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On the cyclotomic elements in K_2 of a rational function field

by

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1. Introduction. Let A be an abelian group and n a positive integer. Write $A_n = \{a \in A \mid a^n = 1\}$. For a field F, let $K_2(F)$ denote the Milnor K_2-group of F (see [8]). Tate [14] proved that if F is a global field containing the nth primitive root of unity ζ_n, then

$$(K_2(F))_n = \{\zeta_n, F^*\}.$$

Suslin [13] generalized Tate's result to any field containing ζ_n. The condition $\zeta_n \in F$ is restrictive. For example, $K_2(\mathbb{Q})$ is a torsion group having elements of any orders by the Dirichlet Theorem, but only elements of order 2 in $K_2(\mathbb{Q})$ can be described by the above result. So, we are led to a question: *For a field F not containing ζ_n, how to describe the elements of $(K_2(F))_n$?*

Browkin [1] considered *cyclotomic elements* in $K_2(F)$, i.e. elements of the form $\{a, \Phi_n(a)\}$, where $\Phi_n(x)$ denotes the nth cyclotomic polynomial. Let

$$G_n(F) = \{\{a, \Phi_n(a)\} \in K_2(F) \mid a, \Phi_n(a) \in F^*\}.$$

It is proved in [1] that $G_n(F) \subseteq (K_2(F))_n$. Now, the question might be whether every element of order n in $K_2(F)$ can be written in the form $\{a, \Phi_n(a)\}$ (up to an element of order 2 if n is even), or for what n the following is true:

$$(K_2(F))_n = G_n(F). \tag{1.1}$$

If $n = 3$, this is true for $F = \mathbb{Q}$ by [1] and for any field F with $\mathrm{ch}(F) \neq 3$ by Urbanowicz [15]; if $n = 4$, it follows from [1] for $F = \mathbb{Q}$ and from [9] for any field F with $\mathrm{ch}(F) \neq 2$ that every element of order 4 in $K_2(F)$ can be written in the form $\{a, \Phi_4(a)\}v$ where $a \in F^*$ and $v \in K_2(F)$ with $v^2 = 1$. Moreover, it is proved in [1] that if $n = 1, 2, 3, 4$ or 6 and $F \neq \mathbb{F}_2$, then $G_n(F)$

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is a subgroup of $K_2(F)$. Browkin [1] proposed the following conjecture (see [11], [17] for more general formulations):

Conjecture ([1]). *For $n \neq 1,2,3,4,6$ and any field F, $G_n(F)$ is not a subgroup of $K_2(F)$, in particular, $G_5(\mathbb{Q})$ is not a subgroup of $K_2(\mathbb{Q})$.*

This conjecture implies that $G_n(F) \subsetneq (K_2(F))_n$, that is, (1.1) is not true in general. In other words, in the n-torsion of $K_2(F)$, there exists at least one element which is not a cyclotomic element. However, the picture is not quite as conjectured: in fact, the conjecture is not true for local fields, i.e. the equality (1.1) does hold for local fields [18], [19], [5].

As for global fields, we pointed out in [18] that the above conjecture should be true. In fact, Qin proved in [10] that neither $G_5(\mathbb{Q})$ nor $G_7(\mathbb{Q})$ is a group and in [9] that $G_{2^n}(\mathbb{Q})$ is a group if and only if $n \leq 2$. Xu [16] found that the conjecture can be reduced to a problem about rational points of curves, and hence could prove by using Faltings' Theorem on the Mordell conjecture [3] that if n is an integer having a square factor and if $n \neq 4,8,12$, then the conjecture is true for any number field F. By using the results of Manin [7], Grauert [4] and Samuel [12] on the Mordell conjecture on function fields, a similar result can be established for function fields over an algebraically closed field (see [17]). See [2] for recent work on this topic.

Unfortunately, the methods used for the above cases do not work for the case of n square-free, in particular, they fail for n being a general prime number ≥ 5 even for the rational number field $\mathbb{Q}$. So the unsolved part of the conjecture, in particular for global fields, seems curiously difficult.

In this paper, for a prime number l, we investigate cyclotomic elements in the l-torsion of $K_2(F)$ for $F = k(x)$, the rational function field over k. We prove that if $l \geq 5$ is a prime number and $\mathrm{ch}(k) \neq 2, l$, and if $\Phi_l(x)$ is irreducible in $k[x]$, then Browkin's conjecture is true (see Theorem 2.1). The proof depends on the fact that the field $k(x)$ has a nontrivial derivation, so it does not carry over to number fields.

2. Main results

Theorem 2.1. *Let $l \geq 5$ be a prime number and let k be a field with $\mathrm{ch}(k) \neq 2, l$. Assume that $\Phi_l(x)$ is irreducible in $k[x]$. Then $G_l(k(x))$ is not a subgroup of $K_2(k(x))$.*

Proof. Suppose that $G_l(k(x))$ is a subgroup of $K_2(k(x))$. Let

$$\beta = \{x, \Phi_l(x)\}\{x+1, \Phi_l(x+1)\}.$$

Then $\beta \in G_l(k(x))$, so there must exist two coprime polynomials f, g in $k[x]$ with $f(x)$ monic such that

$$\beta = \{f/g, \Phi_l(f/g)\}.$$

We will prove that this is impossible.

We use the symbol $\mathfrak{p}$ to denote a prime in $k[x]$. By the definition of the tame symbol τ, we have

$$\begin{aligned}\tau_{\mathfrak{p}}(\{f/g,\Phi_l(f/g)\}) &= (-1)^{v_{\mathfrak{p}}(f/g)v_{\mathfrak{p}}(\Phi_l(f/g))}\frac{(f/g)^{v_{\mathfrak{p}}(\Phi_l(f/g))}}{(\Phi_l(f/g))^{v_{\mathfrak{p}}(f/g)}} \pmod{\mathfrak{p}} \\ &= \begin{cases} (f/g)^{v_{\mathfrak{p}}(\Phi_l(f,g))} \pmod{\mathfrak{p}} & \text{if } v_{\mathfrak{p}}(f)=v_{\mathfrak{p}}(g)=0, \\ 1 \pmod{\mathfrak{p}} & \text{otherwise,}\end{cases}\end{aligned}$$

where $\Phi_l(f,g) := g^{l-1}\Phi_l(f/g)$.

We claim that if $v_{\mathfrak{p}}(f)=v_{\mathfrak{p}}(g)=0$, then

$$\tau_{\mathfrak{p}}(\{f/g,\Phi_l(f/g)\}) = \begin{cases} \not\equiv 1 \pmod{\mathfrak{p}} & \text{if } l\nmid v_{\mathfrak{p}}(\Phi_l(f,g)), \\ 1 \pmod{\mathfrak{p}} & \text{if } l\mid v_{\mathfrak{p}}(\Phi_l(f,g)).\end{cases}$$

In fact, if $l\mid v_{\mathfrak{p}}(\Phi_l(f,g))$, then clearly $\tau_{\mathfrak{p}}(\{f/g,\Phi_l(f/g)\}) \equiv 1 \pmod{\mathfrak{p}}$. Now, suppose that $l\nmid v_{\mathfrak{p}}(\Phi_l(f,g))$. If $\tau_{\mathfrak{p}}(\{f/g,\Phi_l(f/g)\}) \equiv 1 \pmod{\mathfrak{p}}$, that is,

$$(f/g)^{v_{\mathfrak{p}}(\Phi_l(f,g))} \equiv 1 \pmod{\mathfrak{p}},$$

then in virtue of $(l, v_{\mathfrak{p}}(\Phi_l(f,g)))=1$ we know that $f/g\equiv 1 \pmod{\mathfrak{p}}$, that is, $f\equiv g \pmod{\mathfrak{p}}$. Hence, from $l\nmid v_{\mathfrak{p}}(\Phi_l(f,g))$, we know that

$$\mathfrak{p}\mid \Phi_l(f,g)\equiv lg^{l-1} \pmod{\mathfrak{p}}.$$

So $\mathfrak{p}\mid g$, which contradicts $v_{\mathfrak{p}}(g)=0$. Hence, $\tau_{\mathfrak{p}}(\{f/g,\Phi_l(f/g)\}) \not\equiv 1 \pmod{\mathfrak{p}}$.

Secondly, since $\Phi_l(x)$ and $\Phi_l(x+1)$ are both irreducible in $k[x]$, we have

$$\tau_{\mathfrak{p}}(\{f/g,\Phi_l(f/g)\}) = \tau_{\mathfrak{p}}(\beta) = \begin{cases} x \pmod{\Phi_l(x)} & \text{if } \mathfrak{p}=\Phi_l(x), \\ x+1 \pmod{\Phi_l(x+1)} & \text{if } \mathfrak{p}=\Phi_l(x+1), \\ 1 \pmod{\mathfrak{p}} & \text{otherwise.}\end{cases}$$

Comparing the above computations, we find that the value $v_{\mathfrak{p}}(\Phi_l(f,g))$ is either nontrivial at the primes $\Phi_l(x)$ and $\Phi_l(x+1)$, i.e.

$$v_{\mathfrak{p}}(\Phi_l(f,g)) \not\equiv 0 \pmod{l}$$

for $\mathfrak{p}=\Phi_l(x)$, $\Phi_l(x+1)$, or $l\mid v_{\mathfrak{p}}(\Phi_l(f,g))$ for primes $\mathfrak{p}$ other than $\Phi_l(x)$ and $\Phi_l(x+1)$.

Hence, we conclude that there exist integers e_1, e_2 satisfying $1\le e_1, e_2\le l-1$ such that

$$\Phi_l(f,g) = \alpha\Phi_l(x)^{e_1}\Phi_l(x+1)^{e_2}h^l \tag{2.1}$$

for some $\alpha\in k$.

Now, we will determine upper bounds of the degrees of f and g. We write (2.1) as

$$f^l-g^l = \alpha(f-g)\Phi_l(x)^{e_1}\Phi_l(x+1)^{e_2}h^l. \tag{2.2}$$

Differentiating, we get

$$\begin{aligned}(2.3)\quad & l(f'f^{l-1}-g'g^{l-1})\\ &= \alpha\Phi_l(x)^{e_1-1}\Phi_l(x+1)^{e_2-1}h^{l-1}\big[(f'-g')\Phi_l(x)\Phi_l(x+1)h\\ &\quad + e_1(f-g)\Phi_l'(x)\Phi_l(x+1)h + e_2(f-g)\Phi_l(x)\Phi_l'(x+1)h\\ &\quad + l(f-g)\Phi_l(x)\Phi_l(x+1)h'\big].\end{aligned}$$

From (2.2), (2.3) and the equality

$$lg'(f^l-g^l) - g\cdot l(f'f^{l-1}-g'g^{l-1}) = lf^{l-1}(fg'-gf'),$$

we deduce that

$$(2.4)\qquad \Phi_l(x)^{e_1-1}\Phi_l(x+1)^{e_2-1}h^{l-1} \mid fg'-gf',$$

since $(f,g)=1$.

Let $\deg f=\theta$, $\deg g=\eta$ and $\deg h=\lambda$. Since $\Phi_l(x,y)$ is a symmetric polynomial, we can assume that $\theta\ge\eta$.

First, we assume that $fg'-gf'\ne 0$.

Obviously $\deg\Phi_l(f,g)=(l-1)\theta$. Hence from (2.1) and (2.4) we get

$$(2.5)\qquad (l-1)\theta = l\lambda + (e_1+e_2)(l-1),$$

$$(2.6)\qquad (l-1)[\lambda+(e_1-1)+(e_2-1)] \le \theta+\eta-1.$$

From (2.5) it follows that $l-1\mid\lambda$. Let $\lambda=(l-1)\lambda_1$.

Claim. $\lambda=0$.

Eliminating e_1 and e_2 from (2.5) and (2.6) we obtain

$$(2.7)\qquad (l-1)(\theta-2)-\lambda \le \theta+\eta-1.$$

Since $e_1\ge 1$ and $e_2\ge 1$, from (2.6) we deduce that

$$(l-1)\lambda \le \theta+\eta-1.$$

Hence (2.7) implies

$$(l-1)(\theta-2) \le (\theta+\eta-1)\left(1+\frac{1}{l-1}\right) \le (2\theta-1)\frac{l}{l-1}.$$

Consequently,

$$(l-1)^2(\theta-2) \le 2l(\theta-2)+3l,$$

so

$$\theta-2 \le \frac{3l}{(l-1)^2-2l} < 3 \quad \text{for } l\ge 5.$$

It follows that $\theta<5$, and so

$$\lambda_1 = \frac{\lambda}{l-1} \le \frac{2\theta-1}{(l-1)^2} < \frac{9}{(l-1)^2} < 1 \quad \text{for } l\ge 5.$$

Hence $\lambda_1=0$, so $\lambda=0$, as claimed.

Now (2.5) and (2.7) simplify to

$$\theta = e_1 + e_2, \qquad (l-1)(\theta-2) \le \theta + \eta - 1 \le 2\theta - 1. \tag{2.8}$$

CLAIM. *If $\theta > 2$, then $l = 5$ and $\theta = 3$.*

From $e_1, e_2 \ge 1$ it follows that $\theta \ge 2$. If $\theta > 2$, then, by (2.8),

$$l \le 1 + \frac{2\theta - 1}{\theta - 2} = 3 + \frac{3}{\theta - 2} \begin{cases} = 6 & \text{if } \theta = 3, \\ < 5 & \text{if } \theta > 3. \end{cases}$$

This proves the Claim.

CASE $\theta = 2$. Then $e_1 = e_2 = 1$ and (2.1) takes the form

$$\Phi_l(f, g) = \alpha \Phi_l(x) \Phi_l(x+1), \tag{2.9}$$

where $\deg f = 2 \ge \deg g$. Note that $f(\zeta)g(\zeta) \ne 0$ when $\zeta = \zeta_l$. From (2.9), we have

$$\frac{g(\zeta)}{f(\zeta)} = \zeta^r, \qquad \frac{f(\zeta)}{g(\zeta)} = \zeta^{l-r}, \qquad \text{for some } 1 \le r \le l-1.$$

Therefore, ζ is a root of two polynomials:

$$\begin{aligned} F(x) &= x^r f(x) - g(x) \in k[x] && \text{of degree } r+2, \text{ and} \\ G(x) &= f(x) - x^{l-r} g(x) \in k[x] && \text{of degree } \le l - r + 2. \end{aligned}$$

Hence

$$\Phi_l(x) \mid F(x) \quad \text{and} \quad \Phi_l(x) \mid G(x), \tag{2.10}$$

since $\Phi_l(x)$ is irreducible in $k[x]$. Clearly, $F(x) \ne 0$.

If $G(x) = 0$, then $f(x) = x^{l-r} = x^2$ and $g(x) = 1$, since $(f, g) = 1$ and $f(x)$ is monic of degree 2. Consequently,

$$\Phi_l(f, g) = \Phi_l(x^2, 1) = \Phi_l(x^2) = \Phi_l(x)\Phi_l(-x).$$

From (2.9), we have $\Phi_l(-x) = \Phi_l(x+1)$. The computation of both sides leads to the following equalities in k:

$$l + 1 = 0, \qquad l + 1 = 2,$$

so $2 = 0$, which is impossible since $\mathrm{ch}(k) \ne 2$. Therefore $G(x) \ne 0$.

Now, from (2.10), we get

$$l - 1 \le r + 2 \quad \text{and} \quad l - 1 \le l - r + 2. \tag{2.11}$$

Hence $l \le 6$, so $l = 5$.

Substituting $l = 5$ in (2.11) we obtain $2 \le r \le 3$. If $r = 3$, then $l - r = 5 - 3 = 2$, so we get from (2.10) the following divisibilities:

$$\Phi_5(x) \mid x^2 f(x) - g(x) \quad \text{or} \quad \Phi_5(x) \mid f(x) - x^2 g(x). \tag{2.12}$$

From (2.9) it follows that

$$\frac{g(\zeta - 1)}{f(\zeta - 1)} = \zeta^s, \qquad \frac{f(\zeta - 1)}{g(\zeta - 1)} = \zeta^{l-s}, \qquad \text{for some } 1 \le s \le l-1.$$

Proceeding as above we obtain the following divisibilities:

(2.13) $\Phi_5(x) \mid x^2 f(x-1) - g(x-1) \quad \text{or} \quad \Phi_5(x) \mid f(x-1) - x^2 g(x-1).$

When the second divisibilities in (2.12) and (2.13) hold, we have $\deg g = 2$.

Since all polynomials in (2.12) and (213) are of degree 4, and $f(x)$ is monic, we have

(2.14) $\Phi_5(x) = x^2 f(x) - g(x) \quad \text{or} \quad -c\Phi_5(x) = f(x) - x^2 g(x),$

(2.15) $\Phi_5(x) = x^2 f(x-1) - g(x-1) \quad \text{or} \quad -c\Phi_5(x) = f(x-1) - x^2 g(x-1),$

where c is the leading coefficient of $g(x)$.

The formulas (2.14) and (2.15) lead to the following four cases:

1. $\Phi_5(x) = x^2 f(x) - g(x) = x^2 f(x-1) - g(x-1)$.

Hence $x^2(f(x) - f(x-1)) = g(x) - g(x-1)$. This is impossible, since $\deg(g(x) - g(x-1)) < \deg g(x) \le 2$.

2. $-c\Phi_5(x) = f(x) - x^2 g(x) = f(x-1) - x^2 g(x-1)$.

Then $f(x) - f(x-1) = x^2(g(x) - g(x-1))$. This leads to a contradiction, since $\deg f(x) = \deg g(x) = 2$ implies that $f(x) \neq f(x-1)$, $g(x) \neq g(x-1)$ and $\deg(f(x) - f(x-1)) < 2$.

3. $\Phi_5(x) = x^2 f(x) - g(x)$ and $-c\Phi_5(x) = f(x-1) - x^2 g(x-1)$.

From the first equality it follows that $f(x) = x^2 + x + (1-c)$. Then the second equality gives $-c(x+1) \equiv f(x-1) \pmod{x^2}$. This is impossible, since $f(x-1) = x^2 - x + (1-c) \equiv -x + (1-c) \pmod{x^2}$.

4. $\Phi_5(x) = x^2 f(x-1) - g(x-1)$ and $-c\Phi_5(x) = f(x) - x^2 g(x)$.

From the second equality it follows that $g(x) = cx^2 + cx + (c+1)$. Then the first equality implies that $x+1 \equiv -g(x-1) \pmod{x^2}$. This is impossible, since $g(x-1) = cx^2 - cx + (c+1) \equiv -cx + (c+1) \pmod{x^2}$.

Thus, in all four cases we get a contradiction.

Case $\theta = 3,\ l = 5$.

1. Assume that $\eta = \theta = 3$, $e_1 + e_2 = 3$.

1.1. $e_1 = 2$, $e_2 = 1$. Similarly, we get

$$f - \zeta_5 g = \beta(x - \zeta_5^i)^2(x + 1 - \zeta_5^j), \qquad \beta \in k(\zeta_5),$$

where $1 \le i, j \le 4$. Let

$$f = a_1x^3 + b_1x^2 + c_1x + d_1, \qquad g = a_2x^3 + b_2x^2 + c_2x + d_2,$$

where $a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2 \in k$ and $a_1a_2 \neq 0$. Then

$$\begin{aligned}(a_1 - \zeta_5 a_2)x^3 &+ (b_1 - \zeta_5 b_2)x^2 + (c_1 - \zeta_5 c_2)x + (d_1 - \zeta_5 d_2)\\ &= \beta[x^3 + (1 - \zeta_5^j - 2\zeta_5^i)x^2 + (2\zeta_5^{i+j} - 2\zeta_5^i + \zeta_5^{2i})x + (\zeta_5^{2i} - \zeta_5^{2i+j})],\end{aligned}$$

where $1 \le i, j \le 4$. Equating the coefficients, we have

$$(a_1 - b_1) + (b_2 - a_2)\zeta_5 - 2a_1\zeta_5^i - a_1\zeta_5^j + 2a_2\zeta_5^{i+1} + a_2\zeta_5^{j+1} = 0,$$
$$c_1 - c_2\zeta_5 - 2a_1\zeta_5^i - 2a_2\zeta_5^{i+1} - a_1\zeta_5^{2i} - 2a_1\zeta_5^{i+j} + 2a_2\zeta_5^{i+j+1} + a_2\zeta_5^{2i+1} = 0,$$
$$d_1 - d_2\zeta_5 - a_1\zeta_5^{2i} + a_1\zeta_5^{2i+j} + a_2\zeta_5^{2i+1} - a_2\zeta_5^{2i+j+1} = 0,$$

where $1 \le i, j \le 4$.

1.1.1. If $i = j$, then a contradiction is obtained by checking the cases $i = 1, 2, 3, 4$ directly.

1.1.2. If $i \ne j$, we get a similar contradiction.

1.2. $e_1 = 1$, $e_2 = 2$. Then

$$f - \zeta_5 g = \beta(x - \zeta_5^i)(x + 1 - \zeta_5^j)^2, \quad \beta \in k(\zeta_5),$$

where $1 \le i, j \le 4$. Similarly we have

$$(2a_1 - b_1) + (b_2 - 2a_2)\zeta_5 - a_1\zeta_5^i - 2a_1\zeta_5^j + a_2\zeta_5^{i+1} + 2a_2\zeta_5^{j+1} = 0,$$
$$(a_1 - c_1) + (c_2 - a_2)\zeta_5 - 2a_1\zeta_5^i - 2a_1\zeta_5^j + 2a_2\zeta_5^{i+1} + 2a_2\zeta_5^{j+1}$$
$$+ a_1\zeta_5^{2j} + 2a_1\zeta_5^{i+j} - a_2\zeta_5^{2j+1} - 2a_2\zeta_5^{i+j+1} = 0,$$
$$d_1 - d_2\zeta_5 + a_1\zeta_5^i - a_2\zeta_5^{i+1} - 2a_1\zeta_5^{i+j} + 2a_2\zeta_5^{i+j+1} + a_1\zeta_5^{2j+i} - a_2\zeta_5^{2j+i+1} = 0,$$

where $1 \le i, j \le 4$, a similar contradiction.

2. $2 = \eta < \theta = 3$, $e_1 + e_2 = 3$.

2.1. $e_1 = 2$, $e_2 = 1$. Similarly we have

$$f - \zeta_5 g = \beta(x - \zeta_5^i)^2(x + 1 - \zeta_5^j), \quad \beta \in k(\zeta_5),$$

where $1 \le i, j \le 4$. Let

$$f = a_1x^3 + b_1x^2 + c_1x + d_1, \quad g = b_2x^2 + c_2x + d_2,$$

where $a_1, b_1, c_1, d_1, b_2, c_2, d_2 \in k$ and $a_1b_2 \ne 0$. Similarly we have

$$(a_1 - b_1) + b_2\zeta_5 + 2a_1\zeta_5^i - a_1\zeta_5^j = 0,$$
$$c_1 - c_2\zeta_5 + 2a_1\zeta_5^i - 2a_1\zeta_5^{i+j} - a_1\zeta_5^{2i} = 0,$$
$$d_1 - d_2\zeta_5 - a_1\zeta_5^{2i} + a_1\zeta_5^{2i+j} = 0,$$

where $1 \le i, j \le 4$, a similar contradiction.

2.2. $e_1 = 1$, $e_2 = 2$. We have

$$(2a_1 - b_1) + b_2\zeta_5 - a_1\zeta_5^i - 2a_1\zeta_5^j = 0,$$
$$(a_1 - c_1) + c_2\zeta_5 - 2a_1\zeta_5^i - 2a_1\zeta_5^j + a_1\zeta_5^{2j} + 2a_1\zeta_5^{i+j} = 0,$$
$$d_1 - d_2\zeta_5 + a_1\zeta_5^i - 2a_1\zeta_5^{i+j} + a_1\zeta_5^{2j+i} = 0,$$

where $1 \le i, j \le 4$, a similar contradiction.

In summary, the equality (2.1) does not hold if $fg' - gf' \ne 0$, So we conclude that $G_l(k(x))$ is not a subgroup of $K_2(k(x))$ if $fg' - gf' \ne 0$.

Now, we consider the case of $fg' - gf' = 0$. In this case, we must have $\mathrm{ch}(k) \neq 0$. Indeed, if $\mathrm{ch}(k) = 0$, then from $fg' - gf' = 0$ and $(f, g) = 1$, we have $f \mid f'$ and $g \mid g'$. So $f' = g' = 0$, since $\mathrm{ch}(k) = 0$. Thus f and g are both nonzero constants. Hence

$$v_{\Phi_l(x)}(\Phi_l(f, g)) = 0,$$

a contradiction.

Assume that $\mathrm{ch}(k) = p \neq 0$ and $fg' - gf' = 0$. Then from $(f, g) = 1$, we have $f' = g' = 0$, so, as is well known, we have

$$f(x) = f_1(x^p), \qquad g(x) = g_1(x^p), \qquad \text{for some } f_1(x), g_1(x) \in k[x].$$

Hence, differentiating (2.1), we have

$$\begin{aligned} 0 &= lh^{l-1}h'\Phi_l(x)^{e_1}\Phi_l(x+1)^{e_2} \\ &\quad + h^l[e_1\Phi_l(x)^{e_1-1}\Phi_l'(x)\Phi_l(x+1)^{e_2} + e_2\Phi_l(x)^{e_1}\Phi_l(x+1)^{e_2-1}\Phi_l'(x+1)]. \end{aligned}$$

So we get

$$0 = lh'\Phi_l(x)\Phi_l(x+1) + h[e_1\Phi_l'(x)\Phi_l(x+1) + e_2\Phi_l(x)\Phi_l'(x+1)].$$

If $h' \neq 0$, then $\Phi_l(x)\Phi_l(x+1) \mid h$. Let $h = h_1 \cdot \Phi_l(x)\Phi_l(x+1)$. Then

$$\begin{aligned} 0 &= l[h_1'\Phi_l(x)\Phi_l(x+1) + h_1(\Phi_l(x)\Phi_l(x+1))']\Phi_l(x)\Phi_l(x+1) \\ &\quad + h_1\Phi_l(x)\Phi_l(x+1)[e_1\Phi_l'(x)\Phi_l(x+1) + e_2\Phi_l(x)\Phi_l'(x+1)]. \end{aligned}$$

So

$$\begin{aligned} 0 &= l[h_1'\Phi_l(x)\Phi_l(x+1) + h_1(\Phi_l(x)\Phi_l(x+1))'] \\ &\quad + h_1[e_1\Phi_l'(x)\Phi_l(x+1) + e_2\Phi_l(x)\Phi_l'(x+1)] \\ &= lh_1'\Phi_l(x)\Phi_l(x+1) \\ &\quad + h_1[(l+e_1)\Phi_l'(x)\Phi_l(x+1) + (l+e_2)\Phi_l(x)\Phi_l'(x+1)]. \end{aligned}$$

Repeating this procedure, we get a nonzero polynomial $h_m \in k[x]$ such that

$$h = (\Phi_l(x)\Phi_l(x+1))^m h_m \quad \text{with } h_m' = 0,\ m \geq 0,$$

and that

$$0 = (ml + e_1)\Phi_l'(x)\Phi_l(x+1) + (ml + e_2)\Phi_l(x)\Phi_l'(x+1).$$

Since $\Phi_l'(x)\Phi_l(x+1)$ and $\Phi_l(x)\Phi_l'(x+1)$ are linearly independent over k, we have the following equalities in k:

$$ml + e_j = 0, \qquad \text{where } j = 1, 2.$$

So, as integers, we can write

$$ml + e_j = pe_j', \qquad \text{where } j = 1, 2.$$

Hence, the equality (2.1) becomes

$$\Phi_l(f_1(x^p), g_1(x^p)) = \alpha'\Phi_l(x^p)^{e_i'}\Phi_l(x^p + 1)^{e_2'}.$$

Let $X = x^p$. Then

$$\Phi_l(f_1(X), g_1(X)) = \alpha' \Phi_l(X)^{e'_1} \Phi_l(X+1)^{e'_2}.$$

Repeating the above discussion we will stop at the sth step, that is,

$$f(x) = f_s(x^{p^s}), \qquad g(x) = g_s(x^{p^s}),$$

with $f_s(x), g_s(x) \in k[x]$ satisfying $f_s g'_s - f'_s g_s \neq 0$, such that

$$\Phi_l(f_s(x^{p^s}), g_s(x^{p^s})) = \alpha' \Phi_l(x^{p^s})^{e_1^{(s)}} \Phi_l(x^{p^s}+1)^{e_2^{(s)}}.$$

Let $X = x^{p^s}$. Then

$$\Phi_l(f_s(X), g_s(X)) = \alpha' \Phi_l(X)^{e_1^{(s)}} \Phi_l(X+1)^{e_2^{(s)}}, \tag{2.16}$$

with $f_s(X) g'_s(X) - f'_s(X) g_s(X) \neq 0$.

Let $\theta_s = \deg f_s$ and $\eta_s = \deg g_s$. Then

$$\theta_s = e_1^{(s)} + e_2^{(s)}, \tag{2.17}$$

$$(l-1)(e_1^{(s)} + e_2^{(s)} - 2) \leq 2\theta_s - 1. \tag{2.18}$$

From (2.17), (2.18), much as from (2.5), (2.6), we can prove that if $\theta_s > 2$, then $l = 5$ and $\theta_s = 3$. A similar discussion for (2.16) leads to a contradiction. If $\theta_s = 2$, then the discussion is also similar. ∎

Remark 2.2. From Izhboldin [6], we know that if p is a prime number and F is a field with $\mathrm{ch}(F) = p$, then $(K_2(F))_p = 1$. Hence, the condition $\mathrm{ch}(k) \neq l$ in Theorem 2.1 is not unnecessary.

Corollary 2.3. *Let $l \geq 5$ be a prime number. Then $G_l(\mathbb{Q}(x))$ is not a subgroup of $K_2(\mathbb{Q}(x))$.*

Corollary 2.4. *Assume that F is a number field and $l \geq 5$ is a prime. If $F \cap \mathbb{Q}(\zeta_l) = \mathbb{Q}$, then $G_l(F(x))$ is not a subgroup of $K_2(F(x))$.*

Proof. It is well known that

$$[F(\zeta_l) : F] = [F \cdot \mathbb{Q}(\zeta_l) : F] = [\mathbb{Q}(\zeta_l) : F \cap \mathbb{Q}(\zeta_l)] = [\mathbb{Q}(\zeta_l) : \mathbb{Q}] = l - 1.$$

Hence $\Phi_l(x)$ is irreducible over F. ∎

Clearly, if a prime number p does not ramify in F, then $F \cap \mathbb{Q}(\zeta_l) = \mathbb{Q}$. For a cyclic extension $F/\mathbb{Q}$ of degree n, according to Chebotarev's density theorem, the density of rational primes l which are inert in F is $\varphi(n)/n$, where φ is the Euler function.

Corollary 2.5. *Let $l \geq 5$ be a prime and d square-free. If $d \neq l^* := (-1)^{(l-1)/2} l$, then $G_l(\mathbb{Q}(\sqrt{d})(x))$ is not a subgroup of $K_2(\mathbb{Q}(\sqrt{d})(x))$.*

Proof. In $\mathbb{Q}(\zeta_l)$, there is only one quadratic field $\mathbb{Q}(\sqrt{l^*})$, so if $d \neq l^*$, we have $\mathbb{Q}(\sqrt{d}) \cap \mathbb{Q}(\zeta_l) = \mathbb{Q}$. ∎

As for cyclotomic fields, we have

Corollary 2.6. *Let $l \geq 5$ be a prime and m a positive integer. If $l \nmid m$, then $G_l(\mathbb{Q}(\zeta_m)(x))$ is not a subgroup of $K_2(\mathbb{Q}(\zeta_m)(x))$.*

Proof. It follows from $l \nmid m$ that $\mathbb{Q}(\zeta_m) \cap \mathbb{Q}(\zeta_l) = \mathbb{Q}(\zeta_{(m,l)}) = \mathbb{Q}$. ∎

Corollary 2.7. *Let l, p be different odd primes with $l \geq 5$. If p is a primitive root of l, then $G_l(\mathbb{F}_p(x))$ is not a subgroup of $K_2(\mathbb{F}_p(x))$.*

Proof. If l is a primitive root of p, then $p \pmod{l}$ has order $l-1$. As is well known, this implies $[\mathbb{F}_p(\zeta_l) : \mathbb{F}_p] = l-1$, so $\Phi_l(x)$ must be irreducible over $\mathbb{F}_p$. ∎

It is very easy to find concrete primes satisfying the condition of Corollary 2.7. For example, 3 is a primitive root of 5.

Corollary 2.8. *Let $l \geq 5$ be a prime number and let k be a field with $\mathrm{ch}(k) \neq 2, l$. Assume that $\Phi_l(x)$ is irreducible in $k[x]$. Then in the l-torsion of $K_2(k(x))$, there exist at least two elements which are not cyclotomic, in other words, there exist at least two elements in $(K_2(k(x)))_l$ which cannot be written in the form $\{a, \Phi_l(a)\}$, where $a, \Phi_l(a) \in k(x)^*$.*

Proof. Note that $\{a, \Phi_l(a)\}^{-1} = \{a^{-1}, \Phi_l(a^{-1})\}$. ∎

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