Correction to
“Computing Galois groups by means of Newton polygons”
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by

MICHAEL KÖLLE and PETER SCHMID (Tübingen)

The last statement of the (main) theorem in the above joint paper, cited as [KS], is not stated correctly. The error comes from Proposition 4 in [KS], because the residue class field of \( \hat{T} \) merely contains the splitting field of the associated polynomial \( \hat{f}_S \) (page 80). We use the notations and conventions introduced in [KS]. In particular, we assume that the side \( S = S_m \) with slope \( m = h/e \) of the Newton polygon of the normalized polynomial \( f \in K[X] \) with respect to the finite prime \( p \) of the number field \( K \) is regular (\( \hat{f}_S \) separable over the residue class field \( k_p \)) and tame (\( p \nmid e \)). Let then \( \omega = o(Np \mod e) \) be the order of the absolute norm \( Np = |k_p| \) of \( p \) in \( (\mathbb{Z}/e\mathbb{Z})^* \), and let the distinct normalized prime factors of \( \hat{f}_S \) over \( k_p \) have degrees \( d_1, \ldots, d_r \) (so that \( \sum_{i=1}^r d_i = d = \deg(\hat{f}_S) \)).

Recall that by part (iii) of the theorem in [KS] the inertia group \( I_{\mathfrak{P}}^{f,m} \) equals \( \langle \tau \rangle \) where \( \tau \) is the product of \( d \) disjoint \( e \)-cycles on the roots \( \hat{Z}_{f,m} \). Part (iv) should read as follows:

(iv) The constituent \( G_{\mathfrak{P}}^{Z_{f,m}} = \langle \sigma, \tau \rangle \) has just \( r \) orbits of sizes \( d_1e, \ldots, d_re \) and is a metacyclic group, with \( \sigma^{-1} \tau \sigma = \tau^{Np} \). The order of the (cyclic) group \( G_{\mathfrak{P}}^{Z_{f,m}}/I_{\mathfrak{P}}^{Z_{f,m}} \) is divisible by \( \mu = \text{lcm}(\omega, d_1, \ldots, d_r) \), and it is a divisor of \( \mu \cdot e \). This order is equal to \( \mu \) if \( e = 1 \) and \( d = 1 \), and if \( r = 1 \) and \( \gcd(\omega, d) = 1 \).

The first assertion follows from Proposition 2 in [KS] (essentially due to Ore). By parts (i), (ii) of the theorem \( \hat{f}_m \), having the set \( Z_{f,m} \) of zeros, and the factor \( f_m \) to the side \( S \) have the same splitting field \( \hat{L}_m \subseteq \overline{K}_p \), and we may identify \( G_{\mathfrak{P}}^{Z_{f,m}} \) with the Galois group \( G_m = \text{Gal}(\hat{L}_m|K_p) \) acting on

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the roots of \( f_m \). Let \( \hat{T} \) be the maximal subfield of \( \hat{L}_m \) unramified over \( K_p \). We identify \( I_{\mathfrak{p}}^{\hat{f}_m} \) with \( I_m = \text{Gal}(\hat{L}_m/\hat{T}) \) (acting on the roots of \( f_m \)). By assumption \( \hat{L}_m/K_p \) is a tame extension. It is well known that \( G_m/I_m \cong \text{Gal}(\hat{T}/K_p) \) is cyclic, generated by the inverse image \( I_m \sigma \) of the Frobenius automorphism over \( k_p \), and that \( \sigma^{-1} \tau \sigma = \tau^{N_p} \).

Observe that \( \omega = [K_p(\varepsilon) : K_p] \) where \( \varepsilon \) is a primitive \( e \)th root of unity. We assert that \( \varepsilon \in \hat{T} \). Of course \( K_p(\varepsilon) | K_p \) is unramified \( (p \nmid e) \). Recall that \( \deg(f_m) = \ell = de \) equals the length of \( S \) and that \( f_m \) is a polynomial in \( X^e \). Indeed, by construction, or in view of Hensel’s lemma, there is a unique normalized lift \( f_S \in K_p[X] \) of \( \hat{f}_S \) such that

\[
 f_m(X) = \pi^{dh} f_S(\pi^{-h} X^e),
\]

where \( \pi \) is the fixed element of \( K \) with order 1 at \( p \). Hence if \( \theta \) is a root of \( f_m \) (in \( \hat{L}_m \)) so is \( \varepsilon^i \theta \) for each integer \( i \), giving the assertion. Moreover, \( \pi^{-h} \theta^e \) is then a root of \( f_S \). If \( \pi^{-h} \theta^e = \pi^{-h} \beta^e \) for some other root \( \beta \) of \( f_m \), then \( \beta/\theta \) is an \( e \)th root of unity and so \( \beta = \varepsilon^i \theta \) for some integer \( i \). For \( \tau \in I_m \) we have \( (\theta^\tau)^e = (\theta^e)^\tau = \theta^e \). We conclude that \( \{ \varepsilon^i \theta : 1 \leq i \leq e \} \) is the orbit of \( \theta \) under \( I_m \). Since there are just \( d = \deg(f_S) \) such orbits, we deduce that each root of \( f_S \) is of the form \( \pi^{-h} \theta^e \) for some root \( \theta \) of \( f_m \). It follows that \( [\hat{T} : K_p] \) is divisible by \( e \).

Let \( T \) be the (unique) subfield of \( \hat{T} \) such that \( [T : K_p] = \mu \). We know that \( \varepsilon \in T \) and that, for each root \( \theta \) of \( f_m \), we have \( \theta^e = \pi^h u_\theta \) for some unit \( u_\theta \in U_T \) in \( T \), which is a root of \( f_S \) in \( T \). By separability of \( \hat{f}_S = f_S \bmod p \) these \( u_\theta \) belong to \( d \) distinct elements in \( k_T^* \), where \( k_T \) is the residue class field of \( T \). From \( p \nmid e \) we infer that \( U_T/U_T^e \cong k_T^*/k_T^{*e} \) is cyclic (of order \( e \)). Observe that \( \pi \) is a prime in \( T \) and that \( \gcd(e, h) = 1 \). Combining Proposition 2 in [KS] with Abhyankar’s lemma and (abelian) Kummer theory we see that, for any root \( \theta \) of \( f_m \), the polynomial \( X^e - \pi^h u_\theta \) is irreducible over \( T \) and that \( T(\theta)|T \) is a cyclic totally ramified extension of degree \( e \) with \( T(\theta) \hat{T} = \hat{L}_m \). In this manner we recover part (iii) of the theorem in [KS]. Now \( \hat{L}_m/T(\theta) \) is cyclic of degree \( [\hat{T} : T] \), and \( \hat{L}_m \) is the compositum of all these \( T(\theta) \). We conclude that the degree \( [\hat{T} : T] \) is a divisor of \( e \). Hence \( |G_m/I_m| = [\hat{T} : K_p] \) divides \( \mu \cdot e \).

It is obvious that \( T = \hat{T} \) if \( e = 1 \) or \( d = 1 \). Suppose that \( r = 1 \) and \( \gcd(\omega, d) = 1 \). Then \( \hat{f}_S \) is (even) irreducible over the residue class field of \( K_p(\varepsilon) \), which has order \( (Np)^\omega \equiv 1 \pmod{e} \). Hence the roots \( u_\theta = \pi^{-h} \theta^e \) of \( f_S \) are conjugate over \( K_p(\varepsilon) \) and so belong to the same class in \( U_T/U_T^e \). Apply Kummer theory.

Whereas Corollary 1 to the theorem in [KS] is true as it stands \( (e = 1) \), Corollary 2 has to be modified. Here we have \( d = 1 \ (\ell = e) \), so that \( G_m/I_m \)
is cyclic of order $\omega = o(Np \mod e)$. For $e \neq 1 \neq d$ it can happen that $|G_m/I_m| > \mu$; an example with $e = 2 = d$ (and $m = 1/2$) is provided by $f_m = X^4 - 2 \cdot 7^2$ over $\mathbb{Q}_7$ ($\pi = 7$).

NOTE. Let us say that two normalized polynomials $\varphi, \psi$ in $K_p[X]$ (of the same degree) belong to the same Ore class provided their Newton polygon (with respect to $v_p$) is the same and consists of one straight line $S$ such that the associated polynomials $\tilde{\varphi}_S = \tilde{\psi}_S$ agree. This means that the points on the line $S$ resulting from $\varphi, \psi$ are the same and that the corresponding coefficients only differ by principal units in $K_p$. The statements of the theorem, for regular and tame $S$, only depend on the Ore class of the polynomials.

Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
D-72076 Tübingen, Germany
E-mail: peter.schmid@uni-tuebingen.de

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