})$$

since $Q^{9/8}x^{3/4} \le Q^{11/10}x^{4/5}$ for $Q \le x^2$.

Thus (3.2) holds in both cases.

It remains to prove (3.3). Let $MN \asymp x$, $M \le N$, $x^{1/2} \ll N \ll x^{2/3}$; then $M \le N \le Q$. From Lemma 11,

$$\sum_{q\sim Q} \max_{(a,q)=1} |S_{\mathrm{II}}(q,a)| \ll Q^{\varepsilon/2}(Q^{9/8}x^{3/4} + Qx^{11/12}) \ll Q^{\varepsilon/2}(Q^{11/10}x^{4/5} + Qx^{11/12})$$

as above. This establishes (3.3), and (3.1) follows; in particular, we have proved Theorem 2. ∎

4. Proof of Theorem 3. Let β be a small positive constant. We write

$$\pi(x,\beta) = |\{p : x < p \le (1+\beta)x\}|,$$
$$\mathcal{T}(x,\beta) = \{(p_1,p_2,p_3) : x < p_i \le (1+\beta)x\}$$

and

$$A(q;x,\beta) = |\{(p_1,p_2,p_3) \in \mathcal{T}(x,\beta) : A(p_1,p_2,p_3) \equiv 0 \pmod q)\},$$
$$\mathcal{L} = \log x, \qquad \tau(q) = |\{d : d \mid q\}|.$$

We shall show that for $\theta < \theta_1$, and $\beta \le \beta_1(\theta)$, $x > x_1(\theta)$,

$$|\{(p_1,p_2,p_3) \in \mathcal{T}(x,\eta) : P^+(A(p_1,p_2,p_3)) > x^\theta\}| > c'(\theta)\pi(x,\beta)^3 \tag{4.1}$$

where $c'(\theta) > 0$; this suffices for Theorem 3. We draw heavily on the analysis in [4] and indicate briefly the changes in the argument that are needed.

LEMMA 12. *Let $A > 0$, $B > 0$, $q \le \mathcal{L}^A$, $(a,q) = 1$. Then*

$$S_q(a;x,\beta) = \frac{\mu(q)}{\varphi(q)}\,\pi(x,\beta) + O_{A,B}(qx\mathcal{L}^{-B}). \tag{4.2}$$

Proof. This follows at once from [4, (3.13)]. ∎

LEMMA 13. *Let $A > 0$. For $x \ge 2$, $1 \le q \le x^{17/16-\varepsilon}$, we have*

$$A(q;x,\beta) - \prod_{p\mid q}\left(1 - \frac{1}{(p-1)^2}\right)\frac{\pi(x,\beta)^3}{q} \ll_{A,\varepsilon} \Bigg(\mathcal{L}^{-A} + \mathcal{L}^5 \sum_{\substack{t\mid q\\ t\ge \mathcal{L}^{-A}}} \left(\frac{\tau(t)}{t}\right)^{1/2}\Bigg)\frac{\pi(x,\beta)^3}{q}. \tag{4.3}$$

Moreover, for $x \ge 2$, $B > 0$,

$$\sum_{q\le x^{17/16-\varepsilon}} \left| A(q;x,\beta) - \prod_{p\mid q}\left(1 - \frac{1}{(p-1)^2}\right)\frac{\pi(x,\beta)^3}{q}\right| \ll_{B,\varepsilon} \pi(x,\beta)^3\mathcal{L}^{-B}. \tag{4.4}$$

Proof. Let

$A^*(q;x,\beta) = |\{(p_1,p_2,p_3) \in \mathcal{T}(x,\beta) : (p_i,q) = 1,\ A(p_1,p_2,p_3) \equiv 0 \pmod q\}|$.

It is clear that

$$0 \le A(q;x,\beta) - A^*(q;x,\beta) \le 3\omega(q)\pi(x,\beta).$$

Moreover,

$$A^*(q;x,\beta) = \frac{1}{q}\sum_{a=1}^{q} S_q^3(a;x,\beta).$$

Just as in [4, (4.4)] this relation leads to

$$A^*(q;x,\beta) = \mathrm{MT}(q;x,\beta) + O(\mathrm{ET}(q;x,\beta) + \mathcal{L}^3)$$

where

$$\mathrm{MT}(q;x,\beta) = \frac{1}{q}\sum_{t\mid q}\sum_{\substack{b=1\\(b,t)=1}}^{t} S_t^3(b;x,\beta),$$

$$\mathrm{ET}(q;x,\beta) = \frac{\mathcal{L}}{q}\sum_{t\mid q}\sum_{\substack{b=1\\(b,t)=1}}^{t} |S_t(b;x,\beta)|^2.$$

As in [4, (4.5)],

$$\sum_{b=1}^{t} |S_t(b;x,\beta)|^2 \ll x^2 + tx, \tag{4.5}$$

leading to

$$\mathrm{ET}(q;x,\beta) \ll q^{-1}x(x+q)\tau(q)\mathcal{L}.$$

We partition $\mathrm{MT}(q;x,\beta)$ as

$$\mathrm{MT}(q;x,\beta) = M_1(\mathcal{L}^A) + M_2(\mathcal{L}^A),$$

where

$$M_1(\Delta) = \frac{1}{q} \sum_{\substack{t|q \\ t \le \Delta}} S_t^3(b; x, \beta).$$

As in (4.7)–(4.9) of [4], an application of Lemma 12 yields

$$\begin{aligned} M_1(\mathcal{L}^A) &= \frac{1}{q} \sum_{\substack{t|q \\ t \le \mathcal{L}^A}} \frac{\mu(t)}{\varphi^2(t)}\, \pi(x,\beta)^3 + O(q^{-1} x^3 \mathcal{L}^{-A}) \\ &= \frac{1}{q}\bigg(\prod_{p|q}\bigg(1 - \frac{1}{(p-1)^2}\bigg) + O(\mathcal{L}^{-A})\bigg)\pi(x,\beta)^3 + O(q^{-1}x^3\mathcal{L}^{-A}). \end{aligned}$$

For the remainder of the proof of (4.3), we follow the argument below [4, (4.10)], verbatim, using (4.5) (above) along the way. By applying the inequality

$$\sum_{q \sim Q} \sum_{\substack{t|q \\ t \ge L}} (\tau(t)/t)^{1/2} \ll L^{-1/2} Q (\log L)^{\sqrt{2}-1}$$

(see [4, (1.4)]), we deduce (4.4) from (4.3).

We now sharpen Theorem 1.5 of [4], where the corresponding range for q is $[1, x^{14/13-\varepsilon}]$. ∎

THEOREM 4. *Let $B > 0$. Then for $x \ge 2$,*

$$\sum_{q \le x^{13/12-\varepsilon}} \bigg| A(q; x, \beta) - \prod_{p|q}\bigg(1 - \frac{1}{(p-1)^2}\bigg)\frac{\pi(x,\beta)^3}{q}\bigg| \ll \pi(x,\beta)^3 \mathcal{L}^{-B}. \tag{4.6}$$

Proof. By Lemma 13, it suffices to estimate the part of the sum in (4.6) with $q > x$. Let $\eta = \varepsilon/6$. We say that q is (η, x)-*good* if for all divisors $t \mid q$ with $t \ge x$, we have

$$\max_{(b,t)=1} |S_t(b; x, \beta)| \le (t^{1/10} x^{4/5} + x^{11/12}) t^{\eta}. \tag{4.7}$$

Otherwise, we say that q is (η, x)-*bad*.

We claim that for $Q < x^2/4$,

$$|\{q \sim Q : q \text{ is } (\eta, x)\text{-bad}\}| \ll_\varepsilon Q x^{-\eta/2}. \tag{4.8}$$

This is trivial for $Q < x/2$, since $t \mid q \sim Q$ implies $t \le 2Q < x$. Suppose now that $Q > x/2$. For $x \le T \le 2Q$, consider the set of $t \in [T, 2T)$ for which (4.7) fails. By Theorem 2 with $\eta/3$ in place of ε, there are $O_\varepsilon(T^{1-2\eta/3})$ values of t with this property. For each $t \in [T, 2T)$, there are $O(Q/T)$ integers $q \sim Q$ with $t \mid q$. So there are at most $O(Qx^{-2\eta/3})$ values of $q \sim Q$ for which (4.7) fails. Summing over $O(\mathcal{L})$ values of T, we obtain (4.8).

For (η, x)-good values of q, we see from the proof of Lemma 13 that it is enough to estimate $M_2(\mathcal{L}^B)$. The contribution to $M_2(\mathcal{L}^B)$ of those t in

$[1, x)$ is estimated as before (individually for every q). Thus it is enough to prove

$$\sum_{\substack{x \le q \le x^{13/12-\varepsilon} \\ q\ (\eta,x)\text{-good}}} \frac{1}{q} \sum_{\substack{t \mid q \\ t \ge x}} \sum_{\substack{b=1 \\ (b,t)=1}}^{t} |S_t(b; x, \beta)|^3 \ll_B \frac{\pi(x,\beta)^3}{\mathcal{L}^B}$$

in order to obtain a satisfactory contribution to (4.6) from $\{x \le q \le x^{13/12-\varepsilon} : q$ is (η, x)-good$\}$. Using (4.5), (4.7), we get

$$\begin{aligned}
&\sum_{\substack{x \le q \le x^{13/12-\varepsilon} \\ q\ (\eta,x)\text{-good}}} \frac{1}{q} \sum_{\substack{t \mid q \\ t \ge x}} \sum_{\substack{b=1 \\ (b,t)=1}}^{t} |S_t(b; x, \beta)|^3 \\
&\qquad \ll x \sum_{x \le q \le x^{13/12-\varepsilon}} \frac{1}{q} \sum_{\substack{t \mid q \\ t \ge x}} (t^{1/10} x^{4/5} + x^{11/12}) t^{1+\eta} \\
&\qquad \le x \sum_{q \le x^{13/12-\varepsilon}} \tau(q)(q^{1/10} x^{4/5} + x^{11/12}) q^{\eta} \\
&\qquad \ll x((x^{13/12-\varepsilon})^{11/10} x^{4/5} + x^{2-\varepsilon}) x^{3\eta} \ll x^{3-\varepsilon/2}.
\end{aligned}$$

As for the (η, x)-bad values of q, we use a bound from [4] for

$$\rho(n) = |\{(p_1, p_2, p_3) : p_i \sim x,\ A(p_1, p_2, p_3) = x\}|,$$

namely

$$\rho(n) \ll \tau(n) x \mathcal{L}$$

(see [4, (1.6))]. Thus the contribution to (4.6) from (η, x)-bad values of q is

$$\begin{aligned}
&\ll \sum_{\substack{x \le q \le x^{13/12-\varepsilon} \\ q\ (\eta,x)\text{-bad}}} \sum_{\substack{n \ll x^2 \\ n \equiv 0 \ (\mathrm{mod}\ q)}} \rho(n) \ll \sum_{\substack{x \le q \le x^{13/12-\varepsilon} \\ q\ (\eta,x)\text{-bad}}} \frac{x^{3+\eta/4}}{q} \\
&\ll \sum_{\substack{x \le Q \le x^{13/12-\varepsilon} \\ Q = 2^j}} \frac{x^{3+\eta/4}}{Q} Q x^{-\eta/2} \ll x^3 \mathcal{L}^{-B}
\end{aligned}$$

where we use (4.8) in the penultimate bound. This completes the proof of Theorem 4. ∎

Proof of Theorem 3. Consider the 'Chebyshev–Hooley' sum

$$\mathrm{CH}(x) := \sum_{p_i \in (x, (1+\beta)x]} \log A(p_1, p_2, p_3).$$

Since all $A(p_1, p_2, p_3)$ are in $[3x^2, 3(1+\beta)^2 x^2]$, we have

$$\mathrm{CH}(x) \sim 2\mathcal{L}\pi(x,\beta)^3 \quad (x \to \infty). \tag{4.9}$$

Let

$$X := \pi(x,\beta)^3, \quad Y := x^{13/12-\varepsilon}, \quad Z := x^{\theta}.$$

Arguing as in the proof of [4, (4.14)], we have

$$\mathrm{CH}(x) = \sum_{q \ll x^2} \Lambda(q) A(q; x, \beta) = \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3 + \sum\nolimits_4, \tag{4.10}$$

where

$$\sum\nolimits_1 := \sum_{q \le Y} \Lambda(q) A(q;x,\beta), \qquad \sum\nolimits_2 := \sum_{\substack{q>Y\\ q \text{ not prime}}} \Lambda(q) A(q;x,\beta),$$

$$\sum\nolimits_3 := \sum_{\substack{Y<q\le Z\\ q \text{ prime}}} \Lambda(q) A(q;x,\beta), \quad \sum\nolimits_4 := \sum_{\substack{q>Z\\ q \text{ prime}}} \Lambda(q) A(q;x,\beta).$$

Theorem 4 easily yields

$$\sum\nolimits_1 \sim \left(\frac{13}{12} - \varepsilon\right) X\mathcal{L} \quad (x \to \infty), \tag{4.11}$$

while, just as in the argument leading to [4, (4.16)],

$$\sum\nolimits_2 \ll x^2. \tag{4.12}$$

We can follow the proof of [4, (4.17)] to obtain

$$\sum\nolimits_3 \le \sum_{0 \le k \le K_0} \log(2^{k+1}Y) \ \sum\nolimits_3(2^k Y), \tag{4.13}$$

where $K_0 = [\log(Z/Y)/\log 2]$ and

$$\sum\nolimits_3(P) = \sum_{p \sim P} A(p;x,\beta).$$

If rp is an integer counted by $A(p;x,\beta)$ in $\sum_3(P)$, then $rp = A(p_1,p_2,p_3)$ and

$$\frac{3x^2}{2P} \le r \le \frac{3(1+\beta)^2 x^2}{P}. \tag{4.14}$$

For a fixed r satisfying (4.14), let $\mathcal{C}^{(r)}$ be the set of integers $A(p_1,p_2,p_3)/r$ for which $(p_1,p_2,p_3) \in \mathcal{T}(x,\beta)$ and $A(p_1,p_2,p_3) \equiv 0 \pmod r$. We see that for any $z < x$,

$$\sum\nolimits_3(P) \le \sum_{r \text{ satisfies (4.14)}} S(\mathcal{C}^{(r)}, z). \tag{4.15}$$

Here we use the standard notation: $S(\mathcal{C}^{(r)}, z)$ counts the elements of $\mathcal{C}^{(r)}$ coprime to $\prod_{p \le z} p$.

Let

$$\omega(m)=\prod_{p\mid m}(1-(p-1)^{-2}),\qquad X^{(r)}=\frac{\omega(r)}{r}X,$$

$$R(x;m)=A(m;x,\beta)-\frac{\omega(m)}{m}X.$$

Let d denote a squarefree positive integer, and

$$C_d^{(r)}=|\{a\in\mathcal{C}^{(r)}:d\mid a\}|.$$

It is clear that

$$C_d^{(r)}=A(x;dr,\beta).$$

We rewrite this as a 'main term' plus an 'error term':

$$C_d^{(r)}=\frac{\omega(dr)/\omega(r)}{d}X^{(r)}+R(x;dr).$$

Using the theory of the linear sieve just as in [4, (4.20)], we have, with an $O(\ldots)$ error independent of r,z,

(4.16)

$$S(\mathcal{C}^{(r)},z)\le\prod_{p\le z}\left(1-\frac{\omega(pr)/\omega(dr}{p}\right)\left(F\left(\frac{\log D}{\log z}\right)+O((\log D)^{-1/3})\right)X^{(r)}+\sum_{d<D}|R(x;dr)|$$

for any choice of $D\ge 1$. For the sieve function F, we only need the formula

$$F(s)=\frac{2e^{\gamma}}{s}\qquad(0<s\le 3),$$

where γ is Euler's constant.

In view of (4.13), (4.15), we need to give an acceptable upper bound for

$$\mathcal{E}(D):=\sum_{d<D}\sum_{r\le 3(1+\beta)^2x^2/P}|R(x;dr)|.$$

LEMMA 14. *For $Y\le P<Z$ and $D\le PY/x^2$, we have*

$$\mathcal{E}(D)\ll X\mathcal{L}^{-3}.$$

Proof. We follow the proof of [4, Lemma 4.1], substituting Theorem 4 for the corresponding result in [4]. ∎

By (4.15), (4.16) and Lemma 14, we have

$$\sum\nolimits_3(P)\le(1+\varepsilon)X\sum_{r\text{ satisfies (4.14)}}\frac{\omega(r)}{r}\prod_{p\le z}\left(1-\frac{\omega(pr)/\omega(r)}{P}\right)\times F\left(\frac{\log(PYx^{-2})}{\log z}\right)\tag{4.17}$$

for every $\varepsilon > 0$ and for every sufficiently large x and every $z \le x$. We choose

$$z := (PYx^{-2})^{1/2}.$$

As noted in [4],

$$\prod_{p\le z}\left(1-\frac{\omega(pr)/\omega(r)}{p}\right) \le (1+O(z^{-1}))C_0V(z)\prod_{p|r}\left(\frac{1-1/p}{1-\omega(p)/p}\right),$$

where

$$C_0 := \prod_{p\ge 2}\left(\frac{1-\omega(p)/p}{1-1/p}\right), \quad V(z) := \prod_{p\le z}\left(1-\frac{1}{p}\right) \sim \frac{e^{-\gamma}}{\log z} \quad (x\to\infty).$$

Inequality (4.17) now simplifies to the form

$$\sum\nolimits_3(P) \le \frac{(2+\varepsilon)C_0X}{\log(PYx^{-2})}\sum_{r \text{ satisfies (4.14)}}\frac{\nu(r)}{r}, \tag{4.18}$$

where ν is the multiplicative function

$$\nu(r) = \omega(r)\prod_{p|r}\left(\frac{1-1/p}{1-\omega(p)/p}\right).$$

From the analysis in [4], we know that

$$\sum_{r\le R}\frac{\nu(r)}{r} = G(1)\log R + F_0 + O(R^{-\delta_2}),$$

where δ_0 is a positive absolute constant, F_0 is a constant, and $G(s)$ is defined by

$$\sum_{r=1}^{\infty}\frac{\nu(r)}{r^s} = \zeta(s)G(s);$$

G is holomorphic in $\operatorname{Re} s > 1/2$ and $C_0G(1) = 1$. We use this to reduce (4.18) to the form

$$\sum\nolimits_3(P) \le \frac{(2+\varepsilon)X}{\log(PYx^{-2})}\log\{2(1+\beta)^2\}.$$

Combining with (4.13), we obtain

$$\sum\nolimits_3 \le (2+\varepsilon)\pi(x,\beta)^3\log\{2(1+\beta)^2\}\sum_{0\le k\le K_0}\frac{\log(2^kY)}{\log(2^kY^2x^{-2})}. \tag{4.19}$$

Now

(4.20)

$$(\log 2)\sum_{0\le k\le K_0}\frac{\log(2^kY)}{\log(2^kY^2x^{-2})} = \mathcal{L}\left\{\frac{\log 2}{\mathcal{L}}\sum_{0\le k\le K_0}\frac{(\log Y)/\mathcal{L}+(k\log 2)/\mathcal{L}}{\log(Y^2x^{-2})/\mathcal{L}+(k\log 2)/\mathcal{L}}\right\}.$$

As in [4], we only have to consider the expression in brackets as a Riemann sum to obtain the asymptotic formula

$$(\log 2)\sum_{0\le k\le K_0}\frac{\log(2^kY)}{\log(2^kY^2x^{-2})}\sim \mathcal{L}J\qquad (x\to\infty),$$

where

$$\begin{aligned}J&=\int_0^{\log(Z/Y)/\mathcal{L}}\frac{t+(\log Y)/\mathcal{L}}{t+\log(Y^2x^{-2})/\mathcal{L}}\,dt\\&=\frac{\log(Z/Y)}{\mathcal{L}}+\frac{\log(x^2/Y)}{\mathcal{L}}\log\left[\frac{\log(YZx^{-2})}{\log(Y^2x^{-2})}\right]\\&=\theta-\frac{13}{12}+\varepsilon+\left(\frac{11}{12}+\varepsilon\right)\log\frac{12\theta-11-12\varepsilon}{2-24\varepsilon}.\end{aligned}$$

Combining this with (4.19), (4.20), for large x we have

$$\begin{aligned}\textstyle\sum_3 &\le (2+2\varepsilon)\pi(x,\beta)^3\mathcal{L}\left(1+\frac{2\log(1+\beta)}{\log 2}\right)\\&\times\left[\left(\theta-\frac{13}{12}+\varepsilon\right)+\left(\frac{11}{12}+\varepsilon\right)\log\frac{12\theta-11-12\varepsilon}{2-24\varepsilon}\right].\end{aligned}\tag{4.21}$$

Since $\theta<\theta_1$, we may choose positive numbers ε and β so small that the right-hand side of (4.21) is less than $(11/12-\varepsilon)\pi(x,\beta)^3\mathcal{L}$. It now follows from (4.9)–(4.12) and (4.21) that, for large x,

$$\textstyle\sum_4 \gg \pi(x,\beta)^3\mathcal{L}.$$

This completes the proof of Theorem 3. ∎

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