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Theoretical evidence in support of the extended abelian rank one Stark conjecture for the base field $\mathbb{Q}$

by

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1. Introduction. The main goal of this paper is to provide further evidence for the extended abelian rank one Stark conjecture by studying two infinite families of abelian extensions of $\mathbb{Q}$.

The extended abelian rank one Stark conjecture was stated for the first time in [3]. Our approach is based on [15] and we refer to this latter paper for the statement of the conjecture (Conjecture 3.6). See also [2] where arbitrary orders of vanishing are treated. Here, we work exclusively with the S-version of the conjecture. For a given 1-cover S (see §1.2 below for the definitions of a 1-cover and $S_{\min}$), rather than studying the extended abelian rank one Stark conjecture, we study Question 4.2 of [15] (denoted by $\mathrm{St}(K/k, S, v)$ for a given 1-cover S and a place $v \in S_{\min}$). It was shown that an affirmative answer to $\mathrm{St}(K/k, S, v)$ for all $v \in S_{\min}$ implies the extended abelian rank one Stark conjecture (see Proposition 4.3 of [15]).

In this paper, we will be concerned almost exclusively with the case where $v \in S_{\min}$ is a finite prime. In this case, as explained in [15], it is possible to formulate an extension of the classical Brumer–Stark conjecture which, if true, implies that $\mathrm{St}(K/k, S, v)$ has an affirmative answer. This extension of the Brumer–Stark conjecture is recalled in §2 (Question 2.1). Question 2.1 will be the main object of study of this paper. In §1.1, we fix some of our basic notation, and in §1.2, we recall some needed background information regarding the extended abelian rank one Stark conjecture. In §2.1, we formulate local versions of Question 2.1 and we prove some reduction statements similar to the ones given in [6] for the classical Brumer–Stark conjecture. In §3, we use the previous results of §2.1 and some previous work of Greither and Kučera [5] and of Smith [12] in order to study two infinite families of abelian extensions of $\mathbb{Q}$.

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1.1. Notation. Given a finite abelian extension K/k of number fields with Galois group G, the symbol $S(K/k)$ denotes the set of primes in k which are either archimedean or ramified in K/k. If S is any finite set of places of k containing $S(K/k)$, we denote the S-equivariant L-function by $\theta_{K/k,S}(s)$. When necessary, we shall write

$$\theta_{K/k,S}(0) = \theta_{K/k,S(K/k)}(0)\omega_{K/k,S},$$

where

$$\omega_{K/k,S} = \prod_{\mathfrak{p}\in S\smallsetminus S(K/k)} (1-\sigma_{\mathfrak{p}}^{-1}),$$

and $\sigma_{\mathfrak{p}}$ denotes the Frobenius automorphism associated to the unramified prime $\mathfrak{p}$ in K/k. In order to simplify the notation, we write $\theta_{K/k}(0)$ instead of $\theta_{K/k,S(K/k)}(0)$ (i.e. we drop the subscript S if S is minimal).

If the top field K is a CM-field, we denote the complex conjugation in G by j.

The subgroup of $K^{\times}$ consisting of anti-units will be denoted by K^0. We remind the reader that $\alpha \in K^{\times}$ is an anti-unit if $|\alpha|_w = 1$ for all archimedean places w of K. We use the symbol w_K for the number of roots of unity in $K^{\times}$. If $n \mid w_K$, the subgroup of $K^{\times}$ consisting of elements $\lambda \in K^{\times}$ such that $K(\lambda^{1/n})/k$ is an abelian extension of number fields will be denoted by $\mathcal{A}(K/k,n)$. This last property does not depend on the choice of $\lambda^{1/n}$. Both K^0 and $\mathcal{A}(K/k,n)$ are G-modules.

If A is a subgroup of I_K, the group of fractional ideals of K, and if $n \mid w_K$, we define $\mathrm{BrSt}^{(n)}_{\mathbb{Z}[G]}(A)$ to be the set of elements $\alpha \in \mathbb{Z}[G]$ for which, given any $\mathfrak{a} \in A$, there exists $\eta(\mathfrak{a}) \in K^0 \cap \mathcal{A}(K/k,n)$ such that

$$\mathfrak{a}^{\alpha} = (\eta(\mathfrak{a})).$$

The set $\mathrm{BrSt}^{(n)}_{\mathbb{Z}[G]}(A)$ is an ideal of $\mathbb{Z}[G]$. The classical Brumer–Stark conjecture says

$$w_K\theta_{K/k,S}(0) \in \mathrm{BrSt}^{(w_K)}_{\mathbb{Z}[G]}(I_K)$$

for any finite set S of places of k containing $S(K/k)$. It is known to be true for every finite abelian extension of $\mathbb{Q}$. See for instance [9].

If A is a finite abelian group, its p-Sylow subgroup will be denoted by $A\{p\}$.

1.2. Background. Let K/k be a finite abelian extension of number fields and S a finite set of places of k containing $S(K/k)$. One of the hypotheses of the classical abelian rank one Stark conjecture, as stated in [13], requires that there is a split prime in S. This condition implies that the order of vanishing of the S-imprimitive L-function $L_{K/k,S}(s,\chi)$ is at least one for all non-trivial characters χ of $\mathrm{Gal}(K/k)$. This is a consequence of the following theorem.

THEOREM 1.1. *Let K/k be an abelian extension of number fields and S a finite set of primes of k containing the infinite ones. We have*

$$\operatorname{ord}_{s=0} L_{K/k,S}(s,\chi) = \begin{cases} |S|-1 & \text{if } \chi = \chi_1, \\ |\{v \in S \mid G_v \subseteq \operatorname{Ker}(\chi)\}| & \text{if } \chi \neq \chi_1, \end{cases}$$

where G_v denotes the decomposition group associated to the place v in G.

Proof. See [14, p. 24, Proposition 3.4]. ∎

On the other hand, it is not difficult to give examples of abelian extensions K/k and finite sets S of places, not containing any split prime, for which all the S-imprimitive L-functions vanish with order of vanishing at least one as well. Such a set S is called a 1-cover:

DEFINITION 1.2. Let K/k be an abelian extension of number fields with Galois group G and let Λ be any subset of $\widehat{G} = \mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{C}^\times)$. Let S be any finite set of primes of k (perhaps not containing the ramified nor the archimedean primes). The set S is said to be a 1-*cover* for Λ if the following two conditions hold:

(1) For all non-trivial $\chi \in \Lambda$, there exists at least one prime $v \in S$ such that $G_v \subseteq \operatorname{Ker}(\chi)$.
(2) If the trivial character is in Λ, then $|S| \geq 2$.

In the case where $\Lambda = \widehat{G}$, we also say that S is a 1-cover for G (or for K/k) rather than for $\widehat{G}$.

If S is a 1-cover for K/k containing $S(K/k)$, then $\operatorname{ord}_{s=0} L_{K/k,S}(s,\chi) \geq 1$ for all $\chi \in \widehat{G}$ by Theorem 1.1. *Unless otherwise stated, we always suppose that a* 1-*cover S for K/k contains $S(K/k)$*. We need one more definition.

DEFINITION 1.3. Let K/k be an abelian extension of number fields with Galois group G and S a 1-cover for K/k. The set of characters $\chi \in \widehat{G}$ whose S-imprimitive L-functions have order of vanishing precisely one will be denoted by $\widehat{G}_{1,S}$. Moreover, the set $S_{\min}$ is defined to consist of all primes $v \in S$ for which there exists $\chi \in \widehat{G}_{1,S}$, $\chi \neq \chi_1$, such that $G_v \subseteq \operatorname{Ker}(\chi)$.

In other words, $S_{\min}$ precisely consists of the places in S which are responsible for the vanishing of the S-imprimitive L-functions associated to non-trivial characters having order of vanishing exactly one.

It is natural to try to generalize the abelian rank one Stark conjecture to handle these more general sets S consisting of 1-covers. This is the purpose of the extended abelian rank one Stark conjecture of Erickson [3].

In [15], the present author formulated and studied a stronger question involving one $v \in S_{\min}$ at the time (this question is denoted by $\mathrm{St}(K/k, S, v)$, where S is a 1-cover and $v \in S_{\min}$). It was shown that if $\mathrm{St}(K/k, S, v)$ has an affirmative answer for all $v \in S_{\min}$, then the extended abelian rank one Stark conjecture is true. This question seems to be easier to study both theoretically and numerically.

If $v \in S_{\min}$ is a *finite* prime, the present author also formulated in [15] a generalization of the Brumer–Stark conjecture which in the case of a positive answer implies that $\mathrm{St}(K/k, S, v)$ also has an affirmative answer. This generalization of the Brumer–Stark conjecture is the main object of study of this paper. We recall its precise statement in the next section.

2. A generalization of the Brumer–Stark conjecture. The setting is as follows: Let K/k be a finite abelian extension of number fields and S a 1-cover for K/k. Suppose that $|S| \geq 3$ and $S \neq S_{\min}$. Let $\mathfrak{p} \in S_{\min}$ be a finite prime. As in [15], let $L^{(\mathfrak{p})} = K^{G_{\mathfrak{p}}}$, $\Gamma_{\mathfrak{p}} = \mathrm{Gal}(L^{(\mathfrak{p})}/k)$, $n_{\mathfrak{p}} = |G_{\mathfrak{p}}|$ and $R_{\mathfrak{p}} = S \smallsetminus \{\mathfrak{p}\}$. If $\mathfrak{p} \in S_{\min}$ is fixed, then we write L, Γ, R, n instead of $L^{(\mathfrak{p})}, \Gamma_{\mathfrak{p}}, R_{\mathfrak{p}}, n_{\mathfrak{p}}$, and similarly for other notations depending on $\mathfrak{p} \in S_{\min}$.

(1) If $\mathfrak{p} \in S_{\min}$ is ramified in K/k, define the subgroup $A = A_{\mathfrak{p}}$ of I_L to be the subgroup generated by the primes $\mathfrak{P}$ of L lying above $\mathfrak{p}$.
(2) If $\mathfrak{p} \in S_{\min}$ is unramified in K/k, let $A = A_{\mathfrak{p}}$ be the subgroup of I_L generated by the primes $\mathfrak{Q}$ of L lying above primes $\mathfrak{q}$ of k satisfying

$$\left(\frac{K/k}{\mathfrak{q}}\right) = \left(\frac{K/k}{\mathfrak{p}}\right).$$

The extension of the Brumer–Stark conjecture alluded to above is the following (Question 4.7 of [15]).

Question 2.1 (Extension of the Brumer–Stark conjecture). *Let $\mathfrak{p} \in S_{\min}$ be a finite prime. In the setting as above, do we have*

$$w_L \theta_{L/k,R}(0)/n \in \mathbb{Z}[\Gamma]?$$

Moreover, do we have

$$w_L \theta_{L/k,R}(0)/n \in \mathrm{BrSt}^{(w_L)}_{\mathbb{Z}[\Gamma]}(A),$$

where A is the subgroup of I_L defined above?

In all theoretical and numerical examples that have been investigated so far, one can in fact take I_L instead of A in the previous question. Numerical examples have been computed only in the case where the base field is $\mathbb{Q}$ (see [15] for some examples and [16] for more). It would be interesting to do computations with other base fields.

We remark that since we assume $\mathfrak{p} \in S_{\min}$ is a finite prime, the field k is necessarily totally real and K totally complex. Now, since K/k is Galois and k is totally real, every complex embedding of K induces a complex conjugation. If w is a (necessarily complex) place of K, we denote the corresponding complex conjugation by σ_w. The fact that K/k is abelian implies that σ_w depends only on the place v of k lying below w. Thus, it will be denoted by σ_v rather than σ_w. By the *decomposition group* associated to the infinite place v of k, we mean, as usual, $G_v := \langle \sigma_v \rangle$. The field L also has to be totally complex, otherwise $G_v \subseteq G_{\mathfrak{p}}$ for some real place v of k, and this is impossible, since $\mathfrak{p}$ is assumed to be in $S_{\min}$. If we are in the situation where K is a CM-field, then L is also a CM-field as subfields of CM-fields are either totally real or CM-fields.

As another remark, we note that if R is any finite set of primes containing $S(K/k)$ and $\mathfrak{p}$ is a prime of k which splits completely in K/k, then $S = R \cup \{\mathfrak{p}\}$ is automatically a 1-cover. There are only two possibilities: Either $S_{\min}$ is empty or $S_{\min} = \{\mathfrak{p}\}$. In the former case, $\theta_{K/k,R}(0) = 0$ and there is nothing to show, whereas in the latter case $K = L$, $n = 1$ and $A = I_K^{spl}$ is the group of fractional ideals generated by the primes $\mathfrak{P}$ of K lying above primes $\mathfrak{p}$ of k splitting completely in K/k. Since it is known (see the classical book on the subject [14] or Lemma 2.3 below) that

$$w_K \theta_{K/k,R}(0) \in \mathrm{BrSt}^{(w_K)}_{\mathbb{Z}[G]}(P_K),$$

where P_K is the group of principal ideals of K and I_K is generated by P_K and I_K^{spl}, we get back the original Brumer–Stark conjecture.

We will refer to the following hypothesis as the Integrality Property.

Hypothesis 2.2 (Integrality Property). One has

$$\mathrm{Ann}_{\mathbb{Z}[\Gamma]}(\mu_L) \cdot \theta_{L/k,R}(0)/n \subseteq \mathbb{Z}[\Gamma].$$

If the Integrality Property is satisfied, then the first part of Question 2.1 holds true, since $w_L \in \mathrm{Ann}_{\mathbb{Z}[\Gamma]}(\mu_L)$. It was shown in [15, Theorem 4.30] that if $\mathfrak{p} \in S_{\min}$ is unramified in K/k, then the Integrality Property follows from a conjecture of Gross (Conjecture 7.6 of [7]). This case of the Gross conjecture is known to be true, see for instance Lemma 2.3 in [8]. Therefore, the Integrality Property is true if the finite prime $\mathfrak{p} \in S_{\min}$ is unramified in K/k. If the finite prime $\mathfrak{p} \in S_{\min}$ is ramified in K/k, then we cannot say anything in theory, but the numerical computations agree with it so far.

Lemma 2.3. *Suppose that the Integrality Property holds true. Then*

$$(L^\times)^{w_L \theta_{L/k,R}(0)/n} \subseteq L^0 \cap \mathcal{A}(L/k, w_L).$$

Proof. This follows from Lemmas 4.10 and 4.12 of [15]. ∎

2.1. Local version of the extension of the Brumer–Stark conjecture. This section (§2.1) is based on [6] where the corresponding results are proved for the classical Brumer–Stark conjecture. The proofs are similar and we include them here only for the sake of completeness relative to the setting of Question 2.1.

Suppose that $m \mid w_L$. If the Integrality Property holds true then it makes sense to say

$$w_L\theta_{L/k,R}(0)/n \in \mathrm{BrSt}^{(m)}_{\mathbb{Z}[\Gamma]}(A),$$

where A is now a subgroup of Cl_L rather than I_L. Indeed, if $\mathfrak{a}$ is a fractional ideal such that there exists $\eta(\mathfrak{a}) \in L^\times$ satisfying

(1) $\mathfrak{a}^{w_L\theta_{L/k,R}(0)/n} = (\eta(\mathfrak{a}))$,
(2) $|\eta(\mathfrak{a})|_w = 1$ for all infinite places w of L,
(3) the extension $L(\eta(\mathfrak{a})^{1/m})/k$ is abelian,

and $\mathfrak{b} = \lambda \cdot \mathfrak{a}$ for some $\lambda \in L^\times$, then $\mathfrak{b}$ has the same three properties with $\eta(\mathfrak{b}) = \lambda^{w_L\theta_{L/k,R}(0)/n} \cdot \eta(\mathfrak{a})$ by Lemma 2.3.

Proposition 2.4. *Suppose the Integrality Property holds true and let A be any subgroup of Cl_L. Let p be any prime number and set $w_{L,p} = |w_L|_p^{-1}$, i.e. $w_{L,p}$ is the exact power of p dividing w_L. Then*

$$w_L\theta_{L/k,R}(0)/n \in \mathrm{BrSt}^{(w_L)}_{\mathbb{Z}[\Gamma]}(A\{p\})$$

if and only if

$$w_L\theta_{L/k,R}(0)/n \in \mathrm{BrSt}^{(w_{L,p})}_{\mathbb{Z}[\Gamma]}(A\{p\}). \tag{2.1}$$

Proof. One direction is clear. So suppose that (2.1) holds, and write $w_L = m \cdot w_{L,p}$ for some integer m relatively prime to p. Since $(m,p) = 1$, the group morphism $A\{p\} \to A\{p\}$ defined by $[\mathfrak{a}] \mapsto [\mathfrak{a}]^m$, is an isomorphism. Let $[\mathfrak{a}] \in A\{p\}$ and let $[\mathfrak{b}] \in A\{p\}$ be such that $[\mathfrak{b}]^m = [\mathfrak{a}]$. By hypothesis, there exists $\eta(\mathfrak{b}) \in L^0 \cap \mathcal{A}(L/k, w_{L,p})$ such that

$$\mathfrak{b}^{w_L\theta_{L/k,R}(0)/n} = (\eta(\mathfrak{b})).$$

Now, there exists $\lambda \in L^\times$ such that $\mathfrak{a} = \lambda\mathfrak{b}^m$. We then have

$$\mathfrak{a}^{w_L\theta_{L/k,R}(0)/n} = (\lambda^{w_L\theta_{L/k,R}(0)/n}\eta(\mathfrak{b})^m).$$

By Lemma 2.3,

$$\lambda^{w_L\theta_{L/k,R}(0)/n} \in L^0 \cap \mathcal{A}(L/k, w_L),$$

and since $\eta(\mathfrak{b}) \in L^0 \cap \mathcal{A}(L/k, w_{L,p})$, it follows that $\eta(\mathfrak{b})^m \in L^0 \cap \mathcal{A}(L/k, w_L)$. This is what we wanted to show. ∎

This last proposition allows one to study Question 2.1 one prime at a time:

Corollary 2.5. *If p is a prime number, let $n_p = |n|_p^{-1}$. The following statements are equivalent:*

(1) $w_L\theta_{L/k,R}(0)/n \in \mathrm{BrSt}_{\mathbb{Z}[\Gamma]}^{(w_L)}(A)$.

(2) $w_L\theta_{L/k,R}(0)/n \in \mathrm{BrSt}_{\mathbb{Z}[\Gamma]}^{(w_L,p)}(A\{p\})$ *for all prime numbers* p.

(3) $w_L\theta_{L/k,R}(0)/n_p \in \mathrm{BrSt}_{\mathbb{Z}[\Gamma]}^{(w_L,p)}(A\{p\})$ *for all prime numbers* p.

Proof. The equivalence between (1) and (2) follows directly from Proposition 2.4. The proof of the equivalence of (2) and (3) is similar and left to the reader. ∎

The next proposition shows that assuming the usual Brumer–Stark conjecture, Question 2.1 does not say anything new at primes p which are relatively prime to n.

PROPOSITION 2.6. *Suppose that the Integrality Property holds true and also that the usual Brumer–Stark conjecture is true for* L/k. *If* p *is a prime satisfying* $(p,n)=1$, *then*

$$w_L\theta_{L/k,R}(0)/n \in \mathrm{BrSt}_{\mathbb{Z}[\Gamma]}^{(w_L,p)}(\mathrm{Cl}_L\{p\}).$$

Proof. Let $[\mathfrak{a}] \in \mathrm{Cl}_L\{p\}$ and p^e be the exponent of $\mathrm{Cl}_L\{p\}$. Let $\lambda_1 \in K^\times$ be such that $\mathfrak{a}^{p^e} = (\lambda_1)$. Since $(p,n)=1$, there exist $s,t\in\mathbb{Z}$ such that $sp^e+tn=1$. Therefore,

$$\begin{aligned}\mathfrak{a}^{w_L\theta_{L/k,R}(0)/n} &= \mathfrak{a}^{sp^e w_L\theta_{L/k,R}(0)/n}\cdot \mathfrak{a}^{tw_L\theta_{L/k,R}(0)}\\ &= (\lambda_1^{sw_L\theta_{L/k,R}(0)/n})\cdot \mathfrak{a}^{tw_L\theta_{L/k,R}(0)}.\end{aligned}$$

By Lemma 2.3, we have $\lambda_1^{w_L\theta_{L/k,R}(0)/n} \in L^0\cap\mathcal{A}(L/k,w_L)$. Moreover, the usual Brumer–Stark conjecture for the special value $w_L\theta_{L/k,R}(0)$ shows that the ideal $\mathfrak{a}^{w_L\theta_{L/k,R}(0)}$ is principal generated by an element in $L^0\cap\mathcal{A}(L/k,w_L)$. This is what we wanted to show. ∎

Hence, we just have to focus on the primes p dividing n.

PROPOSITION 2.7. *Let* A *be any subgroup of* Cl_L *and suppose that the Integrality Property holds true. Let* p *be a prime number dividing* n. *Suppose moreover that* $(p,w_L)=1$. *Then*

$$w_L\theta_{L/k,R}(0)/n \in \mathrm{BrSt}_{\mathbb{Z}[\Gamma]}^{(w_L,p)}(A\{p\})$$

if and only if

$$w_L\theta_{L/k,R}(0)/n \in \mathrm{Ann}_{\mathbb{Z}[\Gamma]}(A\{p\}). \tag{2.2}$$

Proof. Again, one direction is clear, so suppose that (2.2) holds, and let $[\mathfrak{a}]\in A\{p\}$. Let p^e be the exponent of $A\{p\}$ and let $\lambda\in L^\times$ be such that $\mathfrak{a}^{p^e}=(\lambda)$. Since $(p,w_L)=1$, there exist $s,t\in\mathbb{Z}$ such that $sp^e+tw_L=1$. By hypothesis, there exists $\eta(\mathfrak{a})\in L^\times$ such that

$$\mathfrak{a}^{w_L\theta_{L/k,R}(0)/n} = (\eta(\mathfrak{a})),$$

hence

$$\mathfrak{a}^{w_L\theta_{L/k,R}(0)/n} = (\lambda^{sw_L\theta_{L/k,R}(0)/n}\eta(\mathfrak{a})^{tw_L}).$$

Setting

$$\eta'(\mathfrak{a}) = \lambda^{sw_L\theta_{L/k,R}(0)/n}\eta(\mathfrak{a})^{tw_L},$$

we see that $\eta'(\mathfrak{a}) \in \mathcal{A}(L/k, w_L)$ by Lemma 2.3.

As explained after Question 2.1, the field L is necessarily totally complex. For every (necessarily complex) place w of L, let σ_w be the associated complex conjugation. From Lemma 4.10 in [15], we have

$$(1+\sigma_w)w_L\theta_{L/k,R}(0)/n = 0$$

for all infinite places w of L. Since $1-\sigma_w = 2-(1+\sigma_w)$, we get

$$(1-\sigma_w)w_L\theta_{L/k,R}(0)/n = 2w_L\theta_{L/k,R}(0)/n.$$

Hence, letting $d = [L:\mathbb{Q}]/2$, we have

$$\prod_w (1-\sigma_w)\cdot w_L\theta_{L/k,R}(0)/n = 2^d \cdot w_L\theta_{L/k,R}(0)/n,$$

where the product is over all (necessarily complex) infinite places of L. Since $(p, w_L) = 1$ and w_L is even, we see that there exists $s, t \in \mathbb{Z}$ such that $s2^d + tp^e = 1$. Then

$$\begin{aligned}\mathfrak{a}^{w_L\theta_{L/k,R}(0)/n} &= (\lambda^{tw_L\theta_{L/k,R}(0)/n})\mathfrak{a}^{s2^d w_L\theta_{L/k,R}(0)/n} \\ &= \big(\lambda^{tw_L\theta_{L/k,R}(0)/n}\eta'(\mathfrak{a})^{s\prod_w(1-\sigma_w)}\big).\end{aligned}$$

Letting

$$\eta''(\mathfrak{a}) = \lambda^{tw_L\theta_{L/k,R}(0)/n}\eta'(\mathfrak{a})^{s\prod_w(1-\sigma_w)},$$

we see that $\eta''(\mathfrak{a}) \in L^0 \cap \mathcal{A}(L/k, w_L)$. ∎

If A is a subgroup of the group of fractional ideals I_L and $m \mid w_L$, we define $\mathrm{Ab}^{(m)}_{\mathbb{Z}[\Gamma]}(A)$ to be the set of elements $\alpha \in \mathbb{Z}[\Gamma]$ such that there exists $\eta(\mathfrak{a}) \in \mathcal{A}(L/k, m)$ satisfying

$$\mathfrak{a}^\alpha = (\eta(\mathfrak{a})).$$

That is, we are just dropping the anti-unit condition. The set $\mathrm{Ab}^{(m)}_{\mathbb{Z}[\Gamma]}(A)$ is an ideal of $\mathbb{Z}[\Gamma]$ and we clearly have the inclusion

$$\mathrm{BrSt}^{(m)}_{\mathbb{Z}[\Gamma]}(A) \subseteq \mathrm{Ab}^{(m)}_{\mathbb{Z}[\Gamma]}(A).$$

Again, if the Integrality Property holds true, Lemma 2.3 implies that it makes sense to say

$$w_L\theta_{L/k,R}(0)/n \in \mathrm{Ab}^{(m)}_{\mathbb{Z}[\Gamma]}(A)$$

if now A is a subgroup of Cl_L rather than I_L.

Proposition 2.8. *Let A be any subgroup of Cl_L and suppose that the Integrality Property holds true. Let p be an odd prime number dividing n. Suppose moreover that $p \mid w_L$. Then*

$$w_L \theta_{L/k,R}(0)/n \in \mathrm{BrSt}_{\mathbb{Z}[\Gamma]}^{(w_L,p)}(A\{p\})$$

if and only if

$$w_L \theta_{L/k,R}(0)/n \in \mathrm{Ab}_{\mathbb{Z}[\Gamma]}^{(w_L,p)}(A\{p\}).$$

Proof. The proof is similar to the second part of the proof of Proposition 2.7 and is left to the reader. ∎

We can summarize the previous discussion as follows, always assuming the Integrality Property. If p is a prime number and $(p, n) = 1$, we have nothing to do assuming the usual Brumer–Stark conjecture. Suppose now $p \mid n$. In order to give an affirmative answer to Question 2.1:

(1) If $(p, w_L) = 1$, we just have to show an annihilation statement.
(2) If $p \mid w_L$ and p is odd, we have to show an annihilation statement and the abelian condition.
(3) If $p = 2$, then we have to show an annihilation statement, the abelian condition and the anti-unit part.

Suppose that L is a CM-field and p is an odd prime. Assuming the usual Brumer–Stark conjecture, we just have to deal with the minus part of the class group, as the following proposition shows.

Proposition 2.9. *Suppose that the Integrality Property holds true and that the usual Brumer–Stark conjecture is true for L/k. Suppose also that L is a CM-field and p is an odd prime dividing n. Letting $n_p = |n|_p^{-1}$, we have*

$$w_L \theta_{L/k,R}(0)/n_p \in \mathrm{BrSt}_{\mathbb{Z}[G]}^{(w_L)}(\mathrm{Cl}_L\{p\}^+).$$

Proof. If $[\mathfrak{a}] \in \mathrm{Cl}_L\{p\}^+$, then $\mathfrak{a}^{1-j} = (\lambda_1)$ for some $\lambda_1 \in L^\times$. Moreover, we have the identity

$$(1 - j) w_L \theta_{L/k,R}(0)/n_p = 2 w_L \theta_{L/k,R}(0)/n_p,$$

because $\theta_{L/k,R}(0) \in \mathbb{Q}[G]^-$. Since $(2, n_p) = 1$, there exist $s, t \in \mathbb{Z}$ such that $s2 + t n_p = 1$. Thus,

$$\mathfrak{a}^{w_L \theta_{L/k,R}(0)/n_p} = \mathfrak{a}^{s \cdot 2 w_L \theta_{L/k,R}(0)/n_p} \cdot \mathfrak{a}^{t w_L \theta_{L/k,R}(0)}.$$

Now, we have

$$\mathfrak{a}^{2 w_L \theta_{L/k,R}(0)/n_p} = \mathfrak{a}^{(1-j) w_L \theta_{L/k,R}(0)/n_p} = (\lambda_1^{w_L \theta_{L/k,R}(0)/n_p}),$$

and $\lambda_1^{w_L \theta_{L/k,R}(0)/n_p} \in L^0 \cap \mathcal{A}(L/k, w_L)$ by Lemma 2.3.

Moreover, the usual Brumer–Stark conjecture for $w_L\theta_{L/k,R}(0)$ shows that the ideal $\mathfrak{a}^{w_L\theta_{L/k,R}(0)}$ is principal generated by an element in $L^0 \cap \mathcal{A}(L/k, w_L)$. This is what we wanted to prove. ■

2.2. Further remarks. It seems clear now that a stronger Brumer–Stark type statement for a certain subgroup A of I_L would imply a big part of the extended abelian rank one Stark conjecture. As explained at the beginning of §2, depending on whether or not $\mathfrak{p}$ is ramified in K/k, we proposed two such subgroups which would be enough to have consequences for the extended abelian rank one Stark conjecture.

On the other hand, surprisingly perhaps, in all theoretical and numerical results so far, Question 2.1 holds true with $A = I_L$. The simplest possible type of fields for which Question 2.1 does not reduce to the classical Brumer–Stark conjecture is when K/k is biquadratic.

THEOREM 2.10. *Suppose that K/k is a biquadratic extension of number fields with K totally complex and k totally real. Suppose also that S is a 1-cover which does not contain a split prime, $|S| \geq 3$ and $S \neq S_{\min}$. Let $\mathfrak{p} \in S_{\min}$ be a finite prime, $L = K^{G_{\mathfrak{p}}}$, $\Gamma = \mathrm{Gal}(L/k)$ and $R = S \smallsetminus \{\mathfrak{p}\}$. Note that $|G_{\mathfrak{p}}| = 2$. Then*

$$w_L\theta_{L/k,R}(0)/2 \in \mathbb{Z}[\Gamma],$$

and moreover

$$w_L\theta_{L/k,R}(0)/2 \in \mathrm{BrSt}^{(w_L)}_{\mathbb{Z}[\Gamma]}(I_L).$$

Proof. Proposition 5.2 of [10] implies the following result for quadratic extensions. If M/k is a quadratic extension with Galois group Γ such that $|S(M/k)| \geq 3$, then

$$w_M\theta_{M/k}(0)/2 \in \mathbb{Z}[\Gamma] \quad \text{and} \quad w_M\theta_{M/k}(0)/2 \in \mathrm{BrSt}^{(w_M)}_{\mathbb{Z}[\Gamma]}(I_M).$$

If we come back to our biquadratic extension K/k and subfield L (which is quadratic over k and also a CM-field), there are only two cases not covered by this result:

(1) The base field k is $\mathbb{Q}$ and only one finite prime ramifies in $L/\mathbb{Q}$.
(2) The base field k is a real quadratic number field and the extension L/k is unramified.

As in the proof of Theorem 4.26 of [15], since S is a 1-cover and $S \neq S_{\min}$, we know that $|S| \geq 4$. So in both cases, there is a prime $\mathfrak{q} \in R$ which is unramified in L/k. Since $\mathfrak{p} \in S_{\min}$, the Frobenius automorphism $\sigma_{\mathfrak{q}}$ has to be the complex conjugation $j \in \Gamma$. Thus,

$$\begin{aligned} w_L\theta_{L/k,R}(0) &= w_L\theta_{L/k}(0) \cdot (1-j)\beta \quad \text{for some } \beta \in \mathbb{Z}[\Gamma] \\ &= 2w_L\theta_{L/k}(0)\beta, \end{aligned}$$

and hence $w_L\theta_{L/k,R}(0)/2 \in \mathrm{BrSt}^{(w_L)}_{\mathbb{Z}[\Gamma]}(I_L)$. This ends the proof, since the usual Brumer–Stark conjecture is known to be true for quadratic extensions. ∎

It might be worthwhile to reinterpret a theorem of Erickson in terms of Question 2.1.

THEOREM 2.11 (Erickson). *Let S be a* 1*-cover for K/k. Moreover, suppose there exists a subset $S' \subseteq S$ which consists only of unramified finite primes and is a* 1*-cover for K/k. Then, for every $\mathfrak{p} \in S_{\min}$,*

$$\omega_{L/k,R} \in n\mathbb{Z}[\Gamma].$$

Therefore, there exists $\alpha \in \mathbb{Z}[\Gamma]$ (which depends on $\mathfrak{p}$) such that

$$w_L\theta_{L/k,R}(0)/n = w_L\theta_{L/k}(0)\alpha,$$

and if the usual Brumer–Stark conjecture is true for L/k, then

$$w_L\theta_{L/k,R}(0)/n \in \mathrm{BrSt}^{(w_L)}_{\mathbb{Z}[\Gamma]}(I_L).$$

Proof. See the proof of Theorem 6.1 in [3]. ∎

Let us remark that if this theorem applies, then $S_{\min}$ consists only of finite unramified primes.

2.3. Non-semisimplicity. Question 2.1 involves a "non-semisimple" situation as we now explain. In [15], we made the following definition.

DEFINITION 2.12. Let G be a finite abelian group. A subgroup H is called *cocyclic* if G/H is a cyclic group. Moreover, if H is cocyclic and if $K \subsetneq H$ implies that G/K is not cyclic, then we say that H is a *minimal cocyclic subgroup*.

We refer to §3.3 of [15] for the relation between minimal cocyclic subgroups and 1-covers.

LEMMA 2.13. *Let G be a finite abelian group and let H be a subgroup. Then H is a minimal cocyclic subgroup of G if and only if $H\{p\}$ is a minimal cocyclic subgroup of $G\{p\}$ for all prime p.*

Proof. This is clear using the fact that $G/H \simeq \bigoplus_p G\{p\}/H\{p\}$, where the direct sum is over all prime numbers p. ∎

Let us now come back to the setting of Question 2.1 for some finite prime $\mathfrak{p} \in S_{\min}$.

PROPOSITION 2.14. *Let p be a prime number. If $p \mid n$, then $p \mid [L : k]$.*

Proof. Since $\mathfrak{p} \in S_{\min}$, we see that $G_{\mathfrak{p}}$ is contained in a minimal cocyclic subgroup H of G (Theorems 3.15 and 3.16 of [15]). If $p \nmid [L : k]$, then $G_{\mathfrak{p}}\{p\} = G\{p\}$. Thus

$$G\{p\} = G_{\mathfrak{p}}\{p\} \subseteq H\{p\} \subseteq G\{p\},$$

and we conclude that $H\{p\} = G\{p\}$. This contradicts Lemma 2.13. ■

By Proposition 2.6, we know that if $(p, n) = 1$, then Question 2.1 does not say anything new assuming the usual Brumer–Stark conjecture. Now, if $p \mid n$, then Proposition 2.14 says that we are studying an annihilation statement

$$w_L \theta_{L/k,R}(0)/n \in \mathrm{Ann}_{\mathbb{Z}[\Gamma]}(\mathrm{Cl}_L\{p\}),$$

in the case where $p \mid [L : k]$. This is the so-called "non-semisimple" case.

Remark. Suppose we are in the setting of the question $\mathrm{St}(K/k, S, v)$ of [15] where $v \in S_{\min}$ is a real infinite place of k. The same argument as above shows that we necessarily have $2 \mid [L : k]$ where $L = K^{G_v}$.

3. Two infinite families of abelian extensions of $\mathbb{Q}$

3.1. The first family of abelian extensions of $\mathbb{Q}$ to be studied. The following example is taken from [2] and was studied numerically in some cases in [15] and [16]. Let p, q be two odd prime numbers satisfying

$$p \equiv 1 \pmod 4, \quad q \equiv 3 \pmod 4, \quad \left(\frac{p}{q}\right) = 1.$$

Let $K' = \mathbb{Q}(\zeta_q)^{D_p}$, where D_p is the decomposition group associated to p in $\mathbb{Q}(\zeta_q)/\mathbb{Q}$. We remark that $\mathbb{Q}(\sqrt{-q}) \subseteq K'$. Let K'^{+} be the maximal real subfield of K'. Let ℓ be an odd prime number, different from p and q and satisfying

$$\left(\frac{K'/\mathbb{Q}}{\ell}\right) \neq 1, \quad \left(\frac{K'^{+}/\mathbb{Q}}{\ell}\right) = 1, \quad \left(\frac{p}{\ell}\right) = -1,$$

(i.e. ℓ splits completely in $K'^{+}/\mathbb{Q}$, but does not split completely in $K'/\mathbb{Q}$ nor in $\mathbb{Q}(\sqrt{p})/\mathbb{Q}$). Let $K = K'(\sqrt{p})$. Then the Galois group $G = \mathrm{Gal}(K/\mathbb{Q})$ is isomorphic to

$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z},$$

where $m = (q-1)/2f_p$, and f_p is the inertia index of p in $\mathbb{Q}(\zeta_q)$. Since $q \equiv 3 \pmod 4$ we have $(m, 2) = 1$. Note that $|G_p| = |G_\ell| = 2$ and G_p, G_∞, G_ℓ are precisely the three subgroups of index 2 in G. By Theorems 3.15 and 3.16 of [15], we see that $S = \{\infty, p, q, \ell\}$ is a 1-cover for $K/\mathbb{Q}$ and $S_{\min} = \{\infty, p, \ell\}$.

If $m = 1$, then we are in the setting of Theorem 2.10. Otherwise, the simplest thing that could happen is that m is an odd prime. Here is our main theorem for this family of abelian extensions of $\mathbb{Q}$.

Theorem 3.1. *We have:*

(1) *For the ramified prime $p \in S_{\min}$,*

$$w_L \theta_{L/\mathbb{Q},R}(0)/2 \in \mathbb{Z}[\Gamma].$$

Moreover,

$$w_L\theta_{L/\mathbb{Q},R}(0)/2 \in \mathrm{BrSt}^{(w_L)}_{\mathbb{Z}[\Gamma]}(I_L).$$

Hence, Question 2.1 *has an affirmative answer for* $p \in S_{\min}$.

(2) *For the unramified prime* $\ell \in S_{\min}$,

$$w_L\theta_{L/\mathbb{Q},R}(0)/2 \in \mathbb{Z}[\Gamma].$$

Suppose moreover that m *is an odd prime and that* 2 *is a primitive root modulo* m. *Then*

$$w_L\theta_{L/\mathbb{Q},R}(0)/2 \in \mathrm{BrSt}^{(w_L)}_{\mathbb{Z}[\Gamma]}(I_L).$$

Proof. We start by looking at the ramified prime p.

The ramified prime p. Recall that $L=K^{G_p}$, $\Gamma=\mathrm{Gal}(L/\mathbb{Q})$, $R=S\setminus\{p\}$, and $|G_p| = 2$. Since

$$\omega_R = 1-\sigma_\ell^{-1},$$

and $\sigma_\ell = j$ in Γ, we have

$$w_L\theta_{L/\mathbb{Q},R}(0) = w_L\theta_{L/\mathbb{Q}}(0)\cdot(1-j) = 2w_L\theta_{L/\mathbb{Q}}(0).$$

Therefore,

$$w_L\theta_{L/\mathbb{Q},R}(0)/2 = w_L\theta_{L/\mathbb{Q}}(0) \in \mathbb{Z}[\Gamma] \quad \text{and} \quad w_L\theta_{L/\mathbb{Q},R}(0)/2 \in \mathrm{BrSt}^{(w_L)}_{\mathbb{Z}[\Gamma]}(I_L),$$

because of the usual Brumer–Stark conjecture for the special value $w_L\theta_{L/\mathbb{Q}}(0)$. Hence, point (1) of Theorem 3.1 is now proved.

The unramified prime ℓ. Recall here that $L = K^{G_\ell}$, $\Gamma = \mathrm{Gal}(L/\mathbb{Q})$, $R = S\setminus\{\ell\}$, and $|G_\ell| = 2$. As noted after Hypothesis 2.2, we have

$$w_L\theta_{L/\mathbb{Q},R}(0)/2 \in \mathbb{Z}[\Gamma],$$

since ℓ is unramified in $K/\mathbb{Q}$. Because of Corollary 2.5, Proposition 2.6 and noting that $w_{L,2} = w_L = 2$, we just have to show that

$$w_L\theta_{L/\mathbb{Q},R}(0)/2 \in \mathrm{BrSt}^{(2)}_{\mathbb{Z}[\Gamma]}(\mathrm{Cl}_L\{2\}).$$

If m is an odd prime and 2 is a primitive root modulo m, then this follows from Theorem 5.3 of [12]. (In Smith's notation, we have $K_1 = L$, $K_0 = \mathbb{Q}(\sqrt{-pq})$, $k_1 = (K')^+$, $k_0 = \mathbb{Q}$, his S is our R, $S_1 = \{\infty, p\}$, and $r = 1$.) ∎

Remark. In [12], it was shown that if a certain equation (7) were true, then the conclusion of point (2) of our Theorem 3.1 would hold true if m is any odd prime. In other words, we would not need the hypothesis that 2 is a primitive root modulo m. See Proposition 5.2 of [12].

As for the extended abelian rank one Stark conjecture for this example, here is what we can say. In the case where m is an odd prime and 2 is a primitive root modulo m, Theorem 3.1 implies that $\mathrm{St}(K/\mathbb{Q},S,p)$ and $\mathrm{St}(K/\mathbb{Q},S,\ell)$ have an affirmative answer (Propositions 4.4 and 4.9 of [15]).

We only have to study what is happening at $\infty \in S_{\min}$, i.e. to study $\mathrm{St}(K/\mathbb{Q}, S, \infty)$. Let now $L = K^+ = K^{G_\infty}$, $\Gamma = \mathrm{Gal}(L/\mathbb{Q})$ and fix a place w of L lying above ∞. Assuming the Gross conjecture (Conjecture 7.6 of [7]) in the case where the split prime is ∞, Theorem 4.27 of [15] implies the existence of an $\eta \in L^0$ such that

$$\theta'_{L/\mathbb{Q},S}(0) = -\sum_{\gamma \in \Gamma} \log|\eta^\gamma|_w \cdot \gamma^{-1}.$$

This case of the Gross conjecture when the base field is $\mathbb{Q}$ is known to be true because of classical results (see §1.1 of [1] combined with the usual functorial properties of the Gross conjecture such as top change and enlarging the set S). Because of Proposition 3.8 of [15], the extended abelian rank one Stark conjecture for K/k and S is true if and only if $\eta \in \mathcal{A}(L/\mathbb{Q}, 2)$, i.e. if the abelian condition of $\mathrm{St}(K/\mathbb{Q}, S, \infty)$ holds true.

3.2. The second family of abelian extensions of $\mathbb{Q}$ to be studied. Let ℓ be an odd prime. Fix two absolutely abelian fields K_i ($i = 1, 2$) which have degree ℓ over $\mathbb{Q}$ and prime conductor p_i ($p_1 \neq p_2$). Note that we necessarily have $p_i \equiv 1 \pmod{\ell}$. Suppose also that p_1 splits in K_2 and p_2 splits in K_1. Let $d < 0$ be such that

$$\ell \nmid d, \quad \left(\frac{d}{p_i}\right) = 1, \quad \text{and} \quad d \equiv 0, 1 \pmod{4}.$$

Let F be the imaginary quadratic field $\mathbb{Q}(\sqrt{d})$ and set $K = FK_1K_2$. For future use, we remark that $(\ell, w_K) = 1$. The field K is an abelian extension of $\mathbb{Q}$ with Galois group $G = \mathrm{Gal}(K/\mathbb{Q})$ isomorphic to

$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/\ell\mathbb{Z} \times \mathbb{Z}/\ell\mathbb{Z}.$$

Corollary 3.22 and Theorem 3.23 of [15] imply that there are $\ell + 1$ minimal cocyclic subgroups which are precisely the subgroups of order ℓ of G (see Section 3.3 of [15] for the notion of minimal cocyclic subgroup and its connection with 1-covers). Let us denote these subgroups by H_i for $i = 1, \ldots, \ell + 1$ in such a way that $H_i = G_{p_i}$, the decomposition group of p_i, for $i = 1, 2$. In order to have a 1-cover, we need to take unramified primes $p_3, \ldots, p_{\ell+1}$ satisfying

$$H_i = \langle \sigma_{p_i} \rangle \quad \text{for } i = 3, \ldots, \ell + 1.$$

By Theorems 3.15 and 3.16 of [15], we see that

$$S = \{\infty, p \mid d, p_1, \ldots, p_{\ell+1}\}$$

is a 1-cover for $K/\mathbb{Q}$ and

$$S_{\min} = \{p_1, \ldots, p_{\ell+1}\}.$$

Note that $|G_{p_i}| = \ell$ for all $i = 1, \ldots, \ell + 1$. Here is our main theorem for this family of abelian extensions of $\mathbb{Q}$.

THEOREM 3.2. *For $p \in S_{\min}$, let $L = K^{G_p}$ and $\Gamma = \mathrm{Gal}(L/\mathbb{Q})$. Let also $R = S \smallsetminus \{p\}$. Then Question* 2.1 *has an affirmative answer. More precisely*

$$w_L \theta_{L/\mathbb{Q},R}(0)/\ell \in \mathbb{Z}[\Gamma] \quad \text{and} \quad w_L \theta_{L/\mathbb{Q},R}(0)/\ell \in \mathrm{BrSt}^{(w_L)}_{\mathbb{Z}[\Gamma]}(I_L).$$

Proof. Given any prime $p \in S_{\min}$, letting $L = K^{G_p}$ as usual, we remark that $F \subseteq L$. Set $H = H_p = \mathrm{Gal}(L/F)$; it is a cyclic group of order ℓ. Given $\alpha \in \mathbb{Z}_\ell[\Gamma]^-$, there exists a unique $\widetilde{\alpha} \in \mathbb{Z}_\ell[H]$ such that

$$\alpha = e_- \widetilde{\alpha},$$

where e_- is the usual idempotent

$$e_- = \frac{1-j}{2} \in \mathbb{Z}_\ell[\Gamma].$$

We obtain in this way an isomorphism of rings

$$\psi : \mathbb{Z}_\ell[\Gamma]^- \to \mathbb{Z}_\ell[H], \qquad \alpha \mapsto \psi(\alpha) = \tilde{\alpha},$$

which allows us to identify $\mathbb{Z}_\ell[\Gamma]^-$ with $\mathbb{Z}_\ell[H]$. At times, we will denote the image of $\alpha \in \mathbb{Z}_\ell[\Gamma]^-$ simply by $\widetilde{\alpha}$ rather than $\psi(\alpha)$.

Since $(w_L, \ell) = 1$, one has

$$\theta_{L/\mathbb{Q}}(0) \in \mathbb{Z}_{(\ell)}[\Gamma]^- \subseteq \mathbb{Z}_\ell[\Gamma]^-.$$

The image of $\theta_{L/\mathbb{Q}}(0)$ via ψ will be denoted more simply by $\tilde{\theta}_{L/\mathbb{Q}}$.

The ramified primes p_1 and p_2. Both ramified primes p_1 and p_2 are treated in the same way. Let us take p_1 and set as usual $L = K^{G_{p_1}}$, $R = S \smallsetminus \{p_1\}$, $\Gamma = \mathrm{Gal}(L/\mathbb{Q})$, and $H = \mathrm{Gal}(L/F)$. Fix a generator h of H. Noting that p_2 ramifies in L^+ and splits in F, Theorem 6.1 of [5] implies that

$$\tilde{\theta}_{L/\mathbb{Q}} = \alpha(1-h)$$

for some $\alpha \in \mathbb{Z}_\ell[H]$ (denoted by ϑ_0 in [5]). Since $S(L/\mathbb{Q}) = S(K/\mathbb{Q}) \smallsetminus \{p_1\}$, we have

$$\theta_{L/\mathbb{Q},R}(0) = \theta_{L/\mathbb{Q}}(0) \cdot \omega_R, \quad \text{where} \quad \omega_R \in I_H^{\ell-1}.$$

Therefore,

$$\omega_R = (1-h)^{\ell-1}\beta$$

for some $\beta \in \mathbb{Z}[H]$, again since H is cyclic. Hence, we have

$$\begin{aligned} w_L \theta_{L/\mathbb{Q},R}(0) &= w_L e_- \alpha\beta(1-h)^\ell \\ &= w_L e_- \alpha\beta\ell(1-h)\gamma \quad \text{for some } \gamma \in \mathbb{Z}[H] \\ &= \ell w_L \theta_{L/\mathbb{Q}}(0)\beta\gamma. \end{aligned}$$

In the previous chain of equalities, we used the fact that

$$(1-h)^\ell \in \ell(1-h)\mathbb{Z}[H],$$

as a simple computation, using the binomial theorem, shows. We conclude that

$$w_L\theta_{L/\mathbb{Q},R}(0)/\ell = w_L\theta_{L/\mathbb{Q}}(0)\beta\gamma \in \mathbb{Z}[\Gamma],$$

and the classical Brumer–Stark conjecture for the special value $w_L\theta_{L/\mathbb{Q}}(0)$ implies that Theorem 3.2 is true for the two ramified primes in $S_{\min}$.

The unramified primes $p_3, \dots, p_{\ell+1}$. Again, all unramified primes p_i, for $i = 3, \dots, \ell+1$, are treated in the same way. Fix such an i and let $L = K^{G_{p_i}}$. We already know that

$$w_L\theta_{L/\mathbb{Q},R}(0)/\ell \in \mathbb{Z}[\Gamma],$$

because we are in the situation where $p \in S_{\min}$ is unramified in $K/\mathbb{Q}$ (see the discussion after Hypothesis 2.2). Because of Corollary 2.5 and Propositions 2.6, 2.7 ($(\ell, w_L) = 1$) and 2.9, we just have to show that

$$w_L\theta_{L/\mathbb{Q},R}(0)/\ell \in \mathrm{Ann}_{\mathbb{Z}[\Gamma]}(\mathrm{Cl}_L\{\ell\}^-).$$

Let h be a generator of H. Since both p_1 and p_2 are ramified in $L^+/\mathbb{Q}$, but split in F, Theorem 6.1 of [5] shows that there exists $\alpha \in \mathbb{Z}_\ell[H]$ (in Greither and Kučera's notation, we have $h = \gamma$, $\alpha = \vartheta_0$, $s = 2$, and our H is their Γ) satisfying

$$\tilde{\theta}_{L/\mathbb{Q}} = \alpha(1-h)^2,$$

and such that

$$\alpha(1-h) \in \mathrm{Ann}_{\mathbb{Z}_\ell[H]}(\mathrm{Cl}_L\{\ell\}^-). \tag{3.1}$$

Now, letting $R = S \smallsetminus \{p_i\}$, we have $\omega_R \in I_H^{\ell-2}$. Write

$$\omega_R = \beta(1-h)^{\ell-2}$$

for some $\beta \in \mathbb{Z}[H]$. Then

$$\begin{aligned} w_L\theta_{L/\mathbb{Q},R}(0) &= w_L e_-\alpha(1-h)^\ell\beta \\ &= \ell w_L e_-\alpha(1-h)\gamma\beta \quad \text{for some } \gamma \in \mathbb{Z}[H]. \end{aligned}$$

Therefore,

$$w_L\theta_{L/\mathbb{Q},R}(0)/\ell = w_L e_-\alpha(1-h)\gamma\beta \in \mathrm{Ann}_{\mathbb{Z}[\Gamma]}(\mathrm{Cl}_L\{\ell\}^-). \blacksquare$$

Here are some remarks. Suppose we are in the situation of an unramified prime $p \in S_{\min}$ in the proof of Theorem 3.2. Given a prime $\mathfrak{p}$ of L, the classical Brumer–Stark conjecture gives the existence of an $\varepsilon_R(\mathfrak{p}) \in L^0 \cap \mathcal{A}(L/\mathbb{Q}, w_L)$ such that

$$\mathfrak{p}^{w_L\theta_{L/\mathbb{Q},R}(0)} = (\varepsilon_R(\mathfrak{p})).$$

In the case where $\mathfrak{p}$ is relatively prime to the conductor of L, the algebraic number $\varepsilon_R(\mathfrak{p})$ is a product of normalized Gauss sums, since our base field is $\mathbb{Q}$. Instead of simply using results of [5] as we did, it would be possible to show directly that $\varepsilon_R(\mathfrak{p})$ is an ℓth power of an element in $L^0 \cap \mathcal{A}(L/\mathbb{Q}, w_L)$

using certain complicated identities in group rings and the norm property satisfied by the Euler system of Gauss sums. In fact, this was Greither and Kučera's original approach to related problems (see [4] for instance).

One could ask the following question regarding Question 2.1: If the answer is "yes", do we get new annihilators or not? In the setting of Theorem 3.2, the classical special value

$$w_L \theta_{L/\mathbb{Q},R}(0)$$

is always in the ℓ-Stickelberger ideal in the sense of Sinnott. The motivating question of [5] was to understand whether or not the Sinnott Stickelberger ideal tells the whole story about the annihilator of the class group. In other words, is it possible to find annihilators outside the Sinnott Stickelberger ideal? The authors of [5] actually found such examples. Using their remarks in that paper, we now explain that in some cases, the special value

$$w_L \theta_{L/\mathbb{Q},R}(0)/\ell$$

of Theorem 3.2 is *not* in the ℓ-Sinnott Stickelberger ideal either.

Let us first recall how the Sinnott Stickelberger ideal is defined (for more details see [11]). If both E and F are finite abelian extensions of $\mathbb{Q}$ such that $E \subseteq F$ and if R is a commutative ring of characteristic 0 (such as $\mathbb{Z}, \mathbb{Z}_\ell, \mathbb{Q}$, etc.), then the usual restriction homomorphism defined as $\sigma \mapsto \sigma|_E$, for $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$ induces an R-algebra morphism

$$\mathrm{res}_{F/E} : R[\mathrm{Gal}(F/\mathbb{Q})] \to R[\mathrm{Gal}(E/\mathbb{Q})].$$

We also need the corestriction map

$$\mathrm{cor}_{F/E} : R[\mathrm{Gal}(E/\mathbb{Q})] \to R[\mathrm{Gal}(F/\mathbb{Q})],$$

defined for $\sigma \in \mathrm{Gal}(E/\mathbb{Q})$ as

$$\mathrm{cor}_{F/E}(\sigma) = \sum_{\substack{\tau \in \mathrm{Gal}(F/\mathbb{Q}) \\ \tau|_E = \sigma}} \tau.$$

In contrast to $\mathrm{res}_{F/E}$, the corestriction map is not a morphism of rings, but it is additive. In other words, it is a morphism of R-modules.

For $\mathbb{Q}(\zeta_n)$, where $n \not\equiv 2 \pmod 4$, we have the classical Stickelberger element

$$\theta_n := \sum_{\substack{t=1 \\ (t,n)=1}}^{n} \frac{t}{n} \sigma_t^{-1} \in \mathbb{Q}[G_n],$$

where G_n is the Galois group of $\mathbb{Q}(\zeta_n)/\mathbb{Q}$. Given $E/\mathbb{Q}$ we let

$$N_E := \sum_{\sigma \in \mathrm{Gal}(E/\mathbb{Q})} \sigma.$$

One has

$$\theta_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}(0) = \frac{1}{2} N_{\mathbb{Q}(\zeta_n)} - \theta_n,$$

as a consequence of the knowledge of the value at $s = 0$ of the Hurwitz zeta function. If L is a finite abelian extension of $\mathbb{Q}$ with conductor m, let

$$\theta_L = \mathrm{res}_{\mathbb{Q}(\zeta_m)/L}(\theta_m).$$

Applying the restriction map to the corresponding equality for $\mathbb{Q}(\zeta_m)$, we have

$$\theta_{L/\mathbb{Q}}(0) = \frac{1}{2}[\mathbb{Q}(\zeta_m) : L] N_L - \theta_L.$$

Let us denote the Galois group of $L/\mathbb{Q}$ by Γ. Let S'_L be the submodule of $\mathbb{Q}[\Gamma]$ generated over $\mathbb{Z}[\Gamma]$ by the elements of the form $\mathrm{cor}_{L/L'}(\theta_{L'})$, where L' runs over all subfields of L. The Sinnott Stickelberger ideal is defined to be

$$S_L = S'_L \cap \mathbb{Z}[\Gamma].$$

If p is any prime number, the same construction as above replacing $\mathbb{Z}$ and $\mathbb{Q}$ by $\mathbb{Z}_p$ and $\mathbb{Q}_p$ respectively gives the p-Sinnott Stickelberger ideal. We shall use the notation $S'_{L,p}$ and $S_{L,p}$. Note that by construction, one has

$$S_L \subseteq S_{L,p}$$

for all prime numbers p.

Let us come back to the situation of Theorem 3.2, where $p \in S_{\min}$ is an unramified prime, $|G_p| = \ell$, and $L = K^{G_p}$. Since ℓ is relatively prime to the conductor of L, we have $S'_{L,\ell} = S_{L,\ell}$. Consider the minus ℓ-Sinnott Stickelberger ideal

$$S^-_{L,\ell} = e_- S_{L,\ell}.$$

If L' is a subfield of L which is real with conductor m, then

$$\theta_{L'} = \frac{1}{2}[\mathbb{Q}(\zeta_m) : L'] N_{L'},$$

which lives in the plus part. Hence, $S^-_{L,\ell}$ is generated over $\mathbb{Z}_\ell[\Gamma]$ by the elements of the form $\mathrm{cor}_{L/L'}(\theta_{L'})$, where L' runs over all totally complex subfields of L. Let $\widetilde{S}_L$ be the image of $S^-_{L,\ell}$ under the isomorphism ψ of the beginning of the proof of Theorem 3.2. The ideal $\widetilde{S}_L$ is generated over $\mathbb{Z}_\ell[H]$ by the elements $\mathrm{cor}_{L/L'}(\widetilde{\theta}_{L'})$, as L' runs over all totally complex subfields of L.

As explained on page 1650 of [5] (see comment (2)), the element $\alpha(1-h)$ of equation (3.1) is outside the Sinnott Stickelberger ideal of L. Greither and Kučera's reasoning is as follows. They work in the quotient ring $\mathbb{Z}_\ell[H]/N_H$, which is isomorphic to $\mathbb{Z}_\ell[\zeta_\ell]$ and hence is a discrete valuation ring. The image of $\widetilde{S}_L$ in this quotient ring is generated over $\mathbb{Z}_\ell[H]$ by a unique element, namely

$$\widetilde{\theta}_L = -\widetilde{\theta}_{L/\mathbb{Q}}.$$

If $\alpha(1-h)$ were in this image, then it would be a multiple of $\widetilde{\theta}_{L/\mathbb{Q}} = \alpha(1-h)^2$. We would then conclude that $1 - h$ is a unit in this ring, which is not the case (the element $1 - \zeta_\ell$ is not a unit in $\mathbb{Z}_\ell[\zeta_\ell]$).

For simplicity, let us assume further that $\ell = 3$. In this case, it is simple to check that both β and γ in the proof of Theorem 3.2 for unramified primes in $S_{\min}$ are also units once projected onto the ring

$$\mathbb{Z}_3[H]/N_H \simeq \mathbb{Z}_3[\zeta_3].$$

Therefore, we conclude as well that $w_L\theta_{L/\mathbb{Q},R}(0)/3 \notin S_{L,3}$.

As a corollary to Theorem 3.2, we get:

COROLLARY 3.3. *In the same setting as in Theorem* 3.2*, the extended abelian rank one Stark conjecture is true for* $K/\mathbb{Q}$ *and the* 1*-cover* S.

Proof. If $p \in S_{\min}$ is a ramified prime, Theorem 3.2 implies that the four conditions of Proposition 4.4 of [15] are true; hence, $\mathrm{St}(K/\mathbb{Q}, S, p)$ has an affirmative answer. If $p \in S_{\min}$ is unramified, Theorem 3.2 implies that Question 4.7 of [15] has an affirmative answer. Hence, by Proposition 4.3 of [15], we can conclude that the extended abelian rank one Stark conjecture is true for $K/\mathbb{Q}$ and the 1-cover S. ∎

4. Conclusion. The goal of this paper was to provide further evidence for the extended abelian rank one Stark conjecture by studying two infinite families of finite abelian extensions of $\mathbb{Q}$. Rather than studying the extended abelian rank one Stark conjecture itself, we studied a stronger statement, namely an extension of the Brumer–Stark conjecture (Question 2.1). We believe that this extension of the Brumer–Stark conjecture is interesting in itself. There are some new phenomena proper to Question 2.1. Most notably perhaps, in the case where the statement does not reduce to the classical Brumer–Stark conjecture, we are always in a "non-semisimple" situation as explained in §2.3. Moreover, we saw that if true, the extension of the Brumer–Stark conjecture predicts the existence of annihilators which might be outside the Stickelberger ideal in the sense of Sinnott.

If Question 2.1 is true, it would be interesting to find a proof, in the case where the base field is $\mathbb{Q}$, along the classical line of Stickelberger's theorem.

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