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The complete determination of wide Richaud–Degert types which are not 5 modulo 8 with class number one

by

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1. Introduction and statement of results. Let d be a square free integer and $h(d)$ be the class number of $\mathbb{Q}(\sqrt{d})$. Let $d = n^2 + r$ be a square free integer such that $r \mid 4n$ and $-n < r \le n$. In this case, we call d a *Richaud–Degert type*. If $|r| \ne 1, 4$, then it is called a *wide Richaud–Degert type*, while if $|r| = 1$ or 4, then it is called a *narrow Richaud–Degert type*.

There had been many conjectures about the upper bound of Richaud–Degert types d with $h(d) = 1$. Yokoi [15] conjectured that $h(n^2 + 4) > 1$ if $n > 17$. Chowla [7] conjectured that $h(4n^2 + 1) > 1$ if $n > 13$. Biró [1], [2] proved the above two conjectures. Also Mollin [10] conjectured that $h(n^2 - 4) = 1$ if $n > 21$. Mollin's conjecture was confirmed by Byeon, Kim and the author [5]. This determines all real quadratic fields of narrow Richaud–Degert types with class number 1.

In this paper, we prove the following Mollin–Williams conjecture (see Conjecture 5.4.4 on page 176 of [11]). From this, we show that there are exactly 14 wide R-D types $d \not\equiv 5 \pmod 8$ with $h(d) = 1$.

MOLLIN AND WILLIAMS CONJECTURE. *Let $d = n^2 \pm 2$ be a squarefree integer. Then $h(d) > 1$ if $n > 20$.*

THEOREM 1.1. *Let $d \not\equiv 5 \pmod 8$ be a wide Richaud–Degert type. Then $h(d) = 1$ if and only if*

$$d = 3, 6, 7, 11, 14, 23, 33, 38, 47, 62, 83, 167, 227, 398.$$

2. Computation of special values of the zeta function associated with $\mathbb{Q}(\sqrt{n^2 - 2})$. Let $d = n^2 - 2$ be a positive square free integer, $K = \mathbb{Q}(\sqrt{d})$ and $O(K)$ the ring of integers of K. Then $\epsilon = n^2 - 1 + n\sqrt{n^2 - 2}$ is a fundamental unit of K (see [8]), and $\{1, \omega\}$ is an integral basis for $O(K)$, where $\omega = \sqrt{n^2 - 2}$. For an integral ideal $\mathbf{a}$, let $N(\mathbf{a})$ be the number of

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cosets of $O(K)/\mathbf{a}$, and for an element α of K, let $N_K(\alpha) = \alpha \cdot \bar{\alpha}$. We note that $N_K(\epsilon) = 1$. Let $I(K)$ be the set of nonzero fractional ideals of K. Let K^+ be the set of totally positive elements in K, and $i(K^+)$ be the set of principal fractional ideals generated by the elements in K^+. Let χ be an odd primitive character with conductor q. Then from the fact that $N_K(\epsilon) = 1$ and $N_K(\omega) < 0$, we have the following proposition.

PROPOSITION 2.1 ([9, pp. 242–243]). *If* $h(d) = 1$, *then*

$$I(K) = (q) \cdot i(K^+) \cup (q\omega) \cdot i(K^+). \blacksquare$$

Thus if $h(d) = 1$, then

$$\begin{aligned} \zeta_K(s,\chi) &:= \sum_{\substack{\mathbf{a} \in I(K) \\ \text{integral}}} \frac{\chi(N(\mathbf{a}))}{N(\mathbf{a})^s} \\ &= \sum_{\substack{\mathbf{a} \in (q)\cdot i(K^+) \\ \text{integral}}} \frac{\chi(N(\mathbf{a}))}{N(\mathbf{a})^s} + \sum_{\substack{\mathbf{a} \in (q\omega)\cdot i(K^+) \\ \text{integral}}} \frac{\chi(N(\mathbf{a}))}{N(\mathbf{a})^s}. \end{aligned} \tag{1}$$

By defining

$$R(\mathbf{b}) := \{a+b\epsilon \mid a,b \in \mathbb{Q} \text{ with } 0 < a \le 1,\ 0 \le b < 1 \text{ and } \mathbf{b}\cdot(a+b\epsilon) \subset O(K)\}$$

for an integral ideal $\mathbf{b}$ in K, we have the following propositon.

PROPOSITION 2.2 ([5, Lemma 2.2]). *An integral ideal* $\mathbf{a}$ *of* K *is in* $\mathbf{b} \cdot i(K^+) := \{\mathbf{b} \cdot \mathbf{c} \mid \mathbf{c} \in i(K^+)\}$ *if and only if*

$$\mathbf{a} = \mathbf{b} \cdot (a + b\epsilon + n_1 + n_2\epsilon)$$

for some $a + b\epsilon \in R(\mathbf{b})$ *and nonnegative integers* n_1, n_2. $\blacksquare$

In the following lemma, we find the set of all (x, y) for which $x + y\epsilon \in R((q))$.

LEMMA 2.3.

$$\begin{aligned} &\{(x,y) \mid x + y\epsilon \in R((q))\} \\ &= \left\{ (x,y) \,\middle|\, x = -\frac{r_{C,D}(n)}{q} + \frac{D+qj}{nq} + \sigma_1(j) \text{ and } y = \frac{D+qj}{nq} \right. \\ &\qquad\qquad \left. \text{for } j = 0,1,\ldots,n-1 \text{ and } 0 \le C, D \le q-1 \right\}, \end{aligned}$$

where

$$\sigma_1(j) = \begin{cases} 1 & \text{if } 0 \le j \le \left[\dfrac{nr_{C,D}(n) - D}{q}\right], \\ 0 & \text{if } \left[\dfrac{nr_{C,D}(n) - D}{q}\right] + 1 \le j \le n-1, \end{cases}$$

and

$$r_{C,D}(n) = nD - C - q\left[\frac{nD-C}{q}\right].$$

Proof. Since $\{C + D\omega \mid 0 \le C, D \le q-1\}$ represents all elements in $O(K)/qO(K)$, we have

$$\begin{aligned}
&\{(x,y) \mid x + y\epsilon \in R((q))\}\\
&\quad = \{(x,y) \mid x + y\epsilon \in q^{-1}O(K) \text{ and } 0 < x \le 1,\ 0 \le y < 1\}\\
&\quad = \{(x,y) \mid q(x+y\epsilon) = C + D\omega + q(i + j\omega) \text{ for } 0 \le C, D \le q-1,\\
&\qquad\qquad\qquad\qquad\qquad \text{integers } i, j \text{ and } 0 < x \le 1,\ 0 \le y < 1\}.
\end{aligned}$$

Moreover, the equation

$$\omega = \frac{\epsilon}{n} - \frac{n^2-1}{n}$$

implies that

$$q(x+y\epsilon) = C + qi + (D+qj)\left(\frac{\epsilon}{n} - \frac{n^2-1}{n}\right).$$

Hence

$$\begin{aligned}
y &= \frac{D+qj}{nq},\\
x &= \frac{C}{q} - \frac{(D+qj)(n^2-1)}{nq} + \left[1 + \frac{(D+qj)(n^2-1)}{nq} - \frac{C}{q}\right]
\end{aligned}$$

for $j = 0, 1, \ldots, n-1$. The equation

$$\frac{(D+qj)(n^2-1)}{nq} - \frac{C}{q} = \left[\frac{nD-C}{q}\right] + \frac{r_{C,D}(n)}{q} - \frac{D+qj}{qn} + nj$$

implies

$$\begin{aligned}
&\frac{C}{q} - \frac{(D+qj)(n^2-1)}{nq} + \left[1 + \frac{(D+qj)(n^2-1)}{nq} - \frac{C}{q}\right]\\
&\qquad\qquad = 1 - \frac{r_{C,D}(n)}{q} + \frac{D+qj}{qn} + \left[\frac{r_{C,D}(n)}{q} - \frac{D+qj}{qn}\right].
\end{aligned}$$

Since

$$\left[\frac{r_{C,D}(n)}{q} - \frac{D+qj}{nq}\right] + 1 = \begin{cases} 1 & \text{if } 0 \le j \le \left[\dfrac{nr_{C,D}(n) - D}{q}\right],\\[2ex] 0 & \text{if } 1 + \left[\dfrac{nr_{C,D}(n) - D}{q}\right] \le j \le n-1,\end{cases}$$

the proof is complete. ■

We also find the set of (x, y) with $x + y\epsilon \in R((q\omega))$:

LEMMA 2.4.

$$\{(x, y) \mid x + y\epsilon \in R((q\omega))\}$$
$$= \left\{ (x, y) \,\middle|\, x = 1 + \frac{D}{q} - \frac{(n^2-1)(C+iq)}{qn(n^2-2)} + \sigma_2(i) \text{ and } y = \frac{C+iq}{qn(n^2-2)} \right.$$
$$\left. \text{for } i = 0, 1, \ldots, n(n^2-2) - 1 \text{ and } 0 \leq C, D \leq q-1 \right\},$$

where

$$\sigma_2(i) = k \;\Leftrightarrow\; l(k) \leq i < l(k+1)$$

for $k = -1, 0, 1, \ldots, n^2 - 2$ *and*

$$l(k) = \left\lceil \frac{n(n^2-2)(qk+D)}{q(n^2-1)} - \frac{C}{q} \right\rceil.$$

Proof. The set $\{C + D\omega \mid 0 \leq C, D \leq q-1\}$ represents all elements in $O(K)/qO(K)$. So

$$\begin{aligned}
&\{(x, y) \mid x + y\epsilon \in R((q\omega))\} \\
&= \{(x, y) \mid x + y\epsilon \in (q\omega)^{-1}O(K) \text{ and } 0 < x \leq 1,\ 0 \leq y < 1\} \\
&= \{(x, y) \mid q\omega(x + y\epsilon) = C + D\omega + q(i + j\omega) \text{ for } 0 \leq C, D \leq q-1, \\
&\qquad\qquad \text{integers } i, j \text{ and } 0 < x \leq 1,\ 0 \leq y < 1\}.
\end{aligned}$$

From $\bar{\omega} = -\frac{\epsilon}{n} + \frac{n^2-1}{n}$, we deduce that

$$q\omega(x + y\epsilon) = (C + qi) + \omega(D + qj) \;\Leftrightarrow$$
$$-q(n^2-2)(x + y\epsilon) = -(n^2-2)(D + qj) + \frac{n^2-1}{n}(C + iq) - \frac{C+iq}{n}\epsilon.$$

So

$$y = \frac{C+qi}{qn(n^2-2)},$$
$$x = 1 + \frac{D}{q} - \frac{(n^2-1)(C+iq)}{qn(n^2-2)} + \left\lfloor \frac{(n^2-1)(C+iq)}{qn(n^2-2)} - \frac{D}{q} \right\rfloor$$

for $i = 0, 1, \ldots, n(n^2-2) - 1$. By defining

$$\sigma_2(i) := \left\lfloor \frac{(n^2-1)(C+iq)}{qn(n^2-2)} - \frac{D}{q} \right\rfloor,$$

we see that $\sigma_2(i) = k$ for $k = -1, 0, 1, \ldots, n^2 - 2$ if and only if

$$\left\lceil \frac{n(n^2-2)(qk+D)}{q(n^2-1)} - \frac{C}{q} \right\rceil \leq i < \left\lceil \frac{n(n^2-2)(q(k+1)+D)}{q(n^2-1)} - \frac{C}{q} \right\rceil. \;\blacksquare$$

From Lemmas 2.3 and 2.4, we deduce

PROPOSITION 2.5. *If* $h(n^2-2)=1$, *then*

$$\zeta_K(0,\chi) = \sum_{0\le C,D\le q-1} \chi(C^2-(n^2-2)D^2) \times \Bigg[\sum_{j=0}^{n-1} S\Big(-\frac{r_{C,D}(n)}{q}+\frac{D+qj}{nq}+\sigma_1(j),\frac{D+qj}{nq}\Big) - \sum_{i=0}^{(n^2-2)n-1} S\Big(1-\frac{(C+iq)(n^2-1)}{qn(n^2-2)}+\frac{D}{q}+\sigma_2(i),\frac{C+iq}{qn(n^2-2)}\Big)\Bigg]$$

where $S(x,y)=B_1(x)B_1(y)+\frac{1}{4}(\epsilon+\overline{\epsilon})(B_2(x)+B_2(y))$ *and* B_1, B_2 *are the first and second Bernoulli polynomials.*

Proof. By Proposition 2.2, we have

$$\sum_{\substack{\mathbf{a}\in(q)\cdot i(K^+)\\ \text{integral}}} \frac{\chi(N(\mathbf{a}))}{N(\mathbf{a})^s} = \sum_{x+y\epsilon\in R((q))}\sum_{n_1,n_2=0}^{\infty} \frac{\chi(N((q)\cdot(x+y\epsilon+n_1+n_2\epsilon)))}{N((q)\cdot(x+y\epsilon+n_1+n_2\epsilon))^s} = \sum_{x+y\epsilon\in R((q))} \chi(N_K(q(x+y\epsilon)))\sum_{n_1,n_2=0}^{\infty} N_K(q(x+y\epsilon+n_1+n_2\epsilon))^{-s} \tag{2}$$

and

$$\sum_{\substack{\mathbf{a}\in(q\omega)\cdot i(K^+)\\ \text{integral}}} \frac{\chi(N(\mathbf{a}))}{N(\mathbf{a})^s} = \sum_{x+y\epsilon\in R((q\omega))}\sum_{n_1,n_2=0}^{\infty} \frac{\chi(N((q\omega)\cdot(x+y\epsilon+n_1+n_2\epsilon)))}{N((q\omega)\cdot(x+y\epsilon+n_1+n_2\epsilon))^s} = \sum_{x+y\epsilon\in R((q\omega))} \chi(-N_K(q\omega(x+y\epsilon)))\sum_{n_1,n_2=0}^{\infty} (-N_K(q\omega(x+y\epsilon+n_1+n_2\epsilon)))^{-s}. \tag{3}$$

Now we recall Shintani's result in [13], [14]:

$$\sum_{n_1,n_2=0}^{\infty} N_K(x+y\epsilon+n_1+n_2\epsilon)^{-s}|_{s=0} = S(x,y). \tag{4}$$

We note that for $x+y\epsilon\in R((q))$,

$$N_K(q(x+y\epsilon)) = N_K(C+D\omega+q(i+j\omega)) \equiv N_K(C+D\omega) \pmod q = C^2-(n^2-2)D^2, \tag{5}$$

and for $x + y\epsilon \in R((q\omega))$,

$$(6)\qquad N_K(q\omega(x+y\epsilon)) = N_K(C + D\omega + q(i + j\omega)) \equiv N_K(C + D\omega) \pmod q = C^2 - (n^2-2)D^2.$$

From Lemmas 2.3, 2.4 and the equations (1)–(6), we deduce the assertion immediately. ∎

We observe that $-1 \le \sigma_2(i) \le n^2-2$ for $0 \le i \le n(n^2-2)-1$ and $\sigma_2(i) = k \Leftrightarrow l(k) \le i < l(k+1)$. So we have

$$\sum_{i=0}^{n(n^2-2)-1} \sigma_2(i) = \sum_{i=0}^{l(0)-1}(-1) + \sum_{k=0}^{n^2-3}\sum_{i=l(k)}^{l(k+1)-1} k + (n^2-2)\sum_{i=l(n^2-2)}^{n(n^2-2)-1} 1,$$

$$(7)\qquad \sum_{i=0}^{n(n^2-2)-1} \sigma_2(i)^2 = \sum_{i=0}^{l(0)-1} 1 + \sum_{k=0}^{n^2-3}\sum_{i=l(k)}^{l(k+1)-1} k^2 + (n^2-2)^2\sum_{i=l(n^2-2)}^{n(n^2-2)-1} 1,$$

$$\sum_{i=0}^{n(n^2-2)-1} \sigma_2(i)i = \sum_{i=0}^{l(0)-1}(-i) + \sum_{k=0}^{n^2-3}\sum_{i=l(k)}^{l(k+1)-1} ki + (n^2-2)\sum_{i=l(n^2-2)}^{n(n^2-2)-1} i,$$

where $l(k)$ is computed in the following lemma.

Lemma 2.6. *Let*

$$s_{C,D}(n) = r_{C,D}(n)n - D - q\left[\frac{r_{C,D}(n)n - D}{q}\right], \qquad u_{C,D}(n) = \left[\frac{nD - C}{q}\right],$$

$$v_{C,D}(n) = \left[\frac{r_{C,D}(n)n - D}{q}\right], \qquad w_{C,D}(n) = \left\lceil\frac{s_{C,D}(n)n - r_{C,D}(n)}{q}\right\rceil.$$

Assume $n > q$ and $0 \le k \le n^2-2$ with $k = nk_1 + k_2$ $(k_2 = 0, 1, \ldots, n-1)$. If $r_{C,D}(n) \ne 0$ or $s_{C,D}(n) = 0$, then

$$l(k) = \begin{cases} u_{C,D}(n) + nk - k_1 + 1 & \text{if } k_1 < w_{C,D}(n) \text{ and } k_2 \le v_{C,D}(n),\\ u_{C,D}(n) + nk - k_1 & \text{if } k_1 < w_{C,D}(n) \text{ and } k_2 > v_{C,D}(n),\\ u_{C,D}(n) + nk - k_1 + 1 & \text{if } k_1 \ge w_{C,D}(n) \text{ and } k_2 < v_{C,D}(n),\\ u_{C,D}(n) + nk - k_1 & \text{if } k_1 \ge w_{C,D}(n) \text{ and } k_2 \ge v_{C,D}(n),\end{cases}$$

while if $r_{C,D}(n) = 0$ and $s_{C,D}(n) \ne 0$, then

$$l(k) = \begin{cases} u_{C,D}(n) + nk - k_1 - 1 & \text{if } k_1 \ge w_{C,D}(n) - 1 \text{ and } k_2 = n-1,\\ u_{C,D}(n) + nk - k_1 & \text{otherwise.}\end{cases}$$

Proof. Note that

$$l(k) = \left\lceil \frac{n(n^2-2)(qk+D)}{q(n^2-1)} - \frac{C}{q} \right\rceil = nk + \left\lceil \frac{nD-C}{q} - \frac{k}{n-\frac{1}{n}} - \frac{D}{q\left(n-\frac{1}{n}\right)} \right\rceil. \tag{8}$$

Moreover,

$$\left\lceil \frac{nD-C}{q} - \frac{k}{n-\frac{1}{n}} - \frac{D}{q\left(n-\frac{1}{n}\right)} \right\rceil = \left\lfloor \frac{nD-C}{q} \right\rfloor + \left\lceil \frac{r_{C,D}(n)}{q} - \frac{\frac{D}{q}+\frac{k_1}{n}+k_2}{n-\frac{1}{n}} \right\rceil - k_1. \tag{9}$$

Now we observe that

$$\left\lceil \frac{r_{C,D}(n)}{q} - \frac{\frac{D}{q}+\frac{k_1}{n}+k_2}{n-\frac{1}{n}} \right\rceil = \left\lceil \frac{\left\lfloor \frac{r_{C,D}(n)n-D}{q} \right\rfloor - k_2}{n-\frac{1}{n}} + \frac{\left\lfloor \frac{s_{C,D}(n)n-r_{C,D}(n)}{q} \right\rfloor - k_1}{n\left(n-\frac{1}{n}\right)} + \frac{s_{C,D}(n)n - r_{C,D}(n) - q\left\lfloor \frac{s_{C,D}(n)n-r_{C,D}(n)}{q} \right\rfloor}{qn\left(n-\frac{1}{n}\right)} \right\rceil. \tag{10}$$

We consider the case of $r_{C,D}(n) \neq 0$ or $s_{C,D}(n) = 0$,

$$k_1 < \left\lceil \frac{s_{C,D}(n)n - r_{C,D}(n)}{q} \right\rceil \quad \text{and} \quad k_2 \leq \left\lfloor \frac{r_{C,D}(n)n - D}{q} \right\rfloor. \tag{11}$$

For $n > q$ we have $\frac{s_{C,D}(n)n-r_{C,D}(n)}{q} < n-1$ and $\frac{r_{C,D}(n)n-D}{q} < n-1$. So

$$\left\lfloor \frac{s_{C,D}(n)n - r_{C,D}(n)}{q} \right\rfloor \leq n-2 \quad \text{and} \quad \left\lfloor \frac{r_{C,D}(n)n - D}{q} \right\rfloor \leq n-2. \tag{12}$$

If $s_{C,D}(n)n - r_{C,D}(n) - q\left\lfloor \frac{s_{C,D}(n)n-r_{C,D}(n)}{q} \right\rfloor = 0$, then $\left\lceil \frac{s_{C,D}(n)n-r_{C,D}(n)}{q} \right\rceil = \left\lfloor \frac{s_{C,D}(n)n-r_{C,D}(n)}{q} \right\rfloor$. Thus from (11), (12), we have

$$1 \leq \left\lfloor \frac{s_{C,D}(n)n - r_{C,D}(n)}{q} \right\rfloor - k_1 \leq n-2,$$

$$0 \leq \left\lfloor \frac{r_{C,D}(n)n - D}{q} \right\rfloor - k_2 \leq n-2.$$

Hence

$$\frac{1}{n\left(n-\frac{1}{n}\right)} \leq \frac{\left\lfloor \frac{r_{C,D}(n)n-D}{q} \right\rfloor - k_2}{n-\frac{1}{n}} + \frac{\left\lfloor \frac{s_{C,D}(n)n-r_{C,D}(n)}{q} \right\rfloor - k_1}{n\left(n-\frac{1}{n}\right)} \leq \frac{n^2-n-2}{n^2-1} < 1.$$

So

$$\left\lceil \frac{\left[\frac{r_{C,D}(n)n-D}{q}\right]-k_2}{n-\frac{1}{n}}+\frac{\left[\frac{s_{C,D}(n)n-r_{C,D}(n)}{q}\right]-k_1}{n\left(n-\frac{1}{n}\right)}\right\rceil=1. \tag{13}$$

If $s_{C,D}(n)n-r_{C,D}(n)-q\left[\frac{s_{C,D}(n)n-r_{C,D}(n)}{q}\right]\neq 0$, then $\left\lceil\frac{s_{C,D}(n)n-r_{C,D}(n)}{q}\right\rceil=\left[\frac{s_{C,D}(n)n-r_{C,D}(n)}{q}\right]+1$. From (11), (12), we have

$$\begin{aligned}0&\leq\left[\frac{s_{C,D}(n)n-r_{C,D}(n)}{q}\right]-k_1\leq n-2,\\0&\leq\left[\frac{r_{C,D}(n)n-D}{q}\right]-k_2\leq n-2,\end{aligned}$$

and

$$1\leq s_{C,D}(n)n-r_{C,D}(n)-q\left[\frac{s_{C,D}(n)n-r_{C,D}(n)}{q}\right]\leq q-1.$$

Thus

$$\begin{aligned}0<\frac{1}{qn\left(n-\frac{1}{n}\right)}&\leq\frac{\left[\frac{r_{C,D}(n)n-D}{q}\right]-k_2}{n-\frac{1}{n}}+\frac{\left[\frac{s_{C,D}(n)n-r_{C,D}(n)}{q}\right]-k_1}{n\left(n-\frac{1}{n}\right)}\\&\quad+\frac{s_{C,D}(n)n-r_{C,D}(n)-q\left[\frac{s_{C,D}(n)n-r_{C,D}(n)}{q}\right]}{qn\left(n-\frac{1}{n}\right)}\\&\leq\frac{n^2-n-2}{n^2-1}+\frac{q-1}{qn\left(n-\frac{1}{n}\right)}<1.\end{aligned}$$

So

$$\begin{aligned}&\left\lceil\frac{\left[\frac{r_{C,D}(n)n-D}{q}\right]-k_2}{n-\frac{1}{n}}+\frac{\left[\frac{s_{C,D}(n)n-r_{C,D}(n)}{q}\right]-k_1}{n\left(n-\frac{1}{n}\right)}\right.\\&\qquad\left.+\frac{s_{C,D}(n)n-r_{C,D}(n)-q\left[\frac{s_{C,D}(n)n-r_{C,D}(n)}{q}\right]}{qn\left(n-\frac{1}{n}\right)}\right\rceil=1.\end{aligned}\tag{14}$$

From (8)–(10) and (13)–(14), we have

$$l(k)=\left[\frac{nD-C}{q}\right]+nk-k_1+1.$$

The other cases can be handled similarly. ∎

We can express the summation (7) more explicitly using Lemma 2.6. In case $r_{C,D}(n)\neq 0$ or $s_{C,D}(n)=0$, we have

$$\sum_{k=0}^{n^2-3}\sum_{i=l(k)}^{l(k+1)-1}$$
$$= \sum_{k_1=0}^{w-1}\Big(\sum_{k_2=0}^{v-1}\sum_{i=n(k_1n+k_2)+u-k_1+1}^{n(k_1n+k_2+1)+u-k_1} + \sum_{i=n(k_1n+v)+u-k_1+1}^{n(k_1n+v+1)+u-k_1-1} + \sum_{k_2=v+1}^{n-1}\sum_{i=n(k_1n+k_2)+u-k_1}^{n(k_1n+k_2+1)+u-k_1-1}\Big)$$
$$+ \sum_{k_1=w}^{n-2}\Big(\sum_{k_2=0}^{v-2}\sum_{i=n(k_1n+k_2)+u-k_1+1}^{n(k_1n+k_2+1)+u-k_1} + \sum_{i=n(k_1n+v-1)+u-k_1+1}^{n(k_1n+v)+u-k_1-1} + \sum_{k_2=v}^{n-1}\sum_{i=n(k_1n+k_2)+u-k_1}^{n(k_1n+k_2+1)+u-k_1-1}\Big)$$
$$+ \sum_{k_2=0}^{v-2}\sum_{i=n(n^2-n+k_2)+u-n+2}^{n(n^2-n+k_2+1)+u-n+1} + \sum_{i=n(n^2-n+v-1)+u-n+2}^{n(n^2-n+v)+u-n} + \sum_{k_2=v}^{n-3}\sum_{i=n(n^2-n+k_2)+u-n+1}^{n(n^2-n+k_2+1)+u-n},$$

while in case $r_{C,D}(n) = 0$ and $s_{C,D}(n) \neq 0$, we have

$$\sum_{k=0}^{n^2-3}\sum_{i=l(k)}^{l(k+1)-1}$$
$$= \sum_{k_1=0}^{\omega-2}\Big(\sum_{k_2=0}^{n-2}\sum_{n(k_1n+k_2)+u-k_1}^{n(k_1n+k_2+1)+u-k_1-1} + \sum_{n(k_1n+n-1)+u-k_1}^{n(k_1n+n)+u-k_1-2}\Big)$$
$$+ \sum_{k_1=\omega-1}^{n-2}\Big(\sum_{k_2=0}^{n-3}\sum_{n(k_1n+k_2)+u-k_1}^{n(k_1n+k_2+1)+u-k_1-1} + \sum_{n(k_1n+n-2)+u-k_1}^{n(k_1n+n-1)+u-k_1-2} + \sum_{n(k_1n+n-1)+u-k_1-1}^{n(k_1n+n)+u-k_1-2}\Big)$$
$$+ \sum_{k_2=0}^{n-3}\sum_{n(n^2-n+k_2)+u-n+1}^{n(n^2-n+k_2+1)+u-n}.$$

(In the above two equations, u, v, w are $u_{C,D}(n)$, $v_{C,D}(n)$, $w_{C,D}(n)$ respectively.) From this expression of $l(k)$, we can calculate $\sum \sigma_2(i)$, $\sum i\sigma_2(i)$ and $\sum \sigma_2(i)^2$ by computer. If $r_{C,D}(n) \neq 0$ or $s_{C,D}(n) = 0$ then

$$\sum_{i=0}^{n(n^2-2)-1} \sigma_2(i) = \frac{1}{2}\,[-4n^3 + n^5 - n^2(1 + 2u_{C,D}(n)) - 2n(-2 + v_{C,D}(n)) + 2(1 + u_{C,D}(n) - w_{C,D}(n))],$$

$$\sum_{i=0}^{n(n^2-2)-1} \sigma_2(i)^2 = -3 - \frac{11n^5}{6} + \frac{n^7}{3} - 3u_{C,D}(n) - \frac{1}{2}\,n^4(1 + 2u_{C,D}(n)) - n^3(-3 + v_{C,D}(n)) + n^2(3 + 4u_{C,D}(n) + v_{C,D}(n)) + w_{C,D}(n) - 2v_{C,D}(n)w_{C,D}(n) - n\big(2 - 2v_{C,D}(n) + v_{C,D}(n)^2 - w_{C,D}(n) + w_{C,D}(n)^2\big),$$

$$\sum_{i=0}^{n(n^2-2)-1} i\sigma_2(i) =$$
$$\frac{1}{12}\big[-23n^6+4n^8-6n^5(1+u_{C,D}(n))+n^4(41-6v_{C,D}(n))$$
$$+6n^3(5+4u_{C,D}(n)+v_{C,D}(n))$$
$$-6n\big(6+v_{C,D}(n)+2u_{C,D}(n)(2+v_{C,D}(n))+2v_{C,D}(n)w_{C,D}(n)\big)$$
$$+6\big(u_{C,D}(n)+u_{C,D}(n)^2-2u_{C,D}(n)w_{C,D}(n)+(-1+w_{C,D}(n))w_{C,D}(n)\big)$$
$$-2n^2\big(11+3u_{C,D}(n)^2-6v_{C,D}(n)+3v_{C,D}(n)^2-3w_{C,D}(n)+3w_{C,D}(n)^2\big)\big].$$

If $r_{C,D}(n)=0$ and $s_{C,D}(n)\neq 0$ then

$$\sum_{i=0}^{n(n^2-2)-1} \sigma_2(i) = 1+3n-2n^3+\frac{n^5}{2}-w_{C,D}(n)+u_{C,D}(n) - \frac{1}{2}n^2(1+2u_{C,D}(n)),$$

$$\sum_{i=0}^{n(n^2-2)-1} \sigma_2(i)^2 =$$
$$4n^3-\frac{11n^5}{6}+\frac{n^7}{3}-\frac{n^4}{2}(1+2u_{C,D}(n))+n^2(2+4u_{C,D}(n))$$
$$-3(1+u_{C,D}(n)-w_{C,D}(n))+n(-5+w_{C,D}(n)-w_{C,D}(n)^2),$$

$$\sum_{i=0}^{n(n^2-2)-1} i\sigma_2(i) =$$
$$\frac{1}{12}\big[47n^4-23n^6+4n^8+24n^3(1+u_{C,D}(n))-6n^5(1+u_{C,D}(n))$$
$$-6n(5+2u_{C,D}(n)-2w_{C,D}(n))$$
$$+6(u_{C,D}(n)+u_{C,D}(n)^2-2u_{C,D}(n)w_{C,D}(n)+(-1+w_{C,D}(n))w_{C,D}(n))$$
$$-2n^2(20+3u_{C,D}(n)^2-3w_{C,D}(n)+3w_{C,D}(n)^2)\big].$$

We will use these expressions in the proof of (ii) of the next proposition.

PROPOSITION 2.7. *If $n>q$ then*

(i) $$\sum_{j=0}^{n-1} S\Big(-\frac{r_{C,D}(n)}{q}+\frac{D+qj}{nq}+\sigma_1(j),\frac{D+qj}{nq}\Big)$$

$$= -\frac{1}{2}+\frac{n}{4}+\frac{D^2n}{2q^2}+\frac{r_{C,D}(n)}{q}-\frac{nr_{C,D}(n)^2}{2q^2}-\frac{v_{C,D}(n)}{2}+\frac{r_{C,D}(n)v_{C,D}(n)}{q}$$
$$-\frac{nD}{2q}+\frac{ns_{C,D}(n)^2}{2q^2}-\frac{ns_{C,D}(n)}{2q},$$

(ii) $$\sum_{i=0}^{(n^2-2)n-1} S\left(1-\frac{(C+iq)(n^2-1)}{qn(n^2-2)}+\frac{D}{q}+\sigma_2(i),\frac{C+iq}{qn(n^2-2)}\right)$$

$$= \frac{1}{2}-\frac{n}{4}-\frac{nr_{C,D}(n)C}{q^2}+\frac{nr_{C,D}(n)}{2q}-\frac{3nr_{C,D}(n)^2}{2q^2}+\frac{u_{C,D}(n)}{2}+\frac{Cv_{C,D}(n)}{q}$$
$$+\frac{r_{C,D}(n)v_{C,D}(n)}{q}+\frac{w_{C,D}(n)}{2}-\frac{D}{q}+\frac{3nD^2}{2q^2}-\frac{2u_{C,D}(n)D}{q}+\frac{ns_{C,D}(n)^2}{2q^2}$$
$$-\frac{u_{C,D}(n)s_{C,D}(n)}{q}-\frac{w_{C,D}(n)s_{C,D}(n)}{q}+\frac{ns_{C,D}(n)D}{q^2}$$

where $r_{C,D}(n)$, $s_{C,D}(n)$, $u_{C,D}(n)$, $v_{C,D}(n)$, $w_{C,D}(n)$ *are as in Lemma* 2.6.

Proof. (i) By computer, we obtain the following equation:

$$\sum_{j=0}^{n-1} S\left(-\frac{r_{C,D}(n)}{q}+\frac{D+qj}{nq}+\sigma_1(j),\frac{D+qj}{nq}\right)$$
$$=\frac{nD^2}{q^2}-\frac{n^2r_{C,D}(n)D}{q^2}+\frac{n^3r_{C,D}(n)^2}{2q^2}+\frac{nv_{C,D}(n)D}{q}-\frac{n^2r_{C,D}(n)v_{C,D}(n)}{q}$$
$$+\frac{nv_{C,D}(n)^2}{2}-\frac{n^2r_{C,D}(n)}{2q}+\frac{nv_{C,D}(n)}{2}-\frac{1}{2}+\frac{n}{4}+\frac{r_{C,D}(n)}{q}-\frac{nr_{C,D}(n)^2}{2q^2}$$
$$-\frac{v_{C,D}(n)}{2}+\frac{r_{C,D}(n)v_{C,D}(n)}{q}.$$

From the equation

$$s_{C,D}(n)=nr_{C,D}(n)-D-q\left[\frac{nr_{C,D}(n)-D}{q}\right]=nr_{C,D}(n)-D-qv_{C,D}(n)$$

we have

$$\frac{nD^2}{2q^2}-\frac{n^2r_{C,D}(n)D}{q^2}+\frac{n^3r_{C,D}(n)^2}{2q^2}+\frac{nv_{C,D}(n)D}{q}$$
$$-\frac{n^2r_{C,D}(n)v_{C,D}(n)}{q}+\frac{nv_{C,D}(n)^2}{2}$$
$$=\frac{n(nr_{C,D}(n)-D-qv_{C,D}(n))^2}{2q^2}=\frac{ns_{C,D}(n)^2}{2q^2}$$

and

$$-\frac{n^2r_{C,D}(n)}{2q}+\frac{nv_{C,D}(n)}{2}=\frac{-n(nr_{C,D}(n)-D-qv_{C,D}(n))}{2q}-\frac{nD}{2q}$$
$$=-\frac{ns_{C,D}(n)}{2q}-\frac{nD}{2q}.$$

Hence we can obtain (i) directly.

(ii) From our expressions above for the sums $\sum_{i=0}^{n(n^2-2)-1}\sigma_2(i)$, $\sum_{i=0}^{n(n^2-2)-1}\sigma_2(i)^2$ and $\sum_{i=0}^{n(n^2-2)-1}\sigma_2(i)i$, we can also calculate

$$\sum_{i=0}^{(n^2-2)n-1} S\left(1-\frac{(C+iq)(n^2-1)}{qn(n^2-2)}+\frac{D}{q}+\sigma_2(i),\frac{C+iq}{qn(n^2-2)}\right).$$

Here are the results. In case $r_{C,D}(n)\neq 0$ or $s_{C,D}(n)=0$, we have

$$\begin{aligned}
(15)\qquad &\sum_{i=0}^{(n^2-2)n-1} S\left(1-\frac{(C+iq)(n^2-1)}{qn(n^2-2)}+\frac{D}{q}+\sigma_2(i),\frac{C+iq}{qn(n^2-2)}\right)\\
&=\frac{1}{2}-\frac{n}{4}+\frac{C^2}{nq^2}-\frac{C}{nq}+\frac{2Cr_{C,D}(n)}{nq^2}-\frac{Cnr_{C,D}(n)}{q^2}-\frac{r_{C,D}(n)}{nq}+\frac{nr_{C,D}(n)}{2q}\\
&\quad+\frac{r_{C,D}(n)^2}{nq^2}-\frac{3nr_{C,D}(n)^2}{2q^2}+\frac{n^3r_{C,D}(n)^2}{2q^2}+\frac{u_{C,D}(n)}{2}-\frac{u_{C,D}(n)}{n}\\
&\quad+\frac{Cu_{C,D}(n)}{nq}+\frac{r_{C,D}(n)u_{C,D}(n)}{nq}-\frac{nr_{C,D}(n)u_{C,D}(n)}{q}+\frac{Cv_{C,D}(n)}{q}\\
&\quad+\frac{r_{C,D}(n)v_{C,D}(n)}{q}-\frac{n^2r_{C,D}(n)v_{C,D}(n)}{q}+u_{C,D}(n)v_{C,D}(n)+\frac{nv_{C,D}(n)^2}{2}\\
&\quad+\frac{w_{C,D}(n)}{2}+\frac{Cw_{C,D}(n)}{nq}+\frac{r_{C,D}(n)w_{C,D}(n)}{nq}-\frac{nr_{C,D}(n)w_{C,D}(n)}{q}\\
&\quad+\frac{u_{C,D}(n)w_{C,D}(n)}{q}+v_{C,D}(n)w_{C,D}(n)\\
&=\left(\frac{C^2}{nq^2}+\frac{2Cr_{C,D}(n)}{nq^2}+\frac{r_{C,D}(n)^2}{nq^2}+\frac{2Cu_{C,D}(n)}{nq}+\frac{2r_{C,D}(n)u_{C,D}(n)}{nq}\right)\\
&\quad+\left(-\frac{C}{nq}-\frac{r_{C,D}(n)}{nq}-\frac{u_{C,D}(n)}{n}\right)\\
&\quad+\left(\frac{Cw_{C,D}(n)}{nq}+\frac{r_{C,D}(n)w_{C,D}(n)}{nq}+\frac{u_{C,D}(n)w_{C,D}(n)}{n}\right)\\
&\quad+\left(-\frac{Cu_{C,D}(n)}{nq}-\frac{r_{C,D}(n)u_{C,D}(n)}{nq}\right)\\
&\quad+\left(-\frac{nr_{C,D}(n)w_{C,D}(n)}{q}+\frac{Dw_{C,D}(n)}{q}+v_{C,D}(n)w_{C,D}(n)\right)\\
&\quad+\left(\frac{n^3r_{C,D}(n)^2}{2q^2}-\frac{n^2r_{C,D}(n)v_{C,D}(n)}{q}+\frac{nv_{C,D}(n)^2}{2}+\frac{nD^2}{2q^2}\right)\\
&\quad+\left(-\frac{nr_{C,D}(n)u_{C,D}(n)}{q}+\frac{Du_{C,D}(n)}{q}+u_{C,D}(n)v_{C,D}(n)\right)
\end{aligned}$$

$$- \frac{Dw_{C,D}(n)}{q} - \frac{nD^2}{2q^2} - \frac{Du_{C,D}(n)}{q} - \frac{Cnr_{C,D}(n)}{q^2} + \frac{nr_{C,D}(n)}{2q}$$
$$- \frac{3nr_{C,D}(n)^2}{2q^2} + \frac{u_{C,D}(n)}{2} + \frac{Cv_{C,D}(n)}{q} + \frac{r_{C,D}(n)v_{C,D}(n)}{q} + \frac{w_{C,D}(n)}{2}$$
$$+ \frac{1}{2} - \frac{n}{4}.$$

In the case $r_{C,D}(n) = 0$ and $s_{C,D}(n) \neq 0$, we have

$$\sum_{i=0}^{(n^2-2)n-1} S\left(1 - \frac{(C+iq)(n^2-1)}{qn(n^2-2)} + \frac{D}{q} + \sigma_2(i), \frac{C+iq}{qn(n^2-2)}\right) \tag{16}$$
$$= \frac{1}{2} + \frac{n}{4} + \frac{C^2}{nq^2} - \frac{C}{nq} - \frac{u_{C,D}(n)}{2} - \frac{u_{C,D}(n)}{n} + \frac{Cu_{C,D}(n)}{nq}$$
$$- \frac{C}{q} - \frac{w_{C,D}(n)}{2} + \frac{Cw_{C,D}(n)}{nq} + \frac{u_{C,D}(n)w_{C,D}(n)}{n}.$$

Since $r_{C,D}(n) = 0$ and $s_{C,D}(n) \neq 0$ implies that $v_{C,D}(n) = -1$, the equation (16) is obtained from $r_{C,D}(n) = 0$ and $s_{C,D}(n) \neq 0$ and equation (15). Thus in either case we have (15). We note that $r_{C,D}(n) = nD - C - qu_{C,D}(n)$ implies that

$$\frac{C^2}{nq^2} + \frac{2Cr_{C,D}(n)}{nq^2} + \frac{r_{C,D}(n)^2}{nq^2} + \frac{2Cu_{C,D}(n)}{nq} + \frac{2r_{C,D}(n)u_{C,D}(n)}{nq} \tag{17}$$
$$= \frac{(C + r_{C,D}(n) + qu_{C,D}(n))^2}{nq^2} - \frac{u_{C,D}(n)^2}{n} = \frac{nD^2}{q^2} - \frac{u_{C,D}(n)^2}{n},$$

$$-\frac{C}{nq} - \frac{r_{C,D}(n)}{nq} - \frac{u_{C,D}(n)}{n} = -\frac{C + r_{C,D}(n) + qu_{C,D}(n)}{nq} = -\frac{D}{q}, \tag{18}$$

$$\frac{Cw_{C,D}(n)}{nq} + \frac{r_{C,D}(n)w_{C,D}(n)}{nq} + \frac{u_{C,D}(n)w_{C,D}(n)}{n} \tag{19}$$
$$= \frac{(C + r_{C,D}(n) + qu_{C,D}(n))w_{C,D}(n)}{nq} = \frac{Dw_{C,D}(n)}{q}$$

and

$$- \frac{Cu_{C,D}(n)}{nq} - \frac{r_{C,D}(n)u_{C,D}(n)}{nq} \tag{20}$$
$$= - \frac{(C + r_{C,D}(n) + u_{C,D}(n)q)u_{C,D}(n)}{nq} + \frac{u_{C,D}(n)^2}{n}$$
$$= - \frac{Du_{C,D}(n)}{q} + \frac{u_{C,D}(n)^2}{n}.$$

Also the equation $s_{C,D}(n) = r_{C,D}(n)n - D - qv_{C,D}(n)$ implies that

$$(21)\qquad -\frac{nr_{C,D}(n)w_{C,D}(n)}{q} + \frac{Dw_{C,D}(n)}{q} + v_{C,D}(n)w_{C,D}(n) = -\frac{w_{C,D}(n)(r_{C,D}(n)n - D - qv_{C,D}(n))}{q} = -\frac{w_{C,D}(n)s_{C,D}(n)}{q},$$

$$(22)\qquad \frac{n^3 r_{C,D}(n)^2}{2q^2} - \frac{n^2 r_{C,D}(n)v_{C,D}(n)}{q} + \frac{nv_{C,D}(n)^2}{2} + \frac{nD^2}{2q^2} = \frac{n(nr_{C,D}(n) - D - qv_{C,D}(n))^2}{2q^2} - \frac{nDv_{C,D}(n)}{q} + \frac{n^2 r_{C,D}(n)D}{q^2} = \frac{ns_{C,D}(n)^2}{2q^2} + \frac{nD(r_{C,D}(n)n - D - qv_{C,D}(n))}{q^2} + \frac{nD^2}{q^2} = \frac{ns_{C,D}(n)^2}{2q^2} + \frac{nDs_{C,D}(n)}{q^2} + \frac{nD^2}{q^2}$$

and

$$(23)\qquad -\frac{nr_{C,D}(n)u_{C,D}(n)}{q} + \frac{Du_{C,D}(n)}{q} + u_{C,D}(n)v_{C,D}(n) = -\frac{u_{C,D}(n)(r_{C,D}(n)n - D - qv_{C,D}(n))}{q} = -\frac{u_{C,D}(n)s_{C,D}(n)}{q}.$$

From (17)–(23), we can deduce (ii). ∎

Finally, we have the following theorem.

THEOREM 2.8. *If $h(n^2-2) = 1$ and $n > q$, then*

$$\zeta_K(0,\chi) = \frac{1}{2q^2} \sum_{0 \le C,D \le q-1} \chi(C^2 - (n^2-2)D^2) \times \big[n\big(q^2 - qs_{C,D}(n) + 2r_{C,D}(n)^2 + 2D^2 - 2qD - 2qr_{C,D}(n)\big) + w_{C,D}(n)(-q^2 + 2qs_{C,D}(n)) - 2q^2 - 4Dr_{C,D}(n) + qC + qs_{C,D}(n) - 2s_{C,D}(n)r_{C,D}(n) + 3qr_{C,D}(n) + 3qD - 2CD\big],$$

where $r_{C,D}(n)$, $s_{C,D}(n)$, $u_{C,D}(n)$, $v_{C,D}(n)$, $w_{C,D}(n)$ are as in Lemma 2.6.

Proof. By Propositions 2.5 and 2.7,

$$\zeta_K(0,\chi) = \sum_{0 \le C,D \le q-1} \chi(C^2 - (n^2-2)D^2) \times \Bigg[\sum_{j=0}^{n-1} S\bigg(-\frac{r_{C,D}(n)}{q} + \frac{D+qj}{nq} + \sigma_1(j), \frac{D+qj}{nq}\bigg)$$

$$
\begin{aligned}
&- \sum_{i=0}^{n(n^2-2)-1} S\Big(1 - \frac{(C+iq)(n^2-1)}{qn(n^2-2)} + \frac{D}{q} + \sigma_2(i), \frac{C+iq}{qn(n^2-2)}\Big)\Big] \\
&= \frac{1}{2q^2} \sum_{0\le C,D\le q-1} \chi(C^2-(n^2-2)D^2)\cdot \big[(-2nD^2 + 4qu_{C,D}(n)D) \\
&\quad + (-qnD - q^2u_{C,D}(n)) + (2qu_{C,D}(n)s_{C,D}(n) - 2nDs_{C,D}(n)) \\
&\quad + (-q^2v_{C,D}(n) - qnr_{C,D}(n)) + (2Cnr_{C,D}(n) - 2qCv_{C,D}(n)) \\
&\quad - 2q^2 + nq^2 + 2qr_{C,D}(n) - qns_{C,D}(n) - q^2w_{C,D}(n) \\
&\quad + 2qD + 2qw_{C,D}(n)s_{C,D}(n) + 2nr_{C,D}(n)^2\big].
\end{aligned}
$$

The equation $nD - C - qu_{C,D}(n) = r_{C,D}(n)$ implies that

$$
\begin{aligned}
-2nD^2 + 4qu_{C,D}(n)D &= 4D(qu_{C,D}(n) - nD) + 2nD^2 \\
&= -4Dr_{C,D}(n) - 4DC + 2nD^2,
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
-qnD - q^2u_{C,D}(n) &= q(nD - qu_{C,D}(n)) - 2qnD \\
&= qr_{C,D}(n) + qC - 2qnD
\end{aligned} \tag{25}
$$

and

$$
2qu_{C,D}(n)s_{C,D}(n) - 2nDs_{C,D}(n) = -2s_{C,D}(n)r_{C,D}(n) - 2s_{C,D}(n)C. \tag{26}
$$

Also the equation $r_{C,D}(n)n - D - qv_{C,D}(n) = s_{C,D}(n)$ implies that

$$
\begin{aligned}
-q^2v_{C,D}(n) - qnr_{C,D}(n) &= -q(qv_{C,D}(n) - nr_{C,D}(n)) - 2qnr_{C,D}(n) \\
&= qD + qs_{C,D}(n) - 2qnr_{C,D}(n)
\end{aligned} \tag{27}
$$

and

$$
-2qnv_{C,D}(n) + 2Cnr_{C,D}(n) = 2CD + 2Cs_{C,D}(n). \tag{28}
$$

From (24)–(28), we can deduce the conclusion of the theorem. ∎

COROLLARY 2.9. *If $h(n^2-2) = 1$, $n > q$ and $n = qk + r$ for $0 \le r < q$ then $\zeta_K(0,\chi) = \frac{1}{2q^2}(B_\chi(r)k + A_\chi(r))$, where*

$$
\begin{aligned}
A_\chi(r) = &\sum_{0\le C,D\le q-1} \chi(C^2-(r^2-2)D^2) \\
&\times \big[r\big(q^2 - qs_{C,D}(r) + 2r_{C,D}(r)^2 + 2D^2 - 2qD - 2qr_{C,D}(r)\big) \\
&+ w_{C,D}(r)(-q^2 + 2qs_{C,D}(r)) - 2q^2 - 4Dr_{C,D}(r) + qC \\
&- 2s_{C,D}(r)r_{C,D}(r) + qs_{C,D}(r) + 3qr_{C,D}(r) + 3qD - 2CD\big], \\
B_\chi(r) = &\sum_{0\le C,D\le q-1} \chi(C^2-(r^2-2)D^2)\cdot \big[q^3 - 2q^2s_{C,D}(r) + 2r_{C,D}(r)^2q \\
&+ 2D^2q - 2Dq^2 - 2q^2r_{C,D}(r) + 2qs_{C,D}(r)^2\big].
\end{aligned}
$$

Proof. Since χ has conductor q, we have $\chi(C^2-(n^2-2)D^2)=\chi(C^2-(r^2-2)D^2)$ for $n=qk+r$. We note that $s_{C,D}(qk+r)=s_{C,D}(r)$, $r_{C,D}(qk+r)=r_{C,D}(r)$ and $w_{C,D}(qk+r)=s_{C,D}(r)k+w_{C,D}(r)$. This proves the corollary. ∎

3. Computation of special values of the zeta function associated with $\mathbb{Q}(\sqrt{n^2+2})$. Let $d=n^2+2$ be a positive square free integer, and $K=\mathbb{Q}(\sqrt{d})$. Then the fundamental unit ϵ of K is $n^2+1+n\sqrt{n^2+2}$ (see [8]). Let $\alpha=n+\sqrt{n^2+2}$. We note that $N_K(\epsilon)=1$ and $N_K(\alpha)=-2<0$. So if $h(d)=1$ then

$$I(K)=(q)\cdot i(K^+)\cup(q\alpha)\cdot i(K^+),$$

where $I(K)$ and $i(K^+)$ are defined in the previous section. So we have

$$\zeta_K(s,\chi)=\sum_{\substack{\mathbf{a}\in(q)\cdot i(K^+)\\ \text{integral}}}\frac{\chi(N(\mathbf{a}))}{N(\mathbf{a})^s}+\sum_{\substack{\mathbf{a}\in(q\alpha)\cdot i(K^+)\\ \text{integral}}}\frac{\chi(N(\mathbf{a}))}{N(\mathbf{a})^s}.$$

We define

$$R(\mathbf{b}):=\{a+b\epsilon\mid a,b\in\mathbb{Q}\text{ with }0<a\le 1,\ 0\le b<1\text{ and }\mathbf{b}\cdot(a+b\epsilon)\subset O(K)\}$$

for an integral ideal $\mathbf{b}$ in K. In the following lemma, we find the set of all (x,y) such that $x+y\epsilon\in R((q))$ or $x+y\epsilon\in R((q\alpha))$.

Lemma 3.1.

(i) $$\{(x,y)\mid x+y\epsilon\in R((q))\}=\Big\{(x,y)\,\Big|\,x=\tfrac{C}{q}-\tfrac{D+qj}{qn}+\delta_1(j),\ y=\tfrac{D+qj}{qn}\ \textit{for } j=0,1,\dots,n-1\ \textit{and}\ 0\le C,D\le q-1\Big\},$$

where

$$\delta_1(j)=\begin{cases}1 & \textit{if } \lceil\frac{Cn-D}{q}\rceil\le j\le n-1,\\ 0 & \textit{if } 0\le j\le\lceil\frac{Cn-D}{q}\rceil-1.\end{cases}$$

(ii) $$\{(x,y)\mid x+y\epsilon\in R((q\alpha))\}=\Big\{(x,y)\,\Big|\,x=-\tfrac{C+qi}{2nq}+\tfrac{t_{C,D}(n)}{q}+\delta_2(i),\ y=\tfrac{C+qi}{2qn}\ \textit{for } i=0,1,\dots,2n-1\ \textit{and}\ 0\le C,D\le q-1\Big\},$$

where

$$\delta_2(i)=\begin{cases}0 & \textit{if } 0\le i\le\lceil\frac{2nt_{C,D}(n)-C}{q}\rceil-1,\\ 1 & \textit{if } \lceil\frac{2nt_{C,D}(n)-C}{q}\rceil\le i\le 2n-1,\end{cases}$$

and $t_{C,D}(n)=D-Cn-q\big[\frac{D-Cn}{q}\big]$.

Proof. The set $\{C + D\alpha \mid 0 \le C, D \le q - 1\}$ is a complete set of representatives of O_K/qO_K, since $\{1, \alpha\}$ is a basis for O_K. So for $x + y\epsilon$ in $R((q))$ we have

$$q(x + y\epsilon) = (C + D\alpha) + q(i + j\alpha)$$

and for $x + y\epsilon$ in $R((q\alpha))$ we also have

$$q\alpha(x + y\epsilon) = (C + D\alpha) + q(i + j\alpha),$$

for some $0 \le C, D \le q - 1$ and integers i, j. From the equation $\bar{\alpha} = \frac{2n^2+1-\epsilon}{n}$ and computations similar to those in Lemmas 2.3 and 2.4, we obtain the assertion. ∎

PROPOSITION 3.2. *If* $h(n^2 + 2) = 1$, *then*

$$\zeta_K(0,\chi) = \sum_{0\le C,D\le q-1} \chi(C^2 - 2D^2 + 2nCD)\Bigg[\sum_{j=0}^{n-1} S\Big(\frac{C}{q} - \frac{D+qj}{qn} + \delta_1(j), \frac{D+qj}{qn}\Big) - \sum_{i=0}^{2n-1} S\Big(\frac{t_{C,D}(n)}{q} - \frac{C+iq}{2qn} + \delta_2(i), \frac{C+iq}{2qn}\Big)\Bigg],$$

where δ_1, δ_2 *and* $t_{C,D}$ *are defined in Lemma* 3.1, *and* $S(x, y)$ *in Proposition* 2.5.

Proof. As in Proposition 2.5, we have

$$\zeta_K(0,\chi) = \sum_{x+y\epsilon\in R((q))} \chi(N_K(q(x + y\epsilon)))S(x, y) - \sum_{x+y\epsilon\in R((\alpha q))} \chi(N_K(q\alpha(x + y\epsilon)))S(x, y). \tag{29}$$

For $x + y\epsilon \in R((q))$,

$$q(x + y\epsilon) = C + D\alpha + q(i + j\alpha).$$

Thus

$$N_K(q(x + y\epsilon)) \equiv N_K(C + D\alpha) \pmod q = C^2 - 2D^2 + 2nCD. \tag{30}$$

If $x + y\epsilon$ is in $R((q\alpha))$, then

$$q\alpha(x + y\epsilon) = C + D\alpha + q(i + j\alpha)$$

for some $0 \le C, D \le q - 1$ and integers i, j. So

$$N_K(q\alpha(x + y\epsilon)) \equiv N_K(C + D\alpha) \pmod q = C^2 - 2D^2 + 2nCD. \tag{31}$$

From (29)–(31), we obtain the conclusion immediately. ∎

Finally, we obtain the following theorem.

THEOREM 3.3. *If* $h(n^2+2)=1$, *then*

$$\zeta_K(0,\chi) = \frac{1}{12q^2}\sum_{0\le C,D\le q-1}\chi(C^2-2D^2+2nCD)\big(6q^2A_{C,D}(n)+2q^2n+3C^2n - 12qA_{C,D}(n)C+6D^2n-6qDn-6nt_{C,D}(n)^2-6qnt_{C,D}(n) - 3ne_{C,D}(n)^2+3qne_{C,D}(n)-6q^2B_{C,D}(n)+12qB_{C,D}(n)t_{C,D}(n)+3qCn\big),$$

where $A_{C,D}(n)=\lceil\frac{nC-D}{q}\rceil$, $B_{C,D}(n)=\lceil\frac{2nt_{C,D}(n)-C}{q}\rceil$ *and* $e_{C,D}(n)=C-2nt_{C,D}(n)+qB_{C,D}(n)$.

Proof. The equation $D-nC+qA_{C,D}(n)=t_{C,D}(n)$ implies that

$$(32)\qquad \frac{A_{C,D}(n)^2n}{2}+\frac{D^2n}{2q^2}-\frac{CDn^2}{q^2}+\frac{C^2n^3}{2q^2}+\frac{A_{C,D}(n)Dn}{q}-\frac{A_{C,D}(n)Cn^2}{q} = \frac{n(D-nC+qA_{C,D}(n))^2}{2q^2}=\frac{nt_{C,D}(n)^2}{2q^2}$$

and

$$(33)\qquad \frac{A_{C,D}(n)n}{2}+\frac{Dn}{2q}-\frac{Cn^2}{2q}=\frac{n(D-nC+qA_{C,D}(n))}{2q}=\frac{nt_{C,D}(n)}{2q}.$$

By (32) and (33), we have

$$(34)\qquad \sum_{j=0}^{n-1}S\left(\frac{C}{q}-\frac{D+qj}{nq}+\delta_1(j),\frac{D+qj}{qn}\right)$$
$$=\left(\frac{A_{C,D}(n)^2n}{2}+\frac{D^2n}{2q^2}-\frac{CDn^2}{q^2}+\frac{C^2n^3}{2q^2}+\frac{A_{C,D}(n)Dn}{q}-\frac{A_{C,D}(n)Cn^2}{q}\right) - \left(\frac{A_{C,D}(n)n}{2}+\frac{Dn}{2q}-\frac{Cn^2}{2q}\right)+\frac{A_{C,D}(n)}{2}+\frac{n}{12}+\frac{C^2n}{2q^2}-\frac{A_{C,D}(n)C}{q} + \frac{D^2n}{2q^2}-\frac{Dn}{2q}$$
$$=\frac{nt_{C,D}(n)^2}{2q^2}-\frac{nt_{C,D}(n)}{2q}+\frac{A_{C,D}(n)}{2}+\frac{n}{12}+\frac{C^2n}{2q^2}-\frac{A_{C,D}(n)C}{q}+\frac{D^2n}{2q^2}-\frac{Dn}{2q}.$$

Also $C-2nt_{C,D}(n)+qB_{C,D}(n)=e_{C,D}(n)$ implies that

$$(35)\qquad \frac{B_{C,D}(n)^2n}{4}+\frac{C^2n}{4q^2}+\frac{B_{C,D}(n)Cn}{2q}-\frac{Cn^2t_{C,D}(n)}{q^2}-\frac{B_{C,D}(n)n^2t_{C,D}(n)}{q} + \frac{n^3t_{C,D}(n)^2}{q^2}=\frac{n(C-2nt_{C,D}(n)+qB_{C,D}(n))^2}{4q^2}=\frac{ne_{C,D}(n)^2}{4q^2},$$

$$(36)\qquad \frac{B_{C,D}(n)n}{4}+\frac{Cn}{4q}-\frac{n^2t_{C,D}(n)}{2q}=\frac{n(C-2nt_{C,D}(n)+qB_{C,D}(n))}{4q}=\frac{ne_{C,D}(n)}{4q}.$$

From (35) and (36), we deduce that

$$(37)\qquad \sum_{i=0}^{2n-1}S\Big(\frac{t_{C,D}(n)}{q}-\frac{C+qi}{2nq}+\delta_2(i),\frac{C+iq}{2nq}\Big)$$
$$=\frac{B_{C,D}(n)}{2}-\frac{n}{12}-\frac{B_{C,D}(n)n}{4}+\frac{B_{C,D}(n)^2n}{4}+\frac{C^2n}{2q^2}-\frac{Cn}{2q}+\frac{B_{C,D}(n)Cn}{2q}$$
$$-\frac{Cn^2t_{C,D}(n)}{q^2}-\frac{B_{C,D}(n)t_{C,D}(n)}{q}+\frac{n^2t_{C,D}(n)}{2q}-\frac{B_{C,D}(n)n^2t_{C,D}(n)}{q}$$
$$+\frac{nt_{C,D}(n)^2}{q^2}+\frac{n^3t_{C,D}(n)^2}{q^2}$$
$$=\Big(\frac{B_{C,D}(n)^2n}{4}+\frac{C^2n}{4q^2}+\frac{B_{C,D}(n)Cn}{2q}-\frac{Cn^2t_{C,D}(n)}{q^2}-\frac{B_{C,D}(n)n^2t_{C,D}(n)}{q}$$
$$+\frac{n^3t_{C,D}(n)^2}{q^2}\Big)-\Big(\frac{B_{C,D}(n)n}{4}+\frac{Cn}{4q}-\frac{n^2t_{C,D}(n)}{2q}\Big)+\frac{B_{C,D}(n)}{2}-\frac{n}{12}$$
$$-\frac{B_{C,D}(n)t_{C,D}(n)}{q}+\frac{nt_{C,D}(n)^2}{q^2}+\frac{C^2n}{4q^2}-\frac{Cn}{4q}$$
$$=\frac{ne_{C,D}(n)^2}{4q^2}-\frac{ne_{C,D}(n)}{4q}+\frac{B_{C,D}(n)}{2}-\frac{n}{12}$$
$$-\frac{B_{C,D}(n)t_{C,D}(n)}{q}+\frac{nt_{C,D}(n)^2}{q^2}+\frac{C^2n}{4q^2}-\frac{Cn}{4q}.$$

By combining Proposition 3.2, (34) and (37), we obtain the assertion. ■

Corollary 3.4. *If $h(n^2+2)=1$ and $n=qk+r$ for $r=0,1,\dots,q-1$, then*

$$\zeta_K(0,\chi)=\frac{1}{12q^2}\,(F_\chi(r)k+E_\chi(r)),$$

where

$$E_\chi(r)=\sum_{0\le C,D\le q-1}\chi(C^2-2D^2+2rCD)\big(6q^2A_{C,D}(r)+2q^2r+3C^2r$$
$$-12qA_{C,D}(r)C+6D^2r-6qDr-6rt_{C,D}(r)^2-6qrt_{C,D}(r)-3re_{C,D}(r)^2$$
$$+3qre_{C,D}(r)-6q^2B_{C,D}(r)+12qB_{C,D}(r)t_{C,D}(r)+3qCr\big),$$

$$F_\chi(r)=\sum_{0\le C,D\le q-1}\chi(C^2-2D^2+2rCD)\big(-9C^2q+6D^2q-3e_{C,D}(r)^2q$$
$$+9Cq^2-6Dq^2+3e_{C,D}(r)q^2+2q^3-18q^2t_{C,D}(r)+18qt_{C,D}(r)^2\big)$$

and $A_{C,D}$, $B_{C,D}$, $t_{C,D}$ and $e_{C,D}$ are as in Theorem 3.3.

Proof. We note that $A_{C,D}(qk+r) = Ck + A_{C,D}(r)$, $B_{C,D}(qk+r) = 2t_{C,D}(r)k + B_{C,D}(r)$, $t_{C,D}(qk+r) = t_{C,D}(r)$ and $e_{C,D}(qk+r) = e_{C,D}(r)$. Since the character χ has conductor q, the above equations yield the conclusion. ∎

4. Proof of Theorem 1.1. Let $d = n^2 \pm 2$ be a positive square free integer and $K = \mathbb{Q}(\sqrt{d})$. Let q be a positive integer with $(q,d) = 1$, χ an odd primitive character with conductor q and L_χ a field over $\mathbb{Q}$ generated by the values of $\chi(a)$ for $a = 1, \ldots, q$. Define $m_\chi := \sum_{a=1}^{q} a\chi(a)$. Then from the same argument in Section 2 of [1] and Corollaries 2.9 and 3.4, we infer that if $n = qk + r$ and $h(n^2-2) = 1$ with $n > q$, then

$$B_\chi(r)k + A_\chi(r) \equiv 0 \pmod{I},$$

and if $h(n^2+2) = 1$, then

$$F_\chi(r)k + E_\chi(r) \equiv 0 \pmod{I},$$

for a prime ideal I of L_χ dividing the principal ideal (m_χ). If the integers q and p satisfy condition $(*)$ in [6], then for r such that $B_\chi(r) \notin I$ [resp. $F_\chi(r) \notin I$], there exists a unique $T^\chi_{A,B}(r)$ [resp. $T^\chi_{E,F}(r)$] $\in \{0, 1, \ldots, p-1\}$ such that

$$-q\,\frac{A_\chi(r)}{B_\chi(r)} + r + I = T^\chi_{A,B}(r) + I, \qquad -q\,\frac{E_\chi(r)}{F_\chi(r)} + r + I = T^\chi_{E,F}(r) + I.$$

So we have

$$n \equiv T^\chi_{A,B}(r) \pmod{p} \text{ for } n = qk + r \text{ with } h(n^2-2) = 1 \text{ and } n > q, \tag{38}$$

$$n \equiv T^\chi_{E,F}(r) \pmod{p} \text{ for } n = qk + r \text{ with } h(n^2+2) = 1. \tag{39}$$

We will write $q \to p$ if q and p satisfy condition $(*)$ in [6]. From Section 4 in [1], we have

$$175 \to 61, \qquad 61 \to 1861, \qquad 175 \to 1861.$$

Now, we find other p and q satisfying condition $(*)$. Consider the function $f_{25} : (\mathbb{Z}/25\mathbb{Z})^* \to \mathbb{Z}/20\mathbb{Z}$ for which $2^{f_{25}(a)} \equiv a \pmod{25}$ and the function $g_7 : (\mathbb{Z}/7\mathbb{Z})^* \to \mathbb{Z}/6\mathbb{Z}$ for which $3^{g_7(a)} \equiv a \pmod{7}$. These two functions are well defined, since $(\mathbb{Z}/25\mathbb{Z})^*$ [resp. $(\mathbb{Z}/7\mathbb{Z})^*$] is a cyclic group generated by 2 [resp. 3]. Define

$$\chi_4 : (\mathbb{Z}/175\mathbb{Z})^* \to \mathbb{C}$$

by $\chi_4(a) = \zeta_{30}^{6f_{25}(a_{25})} \cdot \zeta_{30}^{25g_7(a_7)}$, where $a \equiv a_{25} \pmod{25}$, $a \equiv a_7 \pmod{7}$ and ζ_{30} is a primitive 30th root of unity. Then χ_4 is an odd primitive character with conductor 175. Since the order of 450 modulo 601 is 30, $I_4 = (601, \zeta_{30} - 450)$ is the prime ideal in $L_{\chi_4} = \mathbb{Q}(\zeta_{30})$ lying over the rational prime 601 of degree 1 (see page 97 in [1]). From $\zeta_{30} \equiv 450 \pmod{I_4}$, we find that

$$m_{\chi_4} \equiv 0 \pmod{I_4}.$$

So we obtain

(40) $$175 \to 601.$$

Now we define the functions $T^{\chi_i}_{A,B}(r)$ as follows:

$$\begin{aligned} -175\,\frac{A_{\chi_1}(r)}{B_{\chi_1}(r)} + r + I_1 &= T^{\chi_1}_{A,B}(r) + I_1,\\ -61\,\frac{A_{\chi_2}(r)}{B_{\chi_2}(r)} + r + I_2 &= T^{\chi_2}_{A,B}(r) + I_2,\\ -175\,\frac{A_{\chi_3}(r)}{B_{\chi_3}(r)} + r + I_3 &= T^{\chi_3}_{A,B}(r) + I_3,\\ -175\,\frac{A_{\chi_4}(r)}{B_{\chi_4}(r)} + r + I_4 &= T^{\chi_4}_{A,B}(r) + I_4, \end{aligned}$$

where the characters χ_i and ideals I_i are defined in Examples 1, 3 and 2 of Section 4 in [1], respectively, for $i = 1, 2, 3$.

For a residue a_{175} modulo 175 with $B_{\chi_1}(a_{175}) \notin I_1$ [resp. $B_{\chi_3}(a_{175}) \notin I_3$], we define b_{61} [resp. d_{1861}] to be residues modulo 61 [resp. 1861] for which

$$b_{61} = T^{\chi_1}_{A,B}(a_{175}), \qquad d_{1861} = T^{\chi_3}_{A,B}(a_{175}).$$

And for a residue b_{61} modulo 61 with $B_{\chi_2}(b_{61}) \notin I_2$, we define c_{1861} to be a residue modulo 1861 such that

$$c_{1861} = T^{\chi_2}_{A,B}(b_{61}).$$

Let $U_m = \left\{a \in \mathbb{Z} \,\middle|\, \left(\frac{a^2-2}{p}\right) = -1 \text{ for any prime } p \text{ dividing } m\right\}$. By computer we can also check that if $a_{175} \in U_{175}$ then $B_{\chi_1}(a_{175}) \notin I_1$ and $B_{\chi_3}(a_{175}) \notin I_3$, and if $b_{61} = T^{\chi_1}_{A,B}(a_{175})$ for $a_{175} \in U_{175}$ then $B_{\chi_2}(b_{61}) \notin I_2$. Hence it is possible to calculate $T^{\chi_1}_{A,B}(a_{175})$, $T^{\chi_3}_{A,B}(a_{175})$ and $T^{\chi_2}_{A,B}(b_{61})$ for $a_{175} \in U_{175}$. From the computer calculations, we obtain the following table:

$a_{175} \in U_{175}$	b_{61}	c_{1861}	d_{1861}	$a_{175} \in U_{175}$	b_{61}	c_{1861}	d_{1861}
±7	±7	±7	±7	±8	±8	±8	±8
±13	±13	±13	±13	±15	±38	±1266	±1060
±20	±20	±20	±20	±22	±23	±595	±1022
±27	±34	±851	±389	±28	±59	±1859	±962
±35	±51	±1851	±288	±42	±43	±329	±392
±43	±40	±1821	±1353	±48	±34	±851	±306
±50	±16	±1075	±193	±55	±35	±1272	±566
±57	±32	±845	±1559	±62	±6	±301	±1647
±63	±43	±329	±399	±70	±58	±1858	±49
±77	±4	±4	±1760	±78	±9	±1690	±561
±83	±26	±589	±427	±85	±49	±1501	±1072

Also for r with $F_{\chi_i}(r) \notin I_i$, the functions $T^{\chi_i}_{E,F}(r)$ are defined as follows:

$$\begin{aligned}
-175\,\frac{E_{\chi_1}(r)}{F_{\chi_1}(r)} + r + I_1 &= T^{\chi_1}_{E,F}(r) + I_1,\\
-61\,\frac{E_{\chi_2}(r)}{F_{\chi_2}(r)} + r + I_2 &= T^{\chi_2}_{E,F}(r) + I_2,\\
-175\,\frac{E_{\chi_3}(r)}{F_{\chi_3}(r)} + r + I_3 &= T^{\chi_3}_{E,F}(r) + I_3,\\
-175\,\frac{E_{\chi_4}(r)}{F_{\chi_4}(r)} + r + I_4 &= T^{\chi_4}_{E,F}(r) + I_4.
\end{aligned}$$

Then for a residue e_{175} modulo 175 with $F_{\chi_1}(e_{175}) \notin I_1$ [resp. $F_{\chi_3}(e_{175}) \notin I_3$], we define f_{61} [resp. h_{1861}] to be residues modulo 61 [resp. 1861] for which

$$f_{61} = T^{\chi_1}_{E,F}(e_{175}), \qquad h_{1861} = T^{\chi_3}_{E,F}(e_{175}).$$

And for a residue f_{61} modulo 61 with $F_{\chi_2}(f_{61}) \notin I_2$, we define g_{1861} to be a residue modulo 1861 such that

$$g_{1861} = T^{\chi_2}_{E,F}(f_{61}).$$

Let $V_m = \left\{a \in \mathbb{Z} \,\middle|\, \left(\frac{a^2+2}{p}\right) = -1 \text{ for any prime } p \text{ dividing } m\right\}$. Then by computer we can also check that if $e_{175} \in V_{175}$ then $F_{\chi_1}(e_{175}) \notin I_1$ and $F_{\chi_3}(e_{175}) \notin I_3$, and if $f_{61} = T^{\chi_1}_{E,F}(e_{175})$ for $e_{175} \in V_{175}$ then $F_{\chi_2}(f_{61}) \notin I_2$. So we can calculate $T^{\chi_1}_{E,F}(e_{175})$, $T^{\chi_3}_{E,F}(e_{175})$ and $T^{\chi_2}_{E,F}(f_{61})$ for $e_{175} \in V_{175}$. We obtain the following table:

$e_{175} \in V_{175}$	f_{61}	g_{1861}	h_{1861}	$e_{175} \in V_{175}$	f_{61}	g_{1861}	h_{1861}
±1	±1	±1	±1	±5	±5	±5	±5
±6	±6	±6	±6	±9	±9	±9	±9
±15	±15	±15	±15	±16	±50	±491	±935
±19	±19	±244	±1534	±20	±44	±403	±943
±26	±41	±1235	±1243	±29	±19	±244	±1567
±30	±24	±610	±363	±34	±12	±32	±1589
±36	±28	±458	±200	±40	±30	±1654	±578
±41	±21	±804	±1762	±44	±30	±1654	±186
±50	±47	±1124	±213	±51	±45	±728	±181
±54	±54	±778	±1097	±55	±39	±240	±40
±61	±51	±753	±858	±64	±25	±155	±817
±65	±4	±4	±1691	±69	±26	±1280	±784
±71	±27	±190	±339	±75	±60	±1860	±70
±76	±57	±1857	±651	±79	±42	±1617	±1056
±85	±44	±403	±1623	±86	±8	±1448	±1048

To prove our theorem, we need the following class number 1 criteria.

LEMMA 4.1.

(i) $h((2k+1)^2-2)=1 \Rightarrow 4k^2+4k-1-4t^2$ *are primes for* $0 \le t \le k$,
(ii) $h(4k^2-2)=1 \Rightarrow 2k^2-1-2t^2$ *are primes for* $0 \le t \le k-1$,
(iii) $h((2k+1)^2+2)=1 \Rightarrow (2k+1)^2+2-4t^2$ *are primes for* $0 \le t \le k$,
(iv) $h(4k^2+2)=1 \Rightarrow 4k^2+2-(2t-1)^2$ *are primes for* $1 \le t \le k$.

Proof. See Corollaries 3.3–3.6 in [4]. (Remark: $d=(2n+1)^2-2$ in Corollary 3.5 of [4] is a misprint. It should read $d=(2n+1)^2+2$.) ∎

In the following proposition, we find an upper bound for n with $h(n^2-2)=1$.

PROPOSITION 4.2. *Let* n^2-2 *be a square free integer. Then* $h(n^2-2)>1$ *for* $n>1244$.

Proof. If $n \notin U_{175}$ and $n=2i+1$, then there exists an integer $t_0 \in \mathbb{Z}$ such that $(2i+1)^2-2-(2t_0)^2 \equiv 0 \pmod 5$ or $(2i+1)^2-2-(2t_0)^2 \equiv 0 \pmod 7$. Similarly, for $n=2j \notin U_{175}$, there is an integer $s_0 \in \mathbb{Z}$ such that $2j^2-1-2s_0^2 \equiv 0 \pmod 5$ or $2j^2-1-2s_0^2 \equiv 0 \pmod 7$. If we take $t_0=4$ for $i=5l, 5l+4$, $t_0=5$ for $i=7l+4, 7l+2$ and $t_0=7$ for $i=7l+1, 7l+5$, then $(2i+1)^2-2-(2t_0)^2$ is a multiple of 5 or 7. And if we take $s_0=4$ for $j=5l+3, 5l+2$, $s_0=5$ for $j=7l+1, 7l+6$ and $s_0=7$ for $j=7l+5, 7l+2$ then $2j^2-1-2s_0^2$ is a multiple of 5 or 7. Thus from Lemma 4.1(i), (ii), we have

$$h(n^2-2)>1 \quad \text{for } n>15 \text{ with } n \notin U_{175}. \tag{41}$$

Suppose $n \equiv a_{175} \pmod{175}$ for $a_{175} \in U_{175}$ and $a_{175} \neq \pm 7, \pm 8, \pm 13, \pm 20$ and $h(n^2-2)=1$, $n>175$. Then from the table we have $c_{1861} \neq d_{1861}$. This is a contradiction to (38). So

$$h(n^2-2)>1 \quad \text{for } n \not\equiv \pm 7, \pm 8, \pm 13, \pm 20 \pmod{175} \text{ and } n \in U_{175} \text{ with } n>175. \tag{42}$$

By computer, we find $T_{A,B}^{\chi_4}(\pm 20)=\pm 20$. That is, $n \equiv \pm 20 \pmod{601}$ for $n \equiv \pm 20 \pmod{175}$ with $h(n^2-2)=1$ and $n>175$. If we take $t_0=20$, then $(2i+1)^2-2-(2t_0)^2$ is a multiple of 601 for $2i+1 \equiv \pm 20 \pmod{601}$. And if we also take $s_0=621$ then $2j^2-1-2s_0^2$ is a multiple of 601 for $2j \equiv \pm 20 \pmod{601}$. Thus by Lemma 4.1(i), (ii), we have

$$h(n^2-2)>1 \quad \text{for } n \equiv \pm 20 \pmod{175} \text{ with } n>1244. \tag{43}$$

If $n \equiv \pm 7$, ± 8 or $\pm 13 \pmod{175}$ then $n \in U_{175}$ but if we assume $n>175$ and $h(n^2-2)=1$, then $n \equiv \pm 7, \pm 8$ or $\pm 13 \pmod{61}$. So $n \notin U_{61}$. We can find t_0 [resp. s_0] making $(2i+1)^2-2-(2t_0)^2$ [resp. $2j^2-1-2s_0^2$] a multiple of 61 for $2i+1 \equiv \pm 7$, ± 8 or $\pm 13 \pmod{61}$ [resp. $2j \equiv \pm 7$, ± 8 or $\pm 13 \pmod{61}$] as above. They give an upper bound for n such that

$h(n^2-2)>1$, for $n \equiv \pm 7, \pm 8$ or $\pm 13 \pmod{175}$. The upper bound does not exceed 1244. So we have

$$(44) \qquad h(n^2-2)>1 \quad \text{for } n \equiv \pm 7, \pm 8, \pm 13 \pmod{175} \text{ with } n > 1244.$$

By (41)–(44), the proof is complete. ∎

Now, we find an upper bound for n with $h(n^2+2)=1$.

PROPOSITION 4.3. *Let n^2+2 be a square free integer. Then $h(n^2+2)>1$ for $n>1596$.*

Proof. For $n \notin V_{175}$, it is possible to find t_0 [resp. s_0] such that $(2i+1)^2+2-(2t_0)^2$ [resp. $(2j)^2+2-(2s_0-1)^2$] is a multiple of 5 or 7 with $n=2i+1$ [resp. $n=2j$]. If we take $t_0=2$ for $i=5l+3$, $i=7l+3$, $t_0=7$ for $i=5l+1$ and $t_0=6$ for $i=7l+1, 7l+5$, then $(2i+1)^2+2-(2t_0)^2$ is a multiple of 5 or 7. And if we take $s_0=6$ for $j=5l+1, 5l+4, 7l$ and $s_0=5$ for $j=7l+5, 7l+2$ then $2j^2-1-2s_0^2$ is a multiple of 5 or 7. Thus from Lemma 4.1(iii), (v), we have

$$(45) \qquad h(n^2+2)>1 \quad \text{for } n>15 \text{ and } n \notin V_{175}.$$

If $n \equiv e_{175} \pmod{175}$ for $e_{175} \in V_{175}$ and $e_{175} \neq \pm 1, \pm 5, \pm 6, \pm 9, \pm 15$ and $h(n^2+2)=1$, then from the table we have $g_{1861} \neq h_{1861}$. This is a contradiction to (39). So

$$(46) \quad h(n^2+2)>1 \quad \text{for } n \not\equiv \pm 1, \pm 5, \pm 6, \pm 9, \pm 15 \pmod{175} \text{ and } n \in V_{175}.$$

If $n \equiv \pm 6 \pmod{175}$ with $h(n^2+2)=1$, then $n \equiv \pm 6 \pmod{1861}$. Suppose $n \equiv \pm 6 \pmod{1861}$ and $n=2i+1$. Then $i=1861l+933$ or $1861l+927$. Take $t_0=133$. Then $(2i+1)^2+2-(2t_0)^2$ is $1861(7444l^2+7468l+1835)$ or $1861(7444l^2+7420l+1811)$. Suppose $n \equiv \pm 6 \pmod{1861}$ and $n=2j$. Then $j=1861l+3$ or $1861l+1858$. If we take $s_0=798$ then $(2j)^2+2-(2s_0-1)^2=1861(7444l^2+24l-1367)$ or $1861(7444l^2+14864l+6053)$. Thus by Lemma 4.1(iii), (v), we have

$$(47) \qquad h(n^2+2)>1 \quad \text{for } n \equiv \pm 6 \pmod{175} \text{ with } n>1596.$$

If $n \equiv \pm 1, \pm 5, \pm 9 \pmod{175}$ then $n \in V_{175}$ but by assumption of $h(n^2+2)=1$, we have $n = \pm 1, \pm 5, \pm 9 \pmod{61}$. Moreover, $n \notin V_{61}$. So we can find t_0 [resp. s_0] such that $(2i+1)^2+2-(2t_0)^2$ [resp. $(2j)^2+2-(2s_0-1)^2$] are multiples of 61 for $2i+1 \equiv \pm 1, \pm 5, \pm 9 \pmod{61}$ [resp. $2j \equiv \pm 1, \pm 5, \pm 9 \pmod{61}$]. They give an upper bound for n such that $h(n^2+2)=1$ and $n \equiv \pm 1, \pm 5, \pm 9 \pmod{175}$ as above. From this, we have

$$(48) \qquad h(n^2+2)>1 \quad \text{for } n \equiv \pm 1, \pm 5, \pm 9 \pmod{175} \text{ with } n>114.$$

By computer, we find $T^{\chi_4}_{A,B}(\pm 15)=\pm 15$. So if $n \equiv \pm 15 \pmod{175}$ with $h(n^2+2)=1$, then $n \equiv \pm 15 \pmod{601}$. If we take $t_0=105$ [resp. $s_0=196$] then $(2i+1)^2+2-(2t_0)^2$ [resp. $(2j)^2+2-(2s_0-1)^2$] is a multiple of 601 for any $2i+1 \equiv \pm 15 \pmod{601}$ [resp. $2j \equiv \pm 15 \pmod{601}$]. Thus by Lemma 4.1(iii), (v), we have

$$h(n^2+2) > 1 \quad \text{for } n \equiv \pm 15 \pmod{175} \text{ with } n > 392. \tag{49}$$

By (45)–(49), the proof is complete. ∎

Proof of Mollin and Williams' Conjecture. By Propositions 4.2 and 4.3,

$$h(n^2 \pm 2) > 1 \quad \text{for } n > 1596.$$

In [12], Mollin and Williams prove that if $d = n^2 \pm 2$ is a square free integer with $n < 5000$ then

$$h(d) = 1 \quad \text{if and only if} \quad d = 3, 6, 7, 11, 14, 38, 47, 62, 83, 167, 227, 398.$$

By combining the above two results, we obtain the conjecture.

Now, we recall the following proposition.

PROPOSITION 4.4 ([3]). *Let d be a Richaud–Degert type.*

I. *Suppose $d = n^2 + r \equiv 2, 3 \pmod 4$.*
   (i) *If $|r| \neq 1, 4$, then $h(d) > 1$ except for $r = \pm 2$.*
   (ii) *If $|r| = 1$, then $h(d) > 1$ except for $d = 2, 3$.*

II. *Suppose $d = n^2 + r \equiv 1 \pmod 8$.*
   (i) *If $|r| \neq 1, 4$, then $h(d) > 1$ except for $d = 33$.*
   (ii) *If $|r| = 1$, then $h(d) > 1$ except for $d = 17$.*

Since Mollin and Williams' conjecture is true, Theorem 1.1 is a direct consequence of Proposition 4.4.

5. Appendix. In this section, we provide a MATHEMATICA program to evaluate the values $T^{\chi_i}_{A,B}(r)$ and $T^{\chi_i}_{E,F}(r)$ in Section 4.

The function `f[x_,y_]` computes the logarithm of x to base 2 modulo y, and `g[x_,y_]` computes the logarithm of x to base 3 modulo y:

```
f[x_, y_] := ( j = 0; m = Mod[x, y];
          If [Mod[x, y] == 0, Return[0]];
          While[ Mod[m, y] >1, m = Mod[m*2, y] ; j = j + 1];
          Return[y - 1 - j]);
g[x_, y_] := (j = 0; m = Mod[x, y];
      If[Mod[x, y] == 0, Return[0]];
      While[ Mod[m, y] >1, m = Mod[m*3, y]; j = j + 1];
      Return[y - 1 - j]);
g7[x_] := g[x, 7];
f25[x_] := (j = 0; m = Mod[x, 25];
        If[ Mod[m, 5] == 0, Return[0]];
        While[Mod[m, 25] >1, m = Mod[m*2, 25]; j = j + 1];
        Return[20 - j]);
f61[x_] := f[x, 61];
```

The function iv[x_,y] computes the multiplicative inverse of x modulo y:

```
iv[x_, y_] := (
      i = 1;
      While[Mod[ i*x, y] >1, i++];
      Return[i] );
```

The functions chi[a_] compute $\chi_i(a)$ modulo I_i for $i = 1, 2, 3, 4$:

```
ch1[a_] := (If [Mod[a, 5] == 0 || Mod[a, 7] == 0, Return[0]];
      Return[Mod[PowerMod[8, f25[Mod[a, 25]], 61]*
      PowerMod[47, g7[Mod[a, 7] ], 61], 61]]);
ch2[a_] := (If[Mod[a, 61] == 0, Return[0]];
      Return[PowerMod[1833, f61[Mod[a, 61]], 1861]]);
ch3[a_] := (If [Mod[a, 5] == 0 || Mod[a, 7] == 0, Return[0]];
      Return[Mod[PowerMod[380, f25[Mod[a, 25]], 1861]*
      PowerMod[1406, g7[Mod[a, 7] ], 1861], 1861]]);
ch4[a_] :=
     (If [Mod[a, 5] == 0 || Mod[a, 7] == 0, Return[0]];
      Return[Mod[PowerMod[432, f25[Mod[a, 25]], 601]*
                      PowerMod[577, g7[Mod[a, 7] ], 601], 601]]);
```

The following are needed to compute $A_{\chi_i}(r)$ and $B_{\chi_i}(r)$ modulo I_i:

```
u[q_, n_, c_, d_] := Floor[(n d - c)/q]
r[q_, n_, c_, d_] := n d - c - q Floor[(n d - c)/q]
v[q_, n_, c_, d_] := Floor[(r[q, n, c, d]n - d)/q]
s[q_, n_, c_, d_] := r[q, n, c, d]n - d - q Floor[(r[q, n, c, d]n - d)/q]
w[q_, n_, c_, d_] := -Floor[(r[q, n, c, d] - n s[q, n, c, d])/q]

A[q_, n_, c_, d_] := -2 q^2 + n q^2 + 3 q r[q, n, c, d] - q n s[q, n, c, d]
- q^2 w[q, n, c, d] + 3q d + 2 q w[q, n, c, d] s[q, n, c, d]
+ 2n r[q, n, c, d]^2 - 2 s[q, n, c, d]r[q, n, c, d] - 4 d r[q, n, c, d]
- 2 c d + 2 n d^2 + q c - 2 q n d + q s[q, n, c, d] - 2 q n r[q, n, c, d]
B[q_, n_, c_, d_] := q^3 - 2 q^2s[q, n, c, d] + 2 r[q, n, c, d]^2q
+ 2 d^2 q - 2 q^2 d - 2 q^2 r[q, n, c, d] + 2 q s[q, n, c, d]^2

SB1[n_, c_, d_] := Mod[ch1[c^2 - (n^2 - 2)d^2]*B[175, n, c, d], 61]
SB2[n_, c_, d_] := Mod[ch2[c^2 - (n^2 - 2)d^2]*B[61, n, c, d], 1861]
SB3[n_, c_, d_] := Mod[ch3[c^2 - (n^2 - 2)d^2]*B[175, n, c, d], 1861]
SB4[n_, c_, d_] := Mod[ch4[c^2 - (n^2 - 2)d^2]*B[175, n, c, d], 601]
SA1[n_, c_, d_] := Mod[ch1[c^2 - (n^2 - 2)d^2]*A[175, n, c, d], 61]
SA2[n_, c_, d_] := Mod[ch2[c^2 - (n^2 - 2)d^2]*A[61, n, c, d], 1861]
SA3[n_, c_, d_] := Mod[ch3[c^2 - (n^2 - 2)d^2]*A[175, n, c, d], 1861]
SA4[n_, c_, d_] := Mod[ch4[c^2 - (n^2 - 2)d^2]*A[175, n, c, d], 601]
```

The functions RAi[a_] and RBi[a_] compute $A_{\chi_i}(a)$ and $B_{\chi_i}(a)$ modulo I_i respectively, where q is the conductor for the character χ_i for $i = 1, 2, 3, 4$:

```
RB1[a_] :=
 Mod[Sum[Mod[Sum[SB1[a, c, d], {c, 0, 174}], 61], {d, 0, 174}], 61]
```

```
RA1[a_] :=
 Mod[Sum[Mod[Sum[SA1[a, c, d], {c, 0, 174}], 61], {d, 0, 174}], 61]
RB2[a_] :=
 Mod[Sum[Mod[Sum[SB2[a, c,  d], {c, 0, 60}], 1861], {d, 0, 60}], 1861]
RA2[a_] :=
 Mod[Sum[Mod[Sum[SA2[a, c, d], {c, 0, 60}], 1861], {d, 0, 60}], 1861]
RB3[a_] :=
 Mod[Sum[Mod[Sum[SB3[a, c, d], {c, 0, 174}], 1861], {d, 0, 174}],1861]
RA3[a_] :=
 Mod[Sum[Mod[Sum[SA3[a, c, d], {c, 0, 174}], 1861], {d, 0, 174}], 1861]
RB4[a_] :=
 Mod[Sum[Mod[Sum[SB4[a, c, d], {c, 0, 174}], 601], {d, 0, 174}], 601]
RA4[a_] :=
 Mod[Sum[Mod[Sum[SA4[a, c, d], {c, 0, 174}], 601], {d, 0, 174}], 601]
```

The functions `TABi[a_]` compute $T^{\chi_i}_{A,B}(a)$ for $i = 1, 2, 3, 4$:

```
TAB1[a_] := Mod[-RA1[a]*175*iv[RB1[a], 61] + a, 61]
TAB2[a_] := Mod[-RA2[a]*61*iv[RB2[a], 1861] + a, 1861]
TAB3[a_] := Mod[-RA3[a]*175*iv[RB3[a], 1861] + a, 1861]
TAB4[a_] := Mod[-RA4[a]*175*iv[RB4[a], 601] + a, 601]
```

The following are needed to compute $E_{\chi_i}(r)$ and $F_{\chi_i}(r)$ modulo I_i:

```
A[c_, d_, n_, q_] := -Floor[(d - c n)/q]
t[c_, d_, n_, q_] := d - c n - q Floor[(d - n c)/q]
B[c_, d_, n_, q_] := -Floor[(c - 2 n t[c, d, n, q])/q]
e[c_, d_, n_, q_] := c - 2n t[c, d, n, q] + q B[c, d, n, q]

E[c_, d_, n_, q_] := 3 c^2 n + 6 d^2 n - 3 e[c, d, n, q]^2 n
- 12 A[c, d, n, q] c q + 3 c n q - 6 d n q + 3 e[c, d, n, q] n q
+ 6 A[c, d, n, q] q^2 - 6 B[c, d, n, q] q^2 + 2 n q^2
+ 12 B[c, d, n, q] q t[c, d, n, q] - 6 n q t[c, d, n, q]
- 6 n t[c, d, n, q]^2
F[c_, d_, n_, q_] := -9 c^2 q + 6 d^2 q - 3 e[c, d, n, q]^2 q + 9 c q^2
- 6 d q^2 + 3 e[c, d, n, q] q^2 + 2 q^3 - 18 q^2 t[c, d, n, q]
+ 18 q t[c, d, n, q]^2

SF1[n_, c_, d_] := Mod[ch1[c^2-2 d^2+2 n c d]*F[175, n, c, d], 61]
SF2[n_, c_, d_] := Mod[ch2[c^2-2 d^2+2 n c d]*F[61, n, c, d], 1861]
SF3[n_, c_, d_] := Mod[ch3[c^2-2 d^2+2 n c d]*F[175, n, c, d], 1861]
SF4[n_, c_, d_] := Mod[ch4[c^2-2 d^2+2 n c d]*F[175, n, c, d], 601]
SE1[n_, c_, d_] := Mod[ch1[c^2-2 d^2+2 n c d]*E[175, n, c, d], 61]
SE2[n_, c_, d_] := Mod[ch2[c^2-2 d^2+2 n c d]*E[61, n, c, d], 1861]
SE3[n_, c_, d_] := Mod[ch3[c^2-2 d^2+2 n c d]*E[175, n, c, d], 1861]
SE4[n_, c_, d_] := Mod[ch4[c^2-2 d^2+2 n c d]*E[175, n, c, d], 601]
```

The functions `REi[a_]` and `RFi[a_]` compute $E_{\chi_i}(a)$ and $F_{\chi_i}(a)$ modulo I_i respectively, where `q` is the conductor for the character χ_i for $i = 1, 2, 3, 4$:

```
RF1[a_] :=
 Mod[Sum[Mod[Sum[SF1[a, c, d], {c, 0, 174}], 61], {d, 0, 174}], 61]
```

```
RE1[a_] :=
 Mod[Sum[Mod[Sum[SE1[a, c, d], {c, 0, 174}], 61], {d, 0, 174}], 61]
RF2[a_] :=
 Mod[Sum[Mod[Sum[SF2[a, c,  d], {c, 0, 60}], 1861], {d, 0, 60}], 1861]
RE2[a_] :=
 Mod[Sum[Mod[Sum[SE2[a, c, d], {c, 0, 60}], 1861], {d, 0, 60}], 1861]
RF3[a_] :=
 Mod[Sum[Mod[Sum[SF3[a, c, d], {c, 0, 174}], 1861], {d, 0, 174}],1861]
RE3[a_] :=
 Mod[Sum[Mod[Sum[SE3[a, c, d], {c, 0, 174}], 1861], {d, 0, 174}], 1861]
RF4[a_] :=
 Mod[Sum[Mod[Sum[SF4[a, c, d], {c, 0, 174}], 601], {d, 0, 174}], 601]
RE4[a_] :=
 Mod[Sum[Mod[Sum[SE4[a, c, d], {c, 0, 174}], 601], {d, 0, 174}], 601]
```

The functions `TEFi[a_]` compute $T_{E,F}^{\chi_i}(a)$ for $i = 1, 2, 3, 4$:

```
TEF1[a_] := Mod[-RE1[a]*175*iv[RF1[a], 61] + a, 61]
TEF2[a_] := Mod[-RE2[a]*61*iv[RF2[a], 1861] + a, 1861]
TEF3[a_] := Mod[-RE3[a]*175*iv[RF3[a], 1861] + a, 1861]
TEF4[a_] := Mod[-RE4[a]*175*iv[RF4[a], 601] + a, 601]
```

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