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Parametric geometry of numbers and applications

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1. Introduction. A basic result in the Geometry of Numbers is Minkowski's Second Convex Body Theorem. Given a closed symmetric convex body K in $\mathbb{R}^n$ and a lattice Λ in $\mathbb{R}^n$, the ith *successive minimum* λ_i, $1 \leq i \leq n$, with respect to K and Λ is the least number $\lambda > 0$ for which λK contains i linearly independent lattice points. Clearly, $\lambda_1 \leq \cdots \leq \lambda_n$ and Minkowski's Theorem ([8]) says that

$$\frac{2^n}{n!}\frac{\det \Lambda}{\mathrm{Vol}(K)} \leq \lambda_1 \cdots \lambda_n \leq 2^n \frac{\det \Lambda}{\mathrm{Vol}(K)},$$

where $\mathrm{Vol}(K)$ is the volume of K and $\det \Lambda$ the determinant of Λ.

Suppose $\mu_1, \ldots, \mu_n$ are reals with $\mu_1 + \cdots + \mu_n = 0$, and for $Q > 1$ let $T_Q : \mathbb{R}^n \to \mathbb{R}^n$ be the linear map with

$$\mathbf{p} := (p_1, \ldots, p_n) \mapsto (Q^{\mu_1} p_1, \ldots, Q^{\mu_n} p_n).$$

Then a symmetric convex body K gives rise to the bodies $K(Q) := T_Q(K)$ parametrized by Q. We propose to study the successive minima $\lambda_1(Q), \ldots, \lambda_n(Q)$ with respect to $K(Q)$, Λ as functions of Q. Trivially,

$$0 < \lambda_1(Q) \leq \cdots \leq \lambda_n(Q),$$

and since $\mathrm{Vol}(K(Q)) = \mathrm{Vol}(K)$, Minkowski's Theorem gives

$$c_1(K, \Lambda) \leq \lambda_1(Q) \cdots \lambda_n(Q) \leq c_2(K, \Lambda),$$

where $c_1(K, \Lambda)$, $c_2(K, \Lambda)$ depend only on K, Λ.

Our study is inspired by Diophantine Approximation where, beginning with Dirichlet's Theorem, a family of systems of inequalities parametrized by $Q > 1$ comes into play. Suppose $n > 1$,

$$\mu_1 = 1, \quad \mu_2 = \cdots = \mu_n = -1/(n-1),$$

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and let $\xi_1, \dots, \xi_{n-1}$ be real numbers. Then Dirichlet's Theorem on simultaneous approximation asserts that for any $Q > 1$ the system of inequalities

$$\begin{aligned} |x| &\le Q^{\mu_1}, \\ |\xi_1 x - y_1| &\le Q^{\mu_2}, \\ &\vdots \\ |\xi_{n-1} x - y_{n-1}| &\le Q^{\mu_n}, \end{aligned}$$

has a nontrivial solution in integer points $\mathbf{x} = (x, y_1, \dots, y_{n-1})$. When $\mathcal{C}$ is the unit cube of points $\mathbf{p}$ with $|p_i| \le 1$, $1 \le i \le n$, and $\Lambda = \Lambda(\xi)$ the lattice of points $\mathbf{p} = (x, \xi_1 x - y_1, \dots, \xi_{n-1} x - y_{n-1})$ with $\mathbf{x} \in \mathbb{Z}^n$, Dirichlet's Theorem asserts that there is a nonzero lattice point in $\mathcal{C}(Q)$, i.e. that the first minimum $\lambda_1(Q)$ with respect to $\mathcal{C}(Q)$ and Λ is at most 1.

Much work in Diophantine Approximation is implicitly about the function $\lambda_1(Q)$ and we believe that a study of the complete set of functions $\lambda_1(Q), \dots, \lambda_n(Q)$ will lead to new insights.

Suppose now that

$$\mu_1 = -1, \qquad \mu_2 = \dots = \mu_n = 1/(n-1),$$

and let $\Lambda^*(\xi)$ be the dual lattice to $\Lambda(\xi)$, which consists of points $\mathbf{p} = (x - \xi_1 y_1 - \dots - \xi_{n-1} y_{n-1}, y_1, \dots, y_{n-1})$ with $\mathbf{x} \in \mathbb{Z}^n$. Dirichlet's Theorem on linear forms may be interpreted as saying that for any $Q > 1$, the body $\mathcal{C}(Q)$ contains a nonzero point of $\Lambda^*(\xi)$. Thus if $\nu_i(Q)$, $1 \le i \le n$, are the successive minima with respect to $\mathcal{C}(Q)$ and $\Lambda^*(\xi)$, then $\nu_1(Q) \le 1$. Again we will be interested in all the functions $\nu_1(Q), \dots, \nu_n(Q)$. In the special situation where $\xi_i = a^i$, $a \in \mathbb{R} \setminus \{0\}$, see also [7].

The *reciprocal body* $\mathcal{C}^*$ of $\mathcal{C}$ consists of the points $\mathbf{p}$ with $|p_1| + \dots + |p_n| \le 1$. Therefore $\mathcal{C}^* \subseteq \mathcal{C} \subseteq n\mathcal{C}^*$ and the successive minima $\lambda_i^*(Q)$ of $\mathcal{C}^*(Q)$ with respect to $\Lambda^*(\xi)$ have

$$\nu_i(Q) \le \lambda_i^*(Q) \le n\nu_i(Q). \tag{1.1}$$

In the general context formulated at the beginning, we observe that each $\lambda_i(Q)$ is a continuous function of Q since $K(Q)$ is closed. In the next step, we wonder whether for given s, $1 \le s < n$, there are arbitrarily large values of Q with

$$\lambda_s(Q) = \lambda_{s+1}(Q).$$

When $A = \{i_1 < \dots < i_s\} \subseteq \{1, \dots, n\}$, set $\mu_A = \sum_{i \in A} \mu_i$ and let $\pi_A : \mathbb{R}^n \to \mathbb{R}^s$ be the map with

$$\pi_A(\mathbf{p}) = (p_{i_1}, \dots, p_{i_s}) \in \mathbb{R}^s.$$

Our result here is as follows:

THEOREM 1.1. *Suppose for every s-dimensional space S spanned by lattice points (i.e. points of Λ), there is some A of cardinality s with $\mu_A < 0$ and $\pi_A(S) = \mathbb{R}^s$. Then there are arbitrarily large values of Q with $\lambda_s(Q) = \lambda_{s+1}(Q)$.*

It will be seen in Section 2 that when $1, \xi_1, \ldots, \xi_{n-1}$ are linearly independent over $\mathbb{Q}$, Theorem 1.1 applies for each s, $1 \le s < n$, in the context of Dirichlet's Theorem on simultaneous approximation, as well as on linear forms.

Thus there are arbitrarily large values of Q with $\lambda_s(Q) = \lambda_{s+1}(Q)$ as well as arbitrarily large values of Q with $\nu_s(Q) = \nu_{s+1}(Q)$. Also, there are arbitrarily large Q with $\lambda_s^*(Q) = \lambda_{s+1}^*(Q)$.

In general, there is a nonzero lattice point $\mathbf{p}$ in $\lambda_1(Q)K(Q)$. This point has

$$|\mathbf{p}| \ge c_3 > 0 \quad \text{and} \quad |\mathbf{p}| \le c_4 \lambda_1(Q) Q^{\mu},$$

where $\mu = \max(\mu_1, \ldots, \mu_n)$, so that

$$\lambda_1(Q) \ge c_5 Q^{-\mu} > 0, \tag{1.2}$$

and hence by Minkowski's Theorem,

$$\lambda_n(Q) \le c_6 Q^{\mu(n-1)}. \tag{1.3}$$

Next we define $\psi_i(Q)$ for $Q > 1$ by

$$\lambda_i(Q) = Q^{\psi_i(Q)}, \qquad i = 1, \ldots, n.$$

The $\psi_i(Q)$ are again continuous and we have $0 < \psi_1(Q) \le \cdots \le \psi_n(Q)$, as well as

$$|\psi_1(Q) + \cdots + \psi_n(Q)| \le c_7(K, \Lambda)/\log Q \tag{1.4}$$

by Minkowski's Theorem.

The quantities

$$\overline{\psi}_i = \limsup_{Q\to\infty} \psi_i(Q) \quad \text{and} \quad \underline{\psi}_i = \liminf_{Q\to\infty} \psi_i(Q)$$

are finite by (1.2), (1.3) and satisfy the inequalities

$$\overline{\psi}_1 \le \cdots \le \overline{\psi}_n \quad \text{and} \quad \underline{\psi}_1 \le \cdots \le \underline{\psi}_n$$

and also $\overline{\psi}_i \ge \underline{\psi}_i$ for $i = 1, \ldots, n$.

By definition, if $\eta > \overline{\psi}_s$ and Q is large we will have $\psi_s(Q) < \eta$. Given that there are arbitrarily large values of Q with $\lambda_s(Q) = \lambda_{s+1}(Q)$, there will be arbitrarily large values of Q with $\psi_{s+1}(Q) = \psi_s(Q) < \eta$ and therefore

$$\underline{\psi}_{s+1} \le \overline{\psi}_s. \tag{1.5}$$

Theorem 1.2. *For* $1 \leq i \leq n$ *we have*

$$\overline{\psi}_1 + \cdots + \overline{\psi}_{i-1} + \underline{\psi}_i + \overline{\psi}_{i+1} + \cdots + \overline{\psi}_n \geq 0, \tag{1.6a}$$

$$\underline{\psi}_1 + \cdots + \underline{\psi}_{i-1} + \overline{\psi}_i + \underline{\psi}_{i+1} + \cdots + \underline{\psi}_n \leq 0. \tag{1.6b}$$

Suppose now that we are in the context of Dirichlet's Theorem on simultaneous approximation, so that $\Lambda = \Lambda(\xi)$ with $1, \xi_1, \ldots, \xi_{n-1}$ linearly independent over $\mathbb{Q}$, and $K = \mathcal{C}$. In this context we have

Theorem 1.3.

$$n\underline{\psi}_2 \leq (n-1)\overline{\psi}_2 + \underline{\psi}_1, \tag{1.7}$$

$$n\overline{\psi}_{n-1} \geq (n-1)\underline{\psi}_{n-1} + \overline{\psi}_n. \tag{1.8}$$

We will connect $\overline{\psi}_1, \underline{\psi}_1$ and $\overline{\psi}_n, \underline{\psi}_n$ to classical approximation exponents as studied by Khinchin [4], [5], Jarník [3], etc., and most recently by Roy [11] and Bugeaud and Laurent [1], [2].

Given $\xi = (\xi_1, \ldots, \xi_{n-1})$ with $1, \xi_1, \ldots, \xi_{n-1}$ linearly independent over $\mathbb{Q}$, the quantities ω (resp. $\hat{\omega}$) are defined as the supremum of the numbers η such that there are arbitrarily large values of X for which (resp. such that for every large value of X) the system of inequalities

$$|x| \leq X, \qquad |\xi_i x - y_i| \leq X^{-\eta} \quad \text{for } i = 1, \ldots, n-1,$$

has a nontrivial solution $\mathbf{x} = (x, y_1, \ldots, y_{n-1}) \in \mathbb{Z}^n$.

On the other hand, ω^* (resp. $\hat{\omega}^*$) is the supremum of the numbers η such that there are arbitrarily large values of X for which (resp. such that for every large value of X) the system

$$\Big| x - \sum_{i=1}^{n-1} \xi_i y_i \Big| \leq X^{-\eta}, \qquad |y_i| \leq X \quad \text{for } i = 1, \ldots, n-1,$$

has a nontrivial solution in integer n-tuples $\mathbf{x}$. It will not be hard to prove

Theorem 1.4.

$$(\omega + 1)(1 + \underline{\psi}_1) = (\hat{\omega} + 1)(1 + \overline{\psi}_1) = \frac{n}{n-1}, \tag{1.9}$$

$$(\omega^* + 1)\Big(\frac{1}{n-1} - \overline{\psi}_n\Big) = (\hat{\omega}^* + 1)\Big(\frac{1}{n-1} - \underline{\psi}_n\Big) = \frac{n}{n-1}. \tag{1.10}$$

Consequently, $\omega, \hat{\omega}, \omega^*, \hat{\omega}^*$ determine $\overline{\psi}_1, \underline{\psi}_1, \overline{\psi}_n, \underline{\psi}_n$ and *vice versa.* We will show that Khinchin's transference principle between ω and ω^* is equivalent to

$$\underline{\psi}_1 + (n-1)\overline{\psi}_n \geq 0 \quad \text{and} \quad \overline{\psi}_n + (n-1)\underline{\psi}_1 \leq 0,$$

which however is weaker than the linear inequalities of Theorem 1.2.

In the context of Dirichlet's Theorem on simultaneous approximation we now specialize further to the case of dimension two in Jarník's and Laurent's papers, which corresponds to $n = 3$ in our notation.

THEOREM 1.5. *When* $n = 3$, *then*

$$\overline{\psi}_1 + \underline{\psi}_3 + 2\overline{\psi}_1\underline{\psi}_3 = 0. \tag{1.11}$$

THEOREM 1.6. *When* $n = 3$, *then also*

$$2\underline{\psi}_1 + \overline{\psi}_3 \le -\underline{\psi}_3(3 + 2\underline{\psi}_1 + 4\overline{\psi}_3), \tag{1.12}$$

$$2\overline{\psi}_3 + \underline{\psi}_1 \ge -\overline{\psi}_1(3 + 2\overline{\psi}_3 + 4\underline{\psi}_1). \tag{1.13}$$

On account of (1.11) *the relations* (1.12), (1.13) *are equivalent to*

$$2\underline{\psi}_1 + \overline{\psi}_3 \le \overline{\psi}_1(3 + 2\overline{\psi}_3 - 2\underline{\psi}_1), \tag{1.14}$$

$$2\overline{\psi}_3 + \underline{\psi}_1 \ge \underline{\psi}_3(3 + 2\underline{\psi}_1 - 2\overline{\psi}_3). \tag{1.15}$$

Observe that there is some symmetry: (1.11) is invariant under interchanging $\overline{\psi}_1, \underline{\psi}_3$, and (1.12), (1.13) are interchanged if we interchange $\underline{\psi}_1, \overline{\psi}_3$ as well as $\overline{\psi}_1, \underline{\psi}_3$ and reverse inequalities. The same holds for (1.14), (1.15). It will be shown that (1.11) is equivalent to Jarník's relation between ω and $\hat{\omega}$, and that (1.12), (1.13) are equivalent to Laurent's refinement of Khinchin's transference principle as stated in [6].

However, $\overline{\psi}_i$ and $\underline{\psi}_i$, $i = 1, 2, 3$, do not give sufficient information on the *functions* $\psi_1(Q), \psi_2(Q), \psi_3(Q)$. In fact, we will give a rather precise description of them, and consider this description to be the most interesting part of our investigation. It is this description, which we postpone to Section 6, that will provide the tools for the proof of Theorems 1.5 and 1.6, carried out in Sections 7 and 8.

2. Proof of Theorem 1.1. For any Q there is a space $V(Q)$ of dimension s containing s linearly independent lattice points in $\lambda_s(Q)K(Q)$.

LEMMA 2.1. *Suppose* $\lambda_s(Q) < \lambda_{s+1}(Q)$ *for* $Q \ge Q_0$. *Then* $V(Q)$ *is unique for* $Q \ge Q_0$, *and in fact is constant*: $V(Q) =: S$ *for* $Q \ge Q_0$.

Proof. If $V(Q)$ and $V'(Q)$ are two spaces with the above property, their span $V(Q) + V'(Q)$ will contain at least $s + 1$ independent lattice points in $\lambda_s(Q)K(Q)$, so that $\lambda_s(Q) = \lambda_{s+1}(Q)$. Hence by our hypothesis, for $Q \ge Q_0$ there is a unique space $V(Q)$.

We claim the following continuity property: if some sequence $X_1, X_2, \ldots$ tends to X, where $X_l \ge Q_0$ and $V(X_l) = S$ for $l = 1, 2, \ldots$, then $V(X) = S$. For when Q runs through an interval J, then $K(Q)$ will be contained in a bounded region of $\mathbb{R}^n$, and since $\lambda_s(Q)$ is continuous, so will be $\lambda_s(Q)K(Q)$. This region will contain only finitely many lattice points. Replacing $P_1, P_2, \ldots$ by a subsequence if necessary, we may suppose that $V(X_l)$ is spanned by fixed independent lattice points $\mathbf{p}_1, \ldots, \mathbf{p}_s$ lying in $\lambda_s(X_l)K(X_l)$. Since $\lambda_s(Q)$ is continuous and $K(Q)$ is closed and varies continuously with Q, we see that $\mathbf{p}_j \in \lambda_s(X)K(X)$ for $j = 1, \ldots, s$, hence indeed $V(X) = S$.

Given a space S and an interval $J = [Q_0, Q_1]$ where $Q_1 > Q_0$, let $J(S)$ consist of the numbers $Q \in J$ with $V(Q) = S$. The sets $J(S)$ cover J and are disjoint by the uniqueness of $V(Q)$. Moreover, they are closed by the continuity property established above. Since $Q \in J$ is bounded, there are only finitely many spaces S having $S = V(Q)$ for some $Q \in J$, hence only finitely many nonempty sets $J(S_1), \ldots, J(S_l)$. But a finite number of nonempty mutually disjoint closed sets can cover J only if there is just one such set. Therefore $J(S) = J$ for some S, hence $V(Q) = S$ for $Q \in J$. Since $Q_1 > Q_0$ above was arbitrary, $V(Q) = S$ for $Q \geq Q_0$. ∎

Proof of Theorem 1.1. Suppose there was a Q_0 and a space S as in the preceding lemma. Then $\Lambda_S := \Lambda \cap S$ is a lattice in S. Also, $K_S(Q) := K(Q) \cap S$ is a symmetric convex body in S whose s-dimensional volume we denote by $\mathrm{Vol}_S(Q)$. Given $Q \geq Q_0$ so that $V(Q) = S$, then (by the uniqueness of $V(Q)$) S contains independent lattice points $\mathbf{p}_1, \ldots, \mathbf{p}_s$ with $\mathbf{p}_j \in \lambda_j(Q)K(Q)$, hence in fact with $\mathbf{p}_j \in \Lambda_S \cap \lambda_j(Q)K_S(Q)$ for $1 \leq j \leq s$. By Minkowski's Second Convex Body Theorem applied to $\Lambda_S(Q), K_S(Q)$, we have

$$\lambda_1(Q) \cdots \lambda_s(Q) \geq c_1(\Lambda, S, K)/\mathrm{Vol}_S(Q).$$

On the other hand, applying Minkowski's Theorem to Λ, $K(Q)$ yields

$$\lambda_1(Q) \cdots \lambda_n(Q) \leq c_2(\Lambda)/\mathrm{Vol}(K(Q)) = c_2(\Lambda)/\mathrm{Vol}(K) = c_3(\Lambda, K),$$

hence $\lambda_1(Q) \cdots \lambda_s(Q) \leq c_4(\Lambda, K)$, so that finally

$$\mathrm{Vol}_S(Q) \geq c_5(\Lambda, S, K) > 0. \tag{2.1}$$

If A is as in our hypothesis, we have

$$\mathrm{Vol}_S(Q) = \mathrm{Vol}(K_S(Q)) \leq c_6(S)\mathrm{Vol}(\pi_A(K(Q))).$$

But if $\mathbf{p} \in K(Q)$, then $\pi_A(\mathbf{p}) = (p_{i_1}, \ldots, p_{i_s})$ has $|p_{i_j}| \leq c_7(K)Q^{\mu_{i_j}}$ $(1 \leq j \leq s)$, and therefore

$$\mathrm{Vol}(\pi_A(K(Q))) \leq c_8(K)Q^{\mu_A}. \tag{2.2}$$

Now (2.2) and $\mu_A < 0$ yield $\mathrm{Vol}_S(Q) \to 0$ as $Q \to \infty$, a contradiction to (2.1). ∎

COROLLARY 2.2. *Suppose $1, \xi_1, \ldots, \xi_{n-1}$ are linearly independent over $\mathbb{Q}$ and $\mu_1 + \cdots + \mu_n = 0$ with $\mu_i < 0$ for $2 \leq i \leq n$. Let $\lambda_i(Q)$, $1 \leq i \leq n$, be the successive minima with respect to $K(Q)$ and $\Lambda(\xi)$. Then for every $s < n$, there are arbitrarily large values of Q for which $\lambda_s(Q) = \lambda_{s+1}(Q)$.*

Proof. First suppose that $s = n - 1$ and set $A_0 = \{2, \ldots, n\}$, so that $\mu_{A_0} < 0$. By Theorem 1.1 it will suffice to show that any $(n-1)$-dimensional subspace S spanned by lattice points has $\pi_{A_0}(S) = \mathbb{R}^{n-1} = \mathbb{R}^s$. Since Λ consists of points $\mathbf{p} = (x, \xi_1 x - y_1, \ldots, \xi_{n-1}x - y_{n-1})$ with $(x, y_1, \ldots, y_{n-1}) \in \mathbb{Z}^n$, the space S will consist of points $\mathbf{p}$ where $(x, y_1, \ldots, y_{n-1}) \in \mathbb{Z}^n$ satisfies a nontrivial equation

$$cx + c_1 y_1 + \cdots + c_{n-1} y_{n-1} = 0 \tag{2.3}$$

with integer coefficients. We have to show that $\pi_{A_0}(S)$ is not a proper subspace of $\mathbb{R}^{n-1}$, say given by a nontrivial equation

$$d_1(\xi_1 x - y_1) + \cdots + d_{n-1}(\xi_{n-1}x - y_{n-1}) = 0$$

with real coefficients. This equation may be rewritten as

$$(d_1\xi_1 + \cdots + d_{n-1}\xi_{n-1})x - d_1 y_1 - \cdots - d_{n-1}y_{n-1} = 0. \tag{2.4}$$

We have to show that (2.4) is not a consequence of (2.3), i.e. that the respective coefficient vectors are not proportional. If this were the case, we could, after multiplication by a factor, in fact suppose that they were equal, hence $d_i = -c_i$ for $i = 1, \ldots, n-1$ and $c_1\xi_1 + \cdots + c_{n-1}\xi_{n-1} = c$, contradicting the independence of $1, \xi_1, \ldots, \xi_{n-1}$.

Now suppose that $\dim S = s$ with $1 \le s < n-1$. Then S can be embedded in a space S' spanned by lattice points with $\dim S' = n-1$. We saw that $\pi_{A_0}(S') = \mathbb{R}^{n-1}$, and it follows that $T := \pi_{A_0}(S)$ has dimension s. There is some $(p_{i_1}, \ldots, p_{i_s})$-coordinate space of $\mathbb{R}^{n-1}$ such that the projection of T on this space is surjective. Now we have $\pi_A(S) = \mathbb{R}^s$ with $A = \{i_1 < \cdots < i_s\}$. Since $\mu_A < 0$, the corollary follows. ∎

Remarks. (a) Corollary 2.2 applies in particular in the context of Dirichlet's Theorem on simultaneous approximation, where $\mu_1 = 1$ and $\mu_2 = \cdots = \mu_n = -1/(n-1)$.

(b) The assertion of Corollary 2.2 is not true in general when the numbers $1, \xi_1, \ldots, \xi_{n-1}$ are linearly dependent over $\mathbb{Q}$. For instance take $n = 3$, $\mu_1 = 1$, $\mu_2 = \mu_3 = -1/2$ and $\xi_1 = \alpha$, $\xi_2 = \alpha + 1$, where α has bounded partial denominators in its continued fraction expansion. The convergents p_ν/q_ν, $\nu = 0, 1, \ldots$, to α will have $q_{\nu+1} < cq_\nu$ for some c. Given large Q there will be some ν with $c^{-2}Q^{3/4} \le q_\nu < q_{\nu+1} < Q^{3/4}$. Then

$$|q_\nu\alpha - p_\nu| < q_\nu^{-1} \le c^2 Q^{-3/4}, \qquad |q_\nu(\alpha+1) - p_\nu - q_\nu| \le c^2 Q^{-3/4},$$

and the same holds with $\nu + 1$ in place of ν. It easily follows that $\lambda_1(Q) \le \lambda_2(Q) \le c^2 Q^{-1/4}$, hence certainly $\lambda_2(Q) < \lambda_3(Q)$ for large Q. Observe that the points $(q_\nu, q_\nu\alpha - p_\nu, q_\nu(\alpha+1) - p_\nu - q_\nu)$ lie in a space S of dimension 2.

(c) For the existence of arbitrarily large values of Q for which $\lambda_s(Q) = \lambda_{s+1}(Q)$ the fact that two consecutive successive minima are dealt with is crucial: answering a problem raised in [12], Moshchevitin [10] showed that for any $s < n-1$ there exist $\xi_j \in [0, 1)$, $1 \le j \le n-1$, such that $1, \xi_1, \ldots, \xi_{n-1}$ are linearly independent over $\mathbb{Q}$, $\lambda_s(Q) \to 0$ and $\lambda_{s+2}(Q) \to \infty$ as $Q \to \infty$.

Corollary 2.3. *Suppose $1, \xi_1, \ldots, \xi_{n-1}$ are linearly independent over $\mathbb{Q}$ and $\mu_1 + \cdots + \mu_n = 0$ with $\mu_i > 0$ for $2 \le i \le n$. Let $\lambda_i^*(Q)$, $1 \le i \le n$, be the successive minima with respect to $K(Q)$ and $\Lambda^*(\xi)$. Then for every $s < n$, there are arbitrarily large values of Q for which $\lambda_s^*(Q) = \lambda_{s+1}^*(Q)$.*

Proof. Observe that $x - \xi_1 y_1 - \cdots - \xi_{n-1} y_{n-1} \neq 0$ when $(x, y_1, \ldots, y_{n-1})$ lies in $\mathbb{Q}^n \setminus \{\mathbf{0}\}$. So if $A_1 = \{1\}$, every nonzero space S spanned by lattice points has $\pi_{A_1}(S) \neq \mathbf{0}$, in fact $\pi_{A_1}(S) = \mathbb{R}$. Hence when $\dim S = s < n$, there is some $A = \{1 < i_2 < \cdots < i_s\}$ with $\pi_A(S) = \mathbb{R}^s$. Since $\mu_A = -\sum_{i \notin A} \mu_i < 0$, the conclusion follows via Theorem 1.1. ∎

Remark. Corollary 2.3 applies in particular in the context of Dirichlet's Theorem on linear forms, where $\mu_1 = -1$ and $\mu_2 = \cdots = \mu_n = 1/(n-1)$.

3. $\overline{\psi}_i$, $\underline{\psi}_i$ and the proof of Theorem 1.2. For $A \subseteq \{1, \ldots, n\}$ set

$$\psi_A(Q) := \sum_{i \in A} \psi_i(Q), \qquad \overline{\psi}_A := \limsup_{Q \to \infty} \psi_A(Q), \qquad \underline{\psi}_A := \liminf_{Q \to \infty} \psi_A(Q).$$

Lemma 3.1. *When A, B are complementary nonempty subsets of $\{1, \ldots, n\}$, we have*

$$\overline{\psi}_A + \underline{\psi}_B = 0$$

and moreover

$$\sum_{i \in A} \underline{\psi}_i \le \underline{\psi}_A \le |A| \max_{i \in A} \underline{\psi}_i, \qquad |A| \min_{i \in A} \overline{\psi}_i \le \overline{\psi}_A \le \sum_{i \in A} \overline{\psi}_i.$$

Proof. By (1.4) we have $\psi_A(Q) + \psi_B(Q) \to 0$ as $Q \to \infty$. So when $\epsilon > 0$ and Q is large, we have $\psi_A(Q) < \overline{\psi}_A + \epsilon$ and $\psi_B(Q) > -\overline{\psi}_A - 2\epsilon$. Since $\epsilon > 0$ is arbitrary, this yields $\underline{\psi}_B \ge -\overline{\psi}_A$, i.e. $\overline{\psi}_A + \underline{\psi}_B \ge 0$. When $\epsilon > 0$ and Q is large, we have $\psi_B(Q) > \underline{\psi}_B - \epsilon$ and $\psi_A(Q) < -\underline{\psi}_B + 2\epsilon$, hence eventually $\overline{\psi}_A + \underline{\psi}_B \le 0$. This proves the first assertion.

The estimates

$$\sum_{i \in A} \underline{\psi}_i \le \underline{\psi}_A, \qquad \overline{\psi}_A \le \sum_{i \in A} \overline{\psi}_i$$

follow immediately from the definitions.

If $A = \{i_1 < \cdots < i_s\}$, there are for $\epsilon > 0$ arbitrarily large values of Q with $\psi_{i_s}(Q) \le \underline{\psi}_{i_s} + \epsilon$, hence

$$\sum_{i \in A} \psi_i(Q) \le |A|(\underline{\psi}_{i_s} + \epsilon) \le |A| \max_{i \in A} \underline{\psi}_i + |A|\epsilon.$$

On the other hand, for any $\epsilon > 0$ there are arbitrarily large values of Q with $\psi_{i_s}(Q) \ge \overline{\psi}_{i_s} - \epsilon$, hence

$$\sum_{i \in A} \psi_i(Q) \ge |A|(\overline{\psi}_{i_s} - \epsilon) \ge |A| \min_{i \in A} \overline{\psi}_i - |A|\epsilon$$

and the lemma follows. ∎

Proof of Theorem 1.2. Taking $A = \{i\}$, $B = \{1, \dots, n\} \setminus \{i\}$ we have

$$0 = \underline{\psi}_A + \overline{\psi}_B \le \underline{\psi}_i + (\overline{\psi}_1 + \dots + \overline{\psi}_{i-1} + \overline{\psi}_{i+1} + \dots + \overline{\psi}_n),$$
$$0 = \overline{\psi}_A + \underline{\psi}_B \ge \overline{\psi}_i + (\underline{\psi}_1 + \dots + \underline{\psi}_{i-1} + \underline{\psi}_{i+1} + \dots + \underline{\psi}_n). \ \blacksquare$$

4. The functions $L_1, \dots, L_n$ and proof of Theorems 1.3 and 1.4. We now specialize to the situation of Dirichlet's Theorem on simultaneous approximation. Thus $\Lambda(\xi)$ consists of points $\mathbf{p}(\mathbf{x}) = (x, \xi_1 x - y_1, \dots, \xi_{n-1}x - y_{n-1})$ where $1, \xi_1, \dots, \xi_{n-1}$ are linearly independent over $\mathbb{Q}$ and $\mathbf{x} = (x, y_1, \dots, y_{n-1})$ runs through $\mathbb{Z}^n$. Further, $\mathcal{C}(Q)$ consists of points $\mathbf{p} = (p_1, \dots, p_n)$ with $|p_1| \le Q$ and $|p_i| \le Q^{-1/(n-1)}$ $(i = 2, \dots, n)$. Therefore $\mathbf{p}(\mathbf{x})$ lies in $\lambda\mathcal{C}(Q)$ precisely if

$$|x| \le \lambda Q, \quad |\xi_j x - y_j| \le \lambda Q^{-1/(n-1)} \quad \text{for } 2 \le j \le n.$$

Hence $\lambda_i(Q)$ $(i = 1, \dots, n)$ is the least value of λ such that there are i independent points $\mathbf{x}$ for which the above system of inequalities is satisfied. If $\lambda_1(Q) \le c/Q$, then the above $\mathbf{x}$ has $|x| \le c$, and $|\xi_1 x - y_1| \le \lambda_1(Q)Q^{-1/(n-1)} \le cQ^{-n/(n-1)}$ has no solution y_1 for large Q. Therefore

$$\lambda_1(Q) \ge f_1(Q)/Q, \tag{4.1}$$

where $f_1(Q)$ tends to ∞ as $Q \to \infty$.

Given $\mathbf{x} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ we let $\lambda_{\mathbf{x}}(Q)$ be the least $\lambda > 0$ with $\mathbf{p}(\mathbf{x}) \in \lambda\mathcal{C}(Q)$. Then for any $Q > 1$ and $i \in \{1, \dots, n\}$ we have $\lambda_i(Q) = \lambda_{\mathbf{x}}(Q)$ for some $\mathbf{x}$ depending on Q and i. In particular,

$$\lambda_1(Q) = \min_{\mathbf{x} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}} \lambda_{\mathbf{x}}(Q).$$

Clearly,

$$\lambda_{\mathbf{x}}(Q) = \max\{|x|Q^{-1}, \max_{1 \le j \le n-1}\{|\xi_j x - y_j|\}Q^{1/(n-1)}\}.$$

Since $\lambda_{\mathbf{x}}(Q) \ge Q^{1/(n-1)}$ (otherwise, we will restrict to points $\mathbf{x}$ with $x \ne 0$), and since $\lambda_{\mathbf{x}}(Q) = \lambda_{-\mathbf{x}}(Q)$, we may suppose $x > 0$.

It will be convenient to replace Q as well as $\lambda_i(Q), \lambda_{\mathbf{x}}(Q)$ by their logarithms. Accordingly we set

$$q := \log Q, \tag{4.2}$$
$$\chi_i(q) := \psi_i(e^q), \tag{4.3}$$
$$L_i(q) := \log \lambda_i(e^q) = q\chi_i(q), \tag{4.4}$$
$$L_{\mathbf{x}}(q) := \log \lambda_{\mathbf{x}}(e^q), \tag{4.5}$$

and with these notations we obtain

$$L_{\mathbf{x}}(q) = \max\left\{\log|x| - q, \max_{1 \le j \le n-1}\{\log|\xi_j x - y_j|\} + \frac{q}{n-1}\right\}.$$

Thus $L_{\mathbf{x}}(q)$ has its minimum at some $q = q(\mathbf{x})$, and is linear with slope -1 for $q \le q(\mathbf{x})$ and slope $1/(n-1)$ for $q \ge q(\mathbf{x})$. When Q lies in a bounded range, there will be only finitely many $\mathbf{x}$ having $\lambda_i(Q) = \lambda_{\mathbf{x}}(Q)$ for some i. Therefore the functions L_i will be continuous, piecewise linear, with slopes among -1 and $1/(n-1)$.

When $\epsilon > 0$ and q is large, then $\underline{\psi}_i - \epsilon < \chi_i(q) < \overline{\psi}_i + \epsilon$, and therefore

$$q\underline{\psi}_i - o(q) \le L_i(q) \le q\overline{\psi}_i + o(q).$$

Set $\mathbf{x}_1 = (1, y_2, \ldots, y_n)$ with $|\xi_j - y_j| < 1/2$ $(1 \le j < n)$. It is easily seen that in some interval $1 < Q < Q_0$, the function $\lambda_1(Q)$ equals $\lambda_{\mathbf{x}_1}(Q)$, and is decreasing. Since L_1 is piecewise linear with only finitely many pieces within any bounded interval, there will be numbers $0 < q_1 < q_2 < \cdots$ tending to infinity such that L_1 has its local minima precisely at q_k, $k = 1, 2, \ldots$. There will be points $\mathbf{x}_k$ with $L_1(q_k) = L_{\mathbf{x}_k}(q_k)$, and in fact there will be a neighborhood of q_k with $L_1(q) = L_{\mathbf{x}_k}(q)$. More precisely, if $q_{0,1} = 0$ and $q_{k,k+1}$ for $k > 0$ is the q-coordinate of the point of intersection of the graphs of $L_{\mathbf{x}_k}$ and $L_{\mathbf{x}_{k+1}}$, then for $k > 0$,

$$L_1(q) = L_{\mathbf{x}_k}(q) \quad \text{for } q_{k-1,k} \le q \le q_{k,k+1},$$

which also holds for $k = 1$ if we define $L_1(0) = 0$.

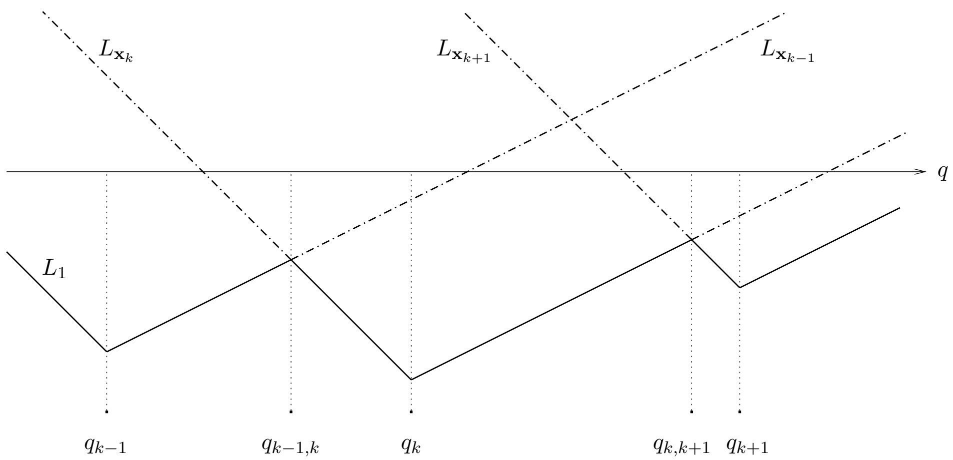
L_1 will decrease with slope -1 in $[q_{k-1,k}, q_k]$ and increase with slope $1/(n-1)$ in $[q_k, q_{k,k+1}]$. Thus L_1 will have its local maxima at $q_{k,k+1}$, $k = 1, 2, \ldots$. Moreover,

$$L_1(q_{k,k+1}) = L_{\mathbf{x}_k}(q_{k,k+1}) = L_{\mathbf{x}_{k+1}}(q_{k,k+1}),$$

and since $\mathbf{x}_k$, $\mathbf{x}_{k+1}$ are easily seen to be linearly independent we have

$$L_2(q_{k,k+1}) = L_1(q_{k,k+1}).$$

So if L_1 has a local maximum at q (hence $q = q_{k,k+1}$ for some k), then $L_1(q) = L_2(q)$.



REMARK. It is well known, and follows from our arguments in Section 2, that for $n \geq 3$ there are arbitrarily large k with $\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{x}_{k+2}$ linearly independent. On the other hand, according to Moshchevitin [9], when $n > 3$, then there are numbers $1, \xi_1, \ldots, \xi_{n-1}$ linearly independent over $\mathbb{Q}$ such that $\mathbf{x}_k, \mathbf{x}_{k+1}, \ldots, \mathbf{x}_{k+n-1}$ for every large k span a space of dimension at most 3.

The function L_1^* will have its local minima at numbers $q_1^* < q_2^* < \cdots$, and $L_1^*(q_l^*) = L_{\mathbf{x}_l^*}^*(q_l^*)$ for certain points $\mathbf{x}_1^*, \mathbf{x}_2^*, \ldots$. For suitably defined $q_{l,l+1}^*$ with $q_l^* < q_{l,l+1}^* < q_{l+1}^*$ we will have

$$L_1^*(q) = L_{\mathbf{x}_l^*}^*(q) \quad \text{for } q_{l-1,l}^* \leq q \leq q_{l,l+1}^*$$

with L_1^* having slope 1 in $[q_{l-1,l}^*, q_l^*]$ and slope $-1/(n-1)$ in $[q_l^*, q_{l,l+1}^*]$. Moreover, $L_2^*(q_{l,l+1}^*) = L_1^*(q_{l,l+1}^*)$. So when L_1^* has a local maximum at q, then $L_1^*(q) = L_2^*(q)$.

Mahler's inequality $\lambda_i^*(Q)\lambda_{n+1-i}(Q) \asymp 1$, where the implied constants depend on n only, gives

$$|L_i^*(q) + L_{n+1-i}(q)| < c \quad (i = 1, \ldots, n) \tag{4.6}$$

where the constant c depends on n only. Further, $L_i(q) = q\psi_i(e^q)$, so (1.4) yields

$$|L_1(q) + \cdots + L_n(q)| < c \tag{4.7}$$

if $c = c(n)$ was chosen large enough, and we also have

$$|L_1^*(q) + \cdots + L_n^*(q)| < c. \tag{4.8}$$

It is not hard to see that $\lambda_1^*(Q) > f_1^*(Q)/Q^{1/(n-1)}$ with $f_1^*(Q)$ tending to infinity with Q, in analogy to (4.1), and therefore by Mahler's inequality $\lambda_n(Q) \asymp \lambda_1^*(Q)^{-1} < Q^{1/(n-1)}/f_1^*(Q)$. But this, together with (4.1), tells us that

$$L_1(q) > -q + g_1(Q) \quad \text{and} \quad L_n(q) < \frac{q}{n-1} - g_n(Q) \tag{4.9}$$

with $g_1(Q), g_n(Q)$ both tending to infinity.

We had

$$\psi_i(Q) = \log \lambda_i(Q)/\log Q \quad \text{and} \quad \chi_i(q) = \psi_i(e^q) = L_i(q)/q,$$

so that

$$\begin{aligned} \overline{\psi}_i &= \limsup \psi_i(Q) = \limsup \chi_i(q), \\ \underline{\psi}_i &= \liminf \psi_i(Q) = \liminf \chi_i(q). \end{aligned} \tag{4.10}$$

We now set

$$\psi_i^*(Q) = \log \lambda_i^*(Q)/\log Q \quad \text{and} \quad \chi_i^*(q) = \psi_i^*(e^q) = L_i^*(q)/q,$$

so that

$$\overline{\psi}_i^* = \limsup \psi_i^*(Q) = \limsup \chi_i^*(q),$$
$$\underline{\psi}_i^* = \liminf \psi_i^*(Q) = \liminf \chi_i^*(q).$$

Inequality (4.6) yields $|\psi_i^*(q) + \psi_{n+1-i}(q)| < c/\log Q$, so that

$$\overline{\psi}_i^* + \underline{\psi}_{n+1-i} = 0 \quad \text{and} \quad \underline{\psi}_i^* + \overline{\psi}_{n+1-i} = 0 \qquad (i = 1, \ldots, n). \tag{4.11}$$

Lemma 4.1. *If $i \neq j$ and L_i has slope -1 in some interval, then for q, q' in that interval,*

$$L_j(q) - L_j(q') = \frac{q - q'}{n-1} + O(1).$$

Here and throughout, the implied constants in $\asymp, O(\ldots), \ll$ depend only on n.

Proof. Say $q > q'$. Each L_l has slopes $-1, 1/(n-1)$ so that

$$L_l(q) - L_l(q') \leq \frac{q - q'}{n-1} \qquad (1 \leq l \leq n). \tag{4.12}$$

By (4.7) and (4.12),

$$\begin{aligned} L_j(q) - L_j(q') &\geq -\sum_{l \neq j} (L_l(q) - L_l(q')) + O(1) \\ &= -(L_i(q) - L_i(q')) - \sum_{l \neq i,j} (L_l(q) - L_l(q')) + O(1) \\ &\geq q - q' - (n-2)\,\frac{q - q'}{n-1} + O(1) = \frac{q - q'}{n-1} + O(1). \end{aligned}$$

The lemma follows from this and from (4.12) with $l = j$. ∎

Proof of Theorem 1.3. For $k \geq 2$ we have

$$L_1(q_{k-1,k}) = L_2(q_{k-1,k}) \leq 0, \qquad L_1(q_k) = L_1(q_{k-1,k}) - (q_k - q_{k-1})$$

and by Lemma 4.1,

$$L_2(q_k) = L_2(q_{k-1,k}) + \frac{q_k - q_{k-1,k}}{n-1} + O(1).$$

Therefore

$$nL_2(q_{k-1,k}) = (n-1)L_2(q_k) + L_1(q_k) + O(1). \tag{4.13}$$

Since both sides are ≤ 0, and $q_{k-1,k} \leq q_k$,

$$n\psi_2(q_{k-1,k}) \leq (n-1)\psi_2(q_k) + \psi_1(q_k) + O(1/q_{k-1,k}),$$

hence we may infer that

$$\begin{aligned} n\underline{\psi}_2 &\leq n \liminf \psi_2(q_{k-1,k}) \leq (n-1) \limsup \psi_2(q_k) + \liminf \psi_1(q_k) \\ &= (n-1)\overline{\psi}_2 + \underline{\psi}_1, \end{aligned}$$

which is (1.7).

The proof of (1.8) is dual to that of (1.7): by (4.11) we need to show that
$$n\underline{\psi}_2^* \le (n-1)\overline{\psi}_2^* + \overline{\psi}_1^*.$$
Lemma 4.1 is replaced by the fact that if $i \ne j$ and L_i^* has slope 1, then
$$L_j^*(q) - L_j^*(q') = \frac{q'-q}{n-1} + O(1).$$
Instead of (4.13) we have
$$nL_2^*(q_{k,k+1}) = (n-1)L_2^*(q_k) + L_1^*(q_k) + O(1);$$
now we proceed as in the first part of the proof. ∎

Proof of Theorem 1.4. If $\theta > \underline{\psi}_1$, then
$$\begin{aligned} |x| &\le Q^{1+\theta}, \\ |\xi_i x - y_i| &\le Q^{-1/(n-1)+\theta} \quad \text{for } i = 1,\dots,n-1 \end{aligned} \tag{4.14}$$
has a nonzero integer solution for certain arbitrarily large values of Q. Setting
$$X = Q^{1+\theta} \quad \text{and} \quad \eta = -1 + \frac{n}{(n-1)(1+\theta)}$$
the system (4.14) becomes
$$|x| \le X, \quad |\xi_i x - y_i| \le X^{-\eta} \quad \text{for } i = 1,\dots,n-1. \tag{4.15}$$
Therefore $\omega \ge \eta$, and since θ is arbitrarily close to $\underline{\psi}_1$, we obtain
$$\omega \ge -1 + \frac{n}{(n-1)(1+\underline{\psi}_1)},$$
hence
$$(1+\omega)(1+\underline{\psi}_1) \ge \frac{n}{n-1}.$$
When $\eta < \omega$, (4.15) has a solution for certain arbitrarily large numbers X. Setting
$$Q = X^{1/(1+\eta)} \quad \text{and} \quad \theta = -1 + \frac{n}{(n-1)(1+\eta)},$$
the system (4.15) becomes (4.14) and hence $\underline{\psi}_1 \le \theta$. As η can be taken arbitrarily close to ω, we obtain
$$\underline{\psi}_1 \le -1 + \frac{n}{(n-1)(1+\omega)} \iff (1+\omega)(1+\underline{\psi}_1) \le \frac{n}{n-1}.$$
Therefore the first expression in (1.9) equals $n/(n-1)$, and the proof for the second one is similar.

We had $\psi_i^*(Q) = \log\lambda_i^*(Q)/\log Q$, hence $\lambda_i^*(Q) = Q^{\psi_i^*(Q)}$, and we now define $\pi_i(Q)$ by $\nu_i(Q) = Q^{\pi_i(Q)}$. By (1.1), $|\pi_i(Q) - \psi_i^*(Q)| < n/\log Q$, so that
$$\overline{\pi}_i := \limsup \pi_i(Q) = \overline{\psi}_i^* \quad \text{and} \quad \underline{\pi}_i := \liminf \pi_i(Q) = \underline{\psi}_i^*.$$

Therefore, by (4.11), $\overline{\psi}_n = -\underline{\psi}_1^* = -\underline{\pi}_1$ and $\underline{\psi}_n = -\overline{\psi}_1^* = -\overline{\pi}_1$, so that in order to establish (1.10) we need to prove

$$(\omega^* + 1)\left(\frac{1}{n-1} + \underline{\pi}_1\right) = (\hat{\omega}^* + 1)\left(\frac{1}{n-1} + \overline{\pi}_1\right) = \frac{n}{n-1}.$$

This is proved similarly to (1.9). ∎

Laurent in [6] gives best possible upper and lower bounds for $\omega, \hat{\omega}, \omega^*, \hat{\omega}^*$. Since

$$\frac{1}{n-1} \le \omega \le \infty \quad \text{and} \quad \frac{1}{n-1} \le \hat{\omega} \le 1$$

are best possible, (1.9) gives the best possible bounds

$$-1 \le \underline{\psi}_1 \le 0 \quad \text{and} \quad -\frac{n-2}{2(n-1)} \le \overline{\psi}_1 \le 0.$$

Further, $n-1 \le \omega^* \le \infty$ and $n-1 \le \hat{\omega}^* \le \infty$, so that by (1.10),

$$0 \le \overline{\psi}_n \le \frac{1}{n-1} \quad \text{and} \quad 0 \le \underline{\psi}_n \le \frac{1}{n-1}.$$

Again these estimates are best possible.

Remarks. (a) Jarník's identity $\hat{\omega} = \hat{\omega}^* - 1/\hat{\omega}^*$ is equivalent to equation (1.11) of Theorem 1.5, for this identity is in turn equivalent to

$$\hat{\omega} = \frac{\hat{\omega}^* - 1}{\hat{\omega}^*} \;\Leftrightarrow\; \hat{\omega} + 1 = 2 - \frac{1}{\hat{\omega}^*} \;\Leftrightarrow\; \frac{3}{2\overline{\psi}_1 + 2} = 2 + \frac{2\underline{\psi}_3 - 1}{2\underline{\psi}_3 + 2}$$

$$\Leftrightarrow\; \frac{1}{\overline{\psi}_1 + 1} = \frac{2\underline{\psi}_3 + 1}{\underline{\psi}_3 + 1} \;\Leftrightarrow\; \overline{\psi}_1 + \underline{\psi}_3 + 2\overline{\psi}_1\underline{\psi}_3 = 0.$$

(b) Khinchin's transference principle ([5]) says that

$$\omega \ge \frac{\omega^*}{(n-2)\omega^* + n - 1} \quad \text{and} \quad \omega^* \ge (n-1)\omega + n - 2.$$

The first of these inequalities yields

$$\omega + 1 \ge \frac{\omega^* + (n-2)\omega^* + n - 1}{(n-2)\omega^* + n - 1} = \frac{n-1}{n-2+1/(\omega^*+1)},$$

hence by Theorem 1.4,

$$\frac{n/(n-1)}{1 + \underline{\psi}_1} \ge \frac{n-1}{n-2+(1/(n-1) - \overline{\psi}_n)(n-1)/n} = \frac{n-1}{n-1-\overline{\psi}_n},$$

so that $n - 1 - \overline{\psi}_n \ge (n-1)(1 + \underline{\psi}_1)$, hence $(n-1)\underline{\psi}_1 + \overline{\psi}_n \le 0$.

The second relation in Khinchin's transference principle leads to $\underline{\psi}_1 + (n-1)\overline{\psi}_n \ge 0$. Conversely, these relations imply Khinchin's transference principle. Observe that they are consequences of Theorem 1.2.

(c) Laurent's refinement of Khinchin's relations between the approximation exponents in [6] states that

$$\frac{\omega^*(\hat{\omega}^*-1)}{\omega^*+\hat{\omega}^*} \le \omega \le \frac{\omega^*-\hat{\omega}^*+1}{\hat{\omega}^*},$$

which is equivalent to the pair of inequalities (1.12), (1.13) of Theorem 1.6. For expressing the classical approximation constants in terms of $\underline{\psi}_1, \underline{\psi}_3$, $\overline{\psi}_3$ using the equations from Theorem 1.4, a lengthy but straightforward computation translates Laurent's inequalities into

$$\frac{3\underline{\psi}_3 - 2\overline{\psi}_3 - 2\underline{\psi}_3\overline{\psi}_3}{1-2\underline{\psi}_3} \le \underline{\psi}_1 \le \frac{-3\underline{\psi}_3 - \overline{\psi}_3 - 4\underline{\psi}_3\overline{\psi}_3}{2+2\underline{\psi}_3},$$

which turns out to be precisely equivalent to (1.14), (1.15) and hence to (1.12), (1.13).

5. The functions $\tilde{L}_1, \tilde{L}_2, \tilde{L}_3$. We now turn to the case $n = 3$ in the context of Dirichlet's Theorem on simultaneous approximation. It would be nice if we had $L_1(q) + L_2(q) + L_3(q) = 0$ in place of the case $n = 3$ of (4.7). We therefore introduce the new triple of functions $(\tilde{L}_1, \tilde{L}_2, \tilde{L}_3)$ with

$$\tilde{L}_1 := L_1, \qquad \tilde{L}_2 := L_1^* - L_1, \qquad \tilde{L}_3 := -L_1^*,$$

whose properties will now be investigated.

LEMMA 5.1. *If c is the constant of* (4.6)–(4.8), *then*:

- (i) $\tilde{L}_1 + \tilde{L}_2 + \tilde{L}_3 = 0$.
- (ii) $|\tilde{L}_i(q) - L_i(q)| < 2c$ *for* $i = 1, 2, 3$.
- (iii) $\tilde{L}_2(q) > \tilde{L}_1(q) - 4c$ *and* $\tilde{L}_3(q) > \tilde{L}_2(q) - 4c$.
- (iv) $\tilde{L}_1, \tilde{L}_3$ *have slopes alternating between* $-1, 1/2$, *whereas* $\tilde{L}_2$ *has slopes among* $-1, 1/2, 2$.
- (v) *An interval where* $\tilde{L}_2$ *has slope* 2 *has length less than* $3c$.
- (vi) *In an interval where* $\tilde{L}_3(q) - \tilde{L}_2(q) > 4c$ *the function* $\tilde{L}_3$ *has no local minimum, and in an interval where* $\tilde{L}_2(q) - \tilde{L}_1(q) > 4c$ *the function* $\tilde{L}_1$ *has no local maximum.*
- (vii) *If* $\{i, j, k\} = \{1, 2, 3\}$ *and* $\tilde{L}_i$ *has slope* $1/2$ *in some interval, then* $\frac{1}{2}(\tilde{L}_j + \tilde{L}_k)$ *will have slope* $-1/4$ *in this interval.* $\tilde{L}_j, \tilde{L}_k$ *will alternate having respective slopes* $1/2, -1$ *and* $-1, 1/2$, *hence their graphs will zigzag around a line with slope* $-1/4$.
- (viii) *Suppose* $i \ne j$ *in* $\{1, 2, 3\}$. *If* $\tilde{L}_i$ *has slope* -1 *in some interval, then*

$$\left| \tilde{L}_j(q) - \tilde{L}_j(q') - \frac{1}{2}(q-q') \right| < 4c$$

for q, q' *in that interval.*

Proof. (i) is obvious from the definition of $\tilde{L}_i$ $(i = 1, 2, 3)$.

For (ii) observe that $\tilde{L}_1 - L_1 = 0$, $\tilde{L}_3 - L_3 = -(L_1^* + L_3)$ and $\tilde{L}_2 - L_2 = L_1^* - L_1 - L_2 = (L_1^* + L_3) - (L_1 + L_2 + L_3)$. Now apply (4.7) and (4.6).

Next, (iii) follows from (ii) and the fact that

$$\tilde{L}_2 - \tilde{L}_1 = \tilde{L}_2 - L_1 = (L_2 - L_1) + (\tilde{L}_2 - L_2) \geq \tilde{L}_2 - L_2,$$
$$\tilde{L}_3 - \tilde{L}_2 = (L_3 - L_2) + (\tilde{L}_3 - L_3) + (L_2 - \tilde{L}_2) \geq (\tilde{L}_3 - L_3) + (L_2 - \tilde{L}_2).$$

Concerning (iv) note that $\tilde{L}_1 = L_1$ has slopes $-1, 1/2$, L_1^* has slopes $1, -1/2$, so that $\tilde{L}_3$ has slopes $-1, 1/2$. The assertion now follows from (i) above.

(v) When $\tilde{L}_2$ has slope 2 in $[q', q]$, then

$$q - q' = \frac{1}{2}\,(\tilde{L}_2(q) - \tilde{L}_2(q')) < \frac{1}{2}\,(L_2(q) - L_2(q')) + 2c \leq \frac{1}{4}\,(q - q') + 2c,$$

hence $q - q' < \frac{8}{3}c < 3c$.

(vi) Suppose $\tilde{L}_3 - \tilde{L}_2 > 4c$ in some interval. Observe that

$$\tilde{L}_3 - \tilde{L}_2 - (L_2^* - L_1^*) = -\tilde{L}_2 - L_2^* = -(L_2 + L_2^*) + (L_2 - \tilde{L}_2).$$

Combining this with (4.6) and (ii) implies $L_2^* - L_1^* \neq 0$, i.e. $L_2^* \neq L_1^*$. Therefore L_1^* has no local maximum, and $\tilde{L}_3 = -L_1^*$ no local minimum in the interval. When $\tilde{L}_2 - \tilde{L}_1 > 4c$, then $L_2(q) - L_1(q) > \tilde{L}_2 - \tilde{L}_1 - 2c > 0$, hence $L_2(q) \neq L_1(q)$, so $\tilde{L}_1 = L_1$ has no local maximum.

(vii) is fairly obvious.

(viii) is essentially the case $n = 3$ of Lemma 4.1. ∎

6. Top and bottom intervals. It may happen that, for all large q,

$$\tilde{L}_3(q) - \tilde{L}_1(q) \leq C,$$

where C is a constant to be specified below. Then $\underline{\psi}_i = \overline{\psi}_i = 0$ for $i = 1, 2, 3$ and (1.11)–(1.13) trivially hold.

The other extreme is when

$$\tilde{L}_3(q) - \tilde{L}_1(q) > C \tag{6.1}$$

for all large q. We will treat this case in detail. It is also possible that all large q lie in a sequence of intervals which alternate between q's in the two extremal cases. In this case $\overline{\psi}_1 = \underline{\psi}_3 = 0$, and this also implies (1.11), (1.12) and (1.13). In large intervals where $\tilde{L}_3(q) - \tilde{L}_1(q) > C$, the situation is essentially the same as if the equality holds for all large q, so we will not elaborate on this case.

We will now suppose (6.1) holds; we set $\gamma = 4c$, where c is the absolute constant of the last section. There are arbitrarily large values of p with $L_3(p) = L_2(p)$ and hence $\tilde{L}_3(p) - \tilde{L}_2(p) < \gamma$, so (6.1) implies $\tilde{L}_2(p) - \tilde{L}_1(p) > C - \gamma > \gamma$, provided $C > 2\gamma$. Also, there are arbitrarily large numbers p^*

with $\tilde{L}_2(p^*)-\tilde{L}_1(p^*)<\gamma$, hence $\tilde{L}_3(p^*)-\tilde{L}_2(p^*)>\gamma$. So by the Intermediate Value Theorem there are arbitrarily large values of p with

$$\tilde{L}_3(p)-\tilde{L}_2(p)=\gamma,$$

as well as arbitrarily large numbers p^* with

$$\tilde{L}_2(p^*)-\tilde{L}_1(p^*)=\gamma.$$

If p, p^* are such numbers, then $\tilde{L}_3(p^*)-\tilde{L}_2(p^*)>C-\gamma$ and

$$\tilde{L}_3(p^*)-\tilde{L}_2(p^*)-(\tilde{L}_3(p)-\tilde{L}_2(p))\le 3|p-p^*|,$$

so that $\tilde{L}_3(p)-\tilde{L}_2(p)>C-\gamma-3|p-p^*|$, which together with $\tilde{L}_3(p)-\tilde{L}_2(p)=\gamma$ yields $3|p-p^*|>C-2\gamma$, hence

$$|p-p^*|>C/3-2\gamma/3>\gamma \tag{6.2}$$

if $C>5\gamma$.

For every p with $\tilde{L}_3(p)-\tilde{L}_2(p)=\gamma$, hence $\tilde{L}_2(p)-\tilde{L}_1(p)>\gamma$, there is a smallest a and a largest b with $a\le p\le b$ such that

$$\tilde{L}_3(a)-\tilde{L}_2(a)=\tilde{L}_3(b)-\tilde{L}_2(b)=\gamma$$

and $\tilde{L}_2(q)-\tilde{L}_1(q)>\gamma$ for $a\le q\le b$. Such an interval $[a,b]$ will be called a *top interval*. It is not required that $\tilde{L}_3(q)-\tilde{L}_2(q)\le\gamma$ in this interval. Also, it may happen that $a=p=b$, so that the interval consists of a single number.

For every p^* with $\tilde{L}_2(p^*)-\tilde{L}_1(p^*)=\gamma$ there is a smallest a^* and a largest b^* with $a^*\le p^*\le b^*$ such that

$$\tilde{L}_2(a^*)-\tilde{L}_1(a^*)=\tilde{L}_2(b^*)-\tilde{L}_1(b^*)=\gamma$$

and $\tilde{L}_3(q)-\tilde{L}_2(q)>\gamma$ for $a^*\le q\le b^*$. Such an interval $[a^*,b^*]$ will be called a *bottom interval*. By (6.2), a top interval has distance greater than γ from a bottom interval.

For intervals $I=[r,s]$ and $I'=[r',s']$ we write $I<I'$ if $s<r'$. We may thus arrange all the top and bottom intervals into a sequence

$$I_1<I_2<I_3<\cdots.$$

There cannot be two adjacent top intervals in this sequence: for if $[a,b]<[a',b']$ were two such intervals, then $\tilde{L}_2(q)-\tilde{L}_1(q)>\gamma$ for $a\le q\le b'$, and b would no longer be the largest number as required in the definition of top intervals. Similarly, there cannot be two adjacent bottom intervals. Hence after a change of notation, our sequence becomes

$$I_1<I_1^*<I_2<I_2^*<\cdots,$$

where, say, each I_j is a top interval and each I_j^* a bottom interval. If $I_j=[a_j,b_j]$ and $I_j^*=[a_j^*,b_j^*]$ then

$$\cdots<a_{j-1}^*\le b_{j-1}^*<a_j\le b_j<a_j^*\le b_j^*<a_{j+1}\le b_{j+1}<\cdots.$$

In $[b^*_{j-1}, a^*_j]$ we have $\tilde{L}_2(q) - \tilde{L}_1(q) > \gamma$ and hence, by Lemma 5.1(vi), the function $\tilde{L}_1$ has no local maximum. There will be some p^*_j in $[b^*_{j-1}, a^*_j]$ such that $\tilde{L}_1$ is decreasing for $b^*_{j-1} \le q \le p^*_j$ and increasing for $p^*_j \le q \le a^*_j$. The cases $p^*_j = b^*_{j-1}$ and $p^*_j = a^*_j$ are not ruled out. Also, there will be a p_j in $[b_j, a_{j+1}]$ such that $\tilde{L}_3$ is increasing for $b_j \le q \le p_j$ and decreasing for $p_j \le q \le a_{j+1}$.

Let now j be fixed and assume j is large, so that $q \in I_j$ is large. We claim that we cannot have both p^*_j, p_j larger than b_j. For if this were so, set $p = \min\{p^*_j, p_j\}$ and observe that $\tilde{L}_3$ is increasing in $[b_j, p] \subset [b_j, p_j]$ with slope $1/2$ and $\tilde{L}_1$ is decreasing in $[b_j, p] \subset [b^*_{j-1}, p_j]$ with slope -1, so that $\tilde{L}_2$ is increasing with slope $1/2$. Hence

$$\tilde{L}_3(q) - \tilde{L}_2(q) = \tilde{L}_3(b_j) - \tilde{L}_2(b_j) = \gamma$$

in $[b_j, p]$, and there is no q in this interval with $\tilde{L}_2(q) - \tilde{L}_1(q) \le \gamma$. This contradicts the maximality property of the right endpoint b_j of $I_j = [a_j, b_j]$. A similar reasoning shows that we cannot have both p^*_j, p_j smaller than a^*_j. This yields the following possibilities. If $p^*_j \le b_j$, then $p_j \ge a^*_j$, and conversely, $p_j \ge a^*_j$ implies $p^*_j \le b_j$. In short,

$$p^*_j \le b_j \quad \text{and} \quad p_j \ge a^*_j. \tag{6.3}$$

The second conceivable possibility is when $p^*_j > b_j$ and $p_j < a^*_j$, but then, by the above remarks and the fact that $p^*_j \le a^*_j$, $p_j \ge b_j$, we have

$$p_j = b_j \quad \text{and} \quad p^*_j = a^*_j. \tag{6.4}$$

In fact, this cannot occur if C is chosen sufficiently large. For if $p_j = b_j$ and $p^*_j = a^*_j$, both $\tilde{L}_1, \tilde{L}_3$ have slope -1 in $[b_j, a^*_j]$ so that $a^*_j - b_j < 3c$ by Lemma 5.1(v). Further, $\tilde{L}_1, \tilde{L}_2$ are close to each other at $q = a^*_j$ and $\tilde{L}_2, \tilde{L}_3$ are close at $q = b_j$, so that $\tilde{L}_3 - \tilde{L}_1$ is small in $[b_j, a^*_j]$, i.e. $\tilde{L}_3 - \tilde{L}_1 \ll c$, which for large C contradicts (6.1).

Let us now assume $p^*_j \le b_j$ and $p_j \ge a^*_j$, so that we have $b^*_j \le p^*_j \le b_j < a^*_j$. Then $\tilde{L}_1$ and $\tilde{L}_3$ are both increasing in $[b_j, a^*_j]$ with slope $1/2$, hence $\tilde{L}_2$ is decreasing with slope -1. In fact, we may suppose that $p^*_j < a_j$, hence

$$b^*_{j-1} \le p^*_j < a_j \le b_j < a^*_j.$$

For otherwise $p^*_j \ge a_j$ and $\tilde{L}_1$ has slope -1 in $[b^*_{j-1}, p^*_j]$, hence in $[b^*_{j-1}, a_j]$. By Lemma 5.1(viii),

$$|\tilde{L}_3(b^*_{j-1}) - L_3(q) - (\tilde{L}_2(b^*_{j-1}) - \tilde{L}_2(q))| < 8c$$

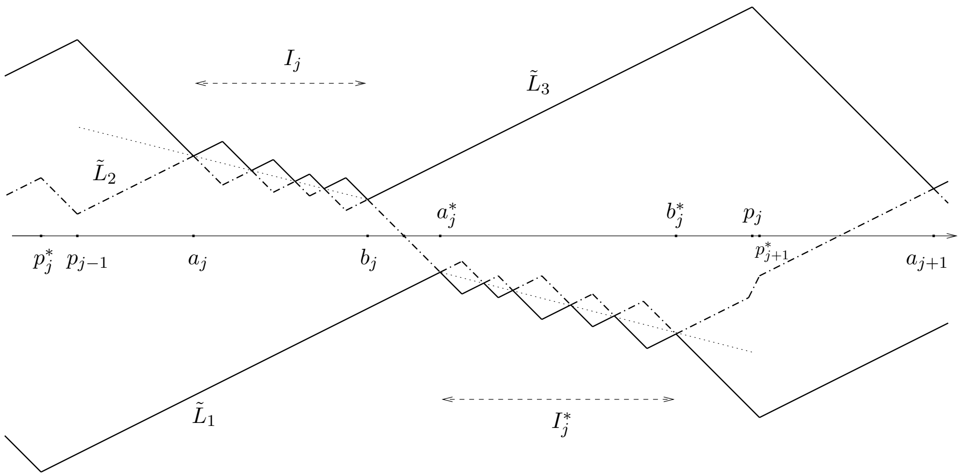
for q in this interval, and since $\tilde{L}_3(a_j) - \tilde{L}_2(a_j) = \gamma$, we deduce that $|\tilde{L}_3(b^*_{j-1}) - \tilde{L}_2(b^*_{j-1})| < 8c + \gamma = 12c$. Moreover, $\tilde{L}_2(b^*_{j-1}) - \tilde{L}_1(b^*_{j-1}) = \gamma$, so that

$$\tilde{L}_3(b^*_{j-1}) - \tilde{L}_1(b^*_{j-1}) < 12c + \gamma = 16c.$$

But this contradicts (6.1) provided C is chosen large enough, i.e. $C > 16c$. Similarly, by a dual argument, we will also have

$$b_j < a^*_j \le b^*_j < p_j \le a_{j+1}.$$

The following figure illustrates a possible behavior of $\tilde{L}_1, \tilde{L}_2, \tilde{L}_3$ for $p^*_j \le q \le a_{j+1}$.



In $[a_j, b_j]$ the function $\tilde{L}_1$ has slope $1/2$, so that by Lemma 5.1(vii), $\tilde{L}_2, \tilde{L}_3$ will zigzag around a line with slope $-1/4$ indicated by dots in the figure. The graphs of the functions are different near p^*_j, p_{j-1} depending on whether $p^*_j < p_{j-1}$ (as shown in the figure) or $p^*_j \ge p_{j-1}$ (which in the figure holds for $j+1$ in place of j, i.e. $p^*_{j+1} \ge p_j$). Note that $p^*_j \ge p_{j-1}$ yields $p^*_j - p_{j-1} < 3c$ by Lemma 5.1(v).

Remarks. (a) In the notation of Section 4, suppose the local minima of $L_1 = \tilde{L}_1$ in $[p^*_j, p^*_{j+1}]$ are assumed at $p^*_j = q_k < q_{k+1} < \cdots < q_l = p^*_{j+1}$. Then the points $\mathbf{x}_k, \mathbf{x}_{k+1}, \ldots, \mathbf{x}_l$ span a 2-dimensional space. For otherwise $l \ge k+2$, and there is an r, $k < r < l$, with $\mathbf{x}_k, \mathbf{x}_r, \mathbf{x}_l$ linearly independent. If $\bar{q}$ is the q-coordinate of the point of intersection of the graphs of $L_{\mathbf{x}_k}, L_{\mathbf{x}_l}$, then it is clear from our figure that both $q_r, \bar{q}$ lie in the interior of $I^*_j = [a^*_j, b^*_j]$; in particular, $a^*_j < b^*_j$. Moreover, $L_{\mathbf{x}_k}(q_k) = L_1(q_k) = \tilde{L}_1(q_k)$, and both $L_{\mathbf{x}_k}$ and $\tilde{L}_1$ have slope $1/2$ in $q_k \le q \le a^*_j$, so that $L_{\mathbf{x}_k}(a^*_j) = \tilde{L}_1(a^*_j) < \tilde{L}_3(a^*_j) - \gamma$. Both $L_{\mathbf{x}_k}$ and $\tilde{L}_3$ have slope $1/2$ in $a^*_j \le q \le \bar{q}$, so that $L_{\mathbf{x}_k}(\bar{q}) < \tilde{L}_3(\bar{q}) - \gamma < L_3(\bar{q})$. But by the independence of $\mathbf{x}_k, \mathbf{x}_r, \mathbf{x}_l$, and since $L_{\mathbf{x}_k}(\bar{q}) = L_{\mathbf{x}_l}(\bar{q}) \ge L_{\mathbf{x}_r}(\bar{q})$, we have $L_3(\bar{q}) \le L_{\mathbf{x}_k}(\bar{q})$, a contradiction.

(b) When $a_j = b_j$ or $a_j^* = b_j^*$, which we believe to be a relatively common event, the figure is much simplified. (In fact, keeping the notations from (a) we have $a_j^* = b_j^*$ if $\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{x}_{k+2}$ are linearly independent.)

(c) We believe that our description of the functions $\tilde{L}_1(q), \tilde{L}_2(q), \tilde{L}_3(q)$, hence implicitly of $\lambda_1(Q), \lambda_2(Q), \lambda_3(Q)$, is close to "best possible". That is, given functions as described in the figure, there are numbers ξ_1, ξ_2 for which the resulting $\tilde{L}_1, \tilde{L}_2, \tilde{L}_3$ are close to the given functions.

7. Proof of Theorem 1.5. We introduce the functions $\tilde{\chi}_i(q) := \tilde{L}_i(q)/q$ for $i = 1, 2, 3$. Then

$$\tilde{\chi}_i(q) - \psi_i(e^q) = \frac{\tilde{L}_i(q)}{q} - \frac{L_i(q)}{q} \ll \frac{1}{q}$$

so that we have

$$\overline{\psi}_i = \limsup \tilde{\chi}_i(q) \quad \text{and} \quad \underline{\psi}_i = \liminf \tilde{\chi}_i(q) \tag{7.1}$$

by (4.10). Moreover, by the case $n = 3$ of (4.9) we have the estimates

$$\tilde{\chi}_i(q) \geq -1 + \frac{g_1(e^q)}{q} \quad \text{and} \quad \tilde{\chi}_i(q) \leq \frac{1}{2} - \frac{g_3(e^q)}{q}, \tag{7.2}$$

hence for large q,

$$-1 < \tilde{\chi}_i(q) < 1/2 \quad \text{for } i = 1, 2, 3. \tag{7.3}$$

Lemma 7.1.

$$\tilde{L}_1(a_j^*) + \tilde{L}_3(b_j) = O(1), \qquad \tilde{\chi}_1(a_j^*) + \tilde{\chi}_3(b_j) = -2\tilde{\chi}_1(a_j^*)\tilde{\chi}_3(b_j) + O(1/b_j),$$

where the implied constants depend only on c, hence are absolute.

Proof. We will drop the subscript j. We have $\tilde{L}_2(a^*) - \tilde{L}_1(a^*) = O(1)$, hence $2\tilde{L}_1(a^*) + \tilde{L}_3(a^*) = O(1)$. Similarly, $2\tilde{L}_3(b) + \tilde{L}_1(b) = O(1)$. Both $\tilde{L}_1, \tilde{L}_3$ have slope $1/2$ in $[b, a^*]$, so that

$$\tilde{L}_3(a^*) - \tilde{L}_3(b) = \frac{1}{2}(a^* - b) = \tilde{L}_1(a^*) - \tilde{L}_1(b),$$

which yields

$$\tilde{L}_1(b) + \tilde{L}_3(a^*) = \tilde{L}_3(b) + \tilde{L}_1(a^*).$$

Therefore

$$\tilde{L}_1(a^*) + \tilde{L}_3(b) = \frac{1}{3}(2\tilde{L}_1(a^*) + \tilde{L}_3(a^*) + 2\tilde{L}_3(b) + \tilde{L}_1(b)) = O(1).$$

Also,

$$\tilde{L}_3(a^*) = -\tilde{L}_1(a^*) - \tilde{L}_2(a^*) = -2\tilde{L}_1(a^*) + O(1) = 2\tilde{L}_3(b) + O(1),$$

so that

$$a^* - b = 2(\tilde{L}_3(a^*) - \tilde{L}_3(b)) = 2\tilde{L}_3(b) + O(1).$$

Therefore

$$
\begin{aligned}
\tilde\chi_1(a^*) + \tilde\chi_3(b) &= \frac{1}{a^*b}\,(b\tilde L_1(a^*) + a^*\tilde L_3(b))\\
&= \frac{1}{a^*b}\,(b\tilde L_1(a^*) - a^*\tilde L_1(a^*)) + O(1/b)\\
&= \frac{1}{a^*b}\,((b-a^*)\tilde L_1(a^*)) + O(1/b)\\
&= -\frac{2}{a^*b}\,\tilde L_1(a^*)\tilde L_3(b) + O(1/b)\\
&= -2\tilde\chi_1(a^*)\tilde\chi_3(b) + O(1/b). \ \blacksquare
\end{aligned}
$$

LEMMA 7.2.

(i) *If some* $\tilde L_i$ *is increasing* (*resp. decreasing*) *in an interval with large endpoints*, *then also* $\tilde\chi_i$ *is increasing* (*resp. decreasing*) *in that interval.*
(ii) *For* $q \in I_j^* = [a_j^*, b_j^*]$ *with large* j *we have* $\tilde\chi_1(q) \le \tilde\chi_1(a_j^*) + O(1/a_j^*)$.
(iii) *For* $q \in I_j = [a_j, b_j]$ *with large* j *we have* $\tilde\chi_3(q) \ge \tilde\chi_3(b_j) - O(1/b_j)$.

Proof. (i) Suppose $q > q'$ are in the interval. If $\tilde L_i$ is increasing, then $\tilde L_i(q) - \tilde L_i(q') \ge \frac{1}{2}(q-q')$, yielding

$$\tilde\chi_i(q) \ge \tilde\chi_i(q')q'/q + \frac{1}{2}\,(1-q'/q) = \tilde\chi_i(q') + \left(\frac{1}{2} - \tilde\chi_i(q')\right)(1-q'/q) > \tilde\chi_i(q')$$

by (7.3). If $\tilde L_i$ is decreasing, then $\tilde L_i(q) - \tilde L_i(q') = -(q-q')$, so that

$$\tilde\chi_i(q) = \tilde\chi_i(q')q'/q - (1-q'/q) = \tilde\chi_i(q') - (1+\tilde\chi_i(q'))(1-q'/q) < \tilde\chi_i(q')$$

by (7.3).

(ii) We have

$$\tilde L_1(a_j^*) \ge \frac{1}{2}\,(\tilde L_1(a_j^*) + \tilde L_2(a_j^*)) - O(1) = -\frac{1}{2}\,\tilde L_3(a_j^*) - O(1),$$

so that for large j,

$$\tilde\chi_1(a_j^*) \ge -\frac{1}{2}\,\tilde\chi_3(a_j^*) - O(1/a_j^*)$$

and hence (7.2) yields

$$\tilde\chi_1(a_j^*) > -\frac{1}{4} + \frac{1}{2}\,g_3(e^{a_j^*})/a_j^* - O(1/a_j^*) > -\frac{1}{4}. \tag{7.4}$$

Moreover

$$
\begin{aligned}
\tilde L_1(q) &\le \frac{1}{2}\,(\tilde L_1(q) + \tilde L_2(q)) + O(1)\\
&= \frac{1}{2}\,(\tilde L_1(a_j^*) + \tilde L_2(a_j^*)) - \frac{1}{4}\,(q-a_j^*) + O(1)\\
&= \tilde L_1(a_j^*) - \frac{1}{4}\,(q-a_j^*) + O(1),
\end{aligned}
$$

hence by (7.4),

$$\begin{aligned}\tilde\chi_1(q) &\le \tilde\chi_1(a_j^*)a_j^*/q - \frac{1}{4}\,(1-a_j^*/q) + O(1/q)\\ &= \tilde\chi_1(a_j^*) - \left(\frac{1}{4}+\tilde\chi_1(a_j^*)\right)(1-a_j^*/q) + O(1/q) \le \tilde\chi_1(a_j^*) + O(1/a_j^*).\end{aligned}$$

(iii) is proved in a dual way. ∎

Set $\mathcal{I}_j := [p_{j-1}, p_j]$, $\mathcal{I}_j^* := [p_j^*, p_{j+1}^*]$ and put

$$\rho_j := \max_{q\in\mathcal{I}_j^*} \tilde\chi_1(q), \qquad \sigma_j := \min_{q\in\mathcal{I}_j} \tilde\chi_3(q).$$

Lemma 7.3.

$$\text{(i)}\ \tilde\chi_1(a_j^*) \le \rho_j < \tilde\chi_1(a_j^*) + O(1/a_j^*),$$
$$\text{(ii)}\ \tilde\chi_3(b_j) \ge \sigma_j > \tilde\chi_3(b_j) - O(1/b_j).$$

Proof. Since $\tilde L_1$ is increasing in $[p_{j-1}, a_j^*]$ and decreasing in $[b_j^*, p_j]$, we see from Lemma 7.2(i) that $\rho_j = \tilde\chi_1(q)$ with $q \in [a_j^*, b_j^*] = I_j^*$. But by Lemma 7.2(ii), in fact $\rho_j < \tilde\chi_1(a_j^*) + O(1/a_j^*)$. Now (i) follows, and (ii) is proved in a dual fashion. ∎

Proof of Theorem 1.5. By combining Lemmas 7.1 and 7.3 we obtain

$$\rho_j + \sigma_j + 2\rho_j\sigma_j = O(1/\min\{b_j, a_j^*\}) = O(1/b_j).$$

For large j the inequalities $-1/3 < \rho_j \le 0$ hold by (7.4) and Lemma 7.3(i), so that $1 + 2\rho_j > 1/3$ and hence

$$\sigma_j = -\frac{\rho_j}{1+2\rho_j} + O(1/b_j).$$

All large reals lie in the union of the intervals $\mathcal{I}_2, \mathcal{I}_3, \ldots$ as well as of the intervals $\mathcal{I}_1^*, \mathcal{I}_2^*, \ldots,$ and as the function $\rho \mapsto \rho/(1+2\rho)$ for $\rho > -1/2$ is increasing, we conclude

$$\underline{\psi}_3 = \liminf \sigma_j = \liminf\left(-\frac{\rho_j}{1+2\rho_j}\right) = -\frac{\limsup \rho_j}{1+2\limsup \rho_j} = -\frac{\overline{\psi}_1}{1+2\overline{\psi}_1},$$

and this is precisely (1.11). ∎

8. Proof of Theorem 1.6. Let $\mathcal{K}_j$ be the interval $[b_{j-1}, b_j]$ and $\mathcal{K}_j^*$ be the interval $[a_j^*, a_{j+1}^*]$. Further, let m_j, m_j^* be numbers in $\mathcal{K}_j$, $\mathcal{K}_j^*$ respectively with

$$\psi_3(m_j) = \max_{q\in\mathcal{K}_j} \psi_3(q) \quad\text{and}\quad \psi_1(m_j^*) = \min_{q\in\mathcal{K}_j^*} \psi_1(q).$$

Lemma 8.1.

$$2\tilde\chi_1(m_j) + \tilde\chi_3(m_j) \le -\tilde\chi_3(b_j)(3 + 2\tilde\chi_1(m_j) + 4\tilde\chi_3(m_j)) + O(1/m_j), \tag{8.1}$$
$$2\tilde\chi_3(m_j^*) + \tilde\chi_1(m_j^*) \ge -\tilde\chi_1(a_j^*)(3 + 2\tilde\chi_3(m_j^*) + 4\tilde\chi_1(m_j^*)) + O(1/m_j^*). \tag{8.2}$$

Proof. Since $\tilde{\chi}_3$ increases in $[b_{j-1}, p_{j-1}]$ by Lemma 7.2, we have $m_j \in [p_{j-1}, b_j]$. Now L_1 increases with slope $1/2$ in $[p_j^*, p_{j-1}]$. Either $p_j^* < p_{j-1}$, or $p_j^* \geq p_{j-1}$ and $p_j^* - p_{j-1} = O(1)$. In either case, since $m_j \in [p_{j-1}, b_j]$, we have

$$\tilde{L}_1(m_j) = \tilde{L}_1(b_j) - \frac{1}{2}(b_j - m_j) + O(1).$$

Since $m_j \leq b_j$ and $\tilde{L}_3(b_j)$ has slopes among $-1, 1/2$,

$$\tilde{L}_3(m_j) \leq \tilde{L}_3(b_j) + (b_j - m_j).$$

Therefore

$$\begin{aligned} 2\tilde{L}_1(m_j) + \tilde{L}_3(m_j) &\leq 2\tilde{L}_1(b_j) + \tilde{L}_3(b_j) + O(1) = -2\tilde{L}_2(b_j) - \tilde{L}_3(b_j) + O(1) \\ &= -3\tilde{L}_3(b_j) + O(1). \end{aligned}$$

Dividing by m_j we obtain

$$2\tilde{\chi}_1(m_j) + \tilde{\chi}_3(m_j) \leq -3\tilde{\chi}_3(b_j)b_j/m_j + O(1/m_j). \tag{8.3}$$

On the other hand,

$$\begin{aligned} 2\tilde{L}_1(m_j) + 4\tilde{L}_3(m_j) &\leq 2\tilde{L}_1(b_j) + 4\tilde{L}_3(b_j) + 3(b_j - m_j) + O(1) \\ &= -2\tilde{L}_2(b_j) + 2\tilde{L}_3(b_j) + 3(b_j - m_j) + O(1) \\ &= 3(b_j - m_j) + O(1). \end{aligned}$$

After division by m_j and addition of 3 we get

$$3 + 2\tilde{\chi}_1(m_j) + 4\tilde{\chi}_3(m_j) \leq 3b_j/m_j + O(1/m_j). \tag{8.4}$$

This, in conjunction with (8.3), gives (8.1).

(8.2) is proved in a dual fashion: there is some symmetry of the graphs of $\tilde{L}_1, \tilde{L}_2, \tilde{L}_3$ about the point $\left(\frac{1}{2}(b_j + a_j^*), 0\right)$. This is not a strict geometric symmetry, but our arguments remain valid if $p_{j-1}, a_j, b_j, a_j^*, b_j^*, p_{j+1}^*$ are respectively replaced by $p_{j+1}^*, b_j^*, a_j^*, b_j, a_j, p_{j-1}$, if $\tilde{L}_1, \tilde{L}_2, \tilde{L}_3$ are respectively replaced by $\tilde{L}_3, \tilde{L}_2, \tilde{L}_1$, and inequalities are reversed. Thus for instance $\tilde{\chi}_3(b_j) = \tilde{L}_3(b_j)/b_j$ is replaced by $\tilde{\chi}_1(a_j^*) = \tilde{L}_1(a_j^*)/a_j^*$.

We now have $m_j^* \in [a_j^*, p_{j+1}^*]$, and (8.3), (8.4) are respectively replaced by

$$\begin{aligned} 2\tilde{\chi}_3(m_j^*) + \tilde{\chi}_1(m_j^*) &\geq -3\tilde{\chi}_1(a_j^*)a_j^*/m_j^* + O(1/m_j^*), \\ 3 + 2\tilde{\chi}_3(m_j^*) + 4\tilde{\chi}_1(m_j^*) &\geq 3a_j^*/m_j^* + O(1/m_j^*). \end{aligned}$$

Thus (8.2) follows. ∎

Proof of Theorem 1.6. For given $\epsilon > 0$ we have $\tilde{\chi}_3(b_j) > \underline{\psi}_3 - \epsilon$, and (8.1) yields

$$2\tilde{\chi}_1(m_j) + \tilde{\chi}_3(m_j) \leq -(\underline{\psi}_3 - \epsilon)(3 + 2\tilde{\chi}_1(m_j) + 4\tilde{\chi}_3(m_j)) + O(1/m_j),$$

hence

$$\tilde{\chi}_3(m_j)(1 + 4\underline{\psi}_3) < -3\underline{\psi}_3 - (2 + 2\underline{\psi}_3)\tilde{\chi}_1(m_j) + O(\epsilon)$$

when j is large. Moreover, $\tilde{\chi}_1(m_j) > \underline{\psi}_1 - \epsilon$, so that

$$\tilde{\chi}_3(m_j)(1+4\underline{\psi}_3) < -3\underline{\psi}_3 - (2+2\underline{\psi}_3)\underline{\psi}_1 + O(\epsilon).$$

Since by definition of m_j,

$$\overline{\psi}_3 = \limsup_{j\to\infty} \tilde{\chi}_3(m_j),$$

we obtain

$$\overline{\psi}_3(1+4\underline{\psi}_3) \le -3\underline{\psi}_3 - (2+2\underline{\psi}_3)\underline{\psi}_1,$$

hence (1.12). The relation (1.13) is established in a dual way. ∎

9. Some consequences

Corollary 9.1. *Again in the context of Dirichlet's Theorem on simultaneous approximation with $n = 3$,*

$$\overline{\psi}_3 \ge \underline{\psi}_3(3+2\underline{\psi}_1) \quad \textit{and} \quad \underline{\psi}_1 \le \overline{\psi}_1(3+2\overline{\psi}_3). \tag{9.1}$$

Proof. (1.12) and (1.15) can be rewritten as

$$3\underline{\psi}_3 + 2\underline{\psi}_1 + \overline{\psi}_3 \le \underline{\psi}_3(-2\underline{\psi}_1 - 4\overline{\psi}_3), \qquad 3\underline{\psi}_3 - \underline{\psi}_1 - 2\overline{\psi}_3 \le \underline{\psi}_3(-2\underline{\psi}_1 + 2\overline{\psi}_3).$$

Adding the first of these inequalities to twice the second, we obtain

$$9\underline{\psi}_3 - 3\overline{\psi}_3 \le -6\underline{\psi}_1\underline{\psi}_3,$$

which gives the first relation of (9.1). We noted that (1.12), ..., (1.15) remain valid if $\underline{\psi}_1, \overline{\psi}_3$ as well as $\overline{\psi}_1, \underline{\psi}_3$ are interchanged and inequalities are reversed. This also holds for the consequences of (1.12), ..., (1.15), and hence the first relation of (9.1) implies the second. ∎

Corollary 9.2. *If $\underline{\psi}_1 = \overline{\psi}_1$ or $\underline{\psi}_2 = \overline{\psi}_2$, then*

$$\underline{\psi}_i = \overline{\psi}_i \quad \textit{for } i = 1,2,3. \tag{9.2}$$

When $\underline{\psi}_3 = \overline{\psi}_3$, then either (9.2) holds or

$$\underline{\psi}_1 = -1, \quad \overline{\psi}_1 = \underline{\psi}_2 = -1/4, \quad \overline{\psi}_2 = \underline{\psi}_3 = \overline{\psi}_3 = 1/2.$$

Proof. Assume $\underline{\psi}_1 = \overline{\psi}_1$. Then we have $\underline{\psi}_1 = \overline{\psi}_1 \le \overline{\psi}_1(3+2\overline{\psi}_3)$ and hence either $\overline{\psi}_1 = \underline{\psi}_1 = 0$ or $1 = 3 + 2\overline{\psi}_3 \ge 3$, which is impossible. Now by (1.5) we have $\underline{\psi}_1 \le \underline{\psi}_2 \le \overline{\psi}_1$, hence $\underline{\psi}_2 = 0$, and $\underline{\psi}_1 + \underline{\psi}_2 + \overline{\psi}_3 \le 0$ yields $\overline{\psi}_2 = \underline{\psi}_3 = \overline{\psi}_3 = 0$ so that (9.2) follows.

If $\underline{\psi}_2 = \overline{\psi}_2$, we have $3\underline{\psi}_2 = 3\overline{\psi}_2 \le 2\overline{\psi}_2 + \underline{\psi}_1$, which implies $\overline{\psi}_2 \le \underline{\psi}_1$ and this is only possible if $\overline{\psi}_2 = \underline{\psi}_1 = 0$. As before, this yields $\overline{\psi}_3 = 0$ as well, and again (9.2) follows.

Now let $\underline{\psi}_3 = \overline{\psi}_3$. Then we have $\overline{\psi}_3 = \underline{\psi}_3 \ge \underline{\psi}_3(3+2\underline{\psi}_1)$ and hence either $\overline{\psi}_3 = \underline{\psi}_3 = 0$ or $1 = 3 + 2\underline{\psi}_1 \ge 3$, which is only possible if $\underline{\psi}_1 = -1$. In the first case $\underline{\psi}_3 \le \overline{\psi}_2 \le \overline{\psi}_3$ (we used (1.5) again) implies $\overline{\psi}_2 = 0$, and

$\underline{\psi}_1 + \overline{\psi}_2 + \overline{\psi}_3 \geq 0$ yields $\underline{\psi}_2 = \overline{\psi}_1 = \underline{\psi}_1 = 0$ and hence (9.2). In the second case $\underline{\psi}_1 = -1$ and in view of $\underline{\psi}_1 + \overline{\psi}_2 + \overline{\psi}_3 \geq 0$ we must have $\overline{\psi}_2 = \underline{\psi}_3 = \overline{\psi}_3 = 1/2$. Then $\overline{\psi}_1 + \underline{\psi}_3 + 2\overline{\psi}_1\underline{\psi}_3 = 0$ yields $\overline{\psi}_1 = -1/4$, and $\underline{\psi}_2 \leq \overline{\psi}_1$ gives $\underline{\psi}_2 = -1/4$, which concludes the proof. ∎

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