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Upper bounds for the L_q norm of Fekete polynomials on subarcs

by

TAMÁS ERDÉLYI (College Station, TX)

1. Introduction. Finding polynomials in the class

$$\mathcal{L}_n := \Big\{ Q : Q(z) = \sum_{k=0}^{n} a_k z^k,\ a_k \in \{-1, 1\} \Big\}$$

with small uniform norm on the unit circle raised the interest of many authors. Observe that the uniform norm of any polynomial in $\mathcal{L}_n$ on the unit circle is always at least $(n+1)^{1/2}$ since the L_2 norm of any such polynomial is $(2\pi(n+1))^{1/2}$ by the Parseval formula. It is difficult to exhibit a polynomial $Q \in \mathcal{L}_n$ with uniform norm at most $C(n+1)^{1/2}$ for all n with an absolute constant C. An example was found by H. S. Shapiro [13] and W. Rudin [12]. A nice account of this and related problems was given by Littlewood in [9, pp. 25–32].

For a prime number p the *pth Fekete polynomial* is defined as

$$f_p(z) := \sum_{k=1}^{p-1} \left(\frac{k}{p}\right) z^k,$$

where

$$\left(\frac{k}{p}\right) = \begin{cases} 1 & \text{if } x^2 \equiv k \pmod p \text{ has a nonzero solution,} \\ 0 & \text{if } p \text{ divides } k, \\ -1 & \text{otherwise} \end{cases}$$

is the usual Legendre symbol. Since f_p has constant coefficient 0, it is not a Littlewood polynomial, but $g_p(z) := f_p(z)/z$ is a Littlewood polynomial, and has the same modulus as f_p has on the unit circle. Fekete polynomials are examined in detail in [1] and [4].

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Let $\alpha < \beta$ be real numbers. The *Mahler measure* $M_0(Q, [\alpha, \beta])$ is defined for bounded measurable functions $Q(e^{it})$ defined on $[\alpha, \beta]$ as

$$M_0(Q, [\alpha, \beta]) := \exp\left(\frac{1}{\beta - \alpha} \int_\alpha^\beta \log |Q(e^{it})|\, dt\right).$$

It is well known that

$$M_0(Q, [\alpha, \beta]) = \lim_{q \to 0+} M_q(Q, [\alpha, \beta]),$$

where

$$M_q(Q, [\alpha, \beta]) := \left(\frac{1}{\beta - \alpha} \int_\alpha^\beta |Q(e^{it})|^q\, dt\right)^{1/q}, \qquad q > 0.$$

It is a simple consequence of the Jensen formula that

$$M_0(Q, [0, 2\pi]) = |c| \prod_{k=1}^n \max\{1, |z_k|\}$$

for every polynomial of the form

$$Q(z) = c \prod_{k=1}^n (z - z_k), \qquad c, z_k \in \mathbb{C}.$$

In [11] Montgomery proved that there is an absolute constant c such that

$$\max_{t \in [0, 2\pi]} |f_p(e^{it})| \le c p^{1/2} \log p$$

for all primes p. In fact a closer look at his argument shows that combining Lemma 1.1 (see below) due to Gauss and the upper bound for the Lebesgue constant for trigonometric interpolation on equidistant nodes given in [3, Theorem 1] implies that

$$\max_{t \in [0, 2\pi]} |f_p(e^{it})| \le p^{1/2} \left(\frac{5}{3} + \frac{2}{\pi} \log \frac{p-1}{2}\right).$$

Montgomery [11] also showed that the lower bound

$$\frac{2}{\pi} p^{1/2} \log \log p < \max_{t \in [0, 2\pi]} |f_p(e^{it})|$$

holds for all sufficiently large primes p. No better upper or lower bounds than those of Montgomery are known even today.

In [7] we proved that for every $\varepsilon > 0$ there is a constant c_ε such that

$$M_0(f_p, [0, 2\pi]) \ge \left(\frac{1}{2} - \varepsilon\right) p^{1/2} \tag{1.1}$$

for all primes $p \ge c_\varepsilon$. One of the key lemmas in the proof of the above theorem formulates a remarkable property of the Fekete polynomials. A simple proof of it is given in [1, pp. 37–38].

LEMMA 1.1 (Gauss). *We have*

$$f_p(z_p^j) = \varepsilon_p \left(\frac{j}{p}\right) p^{1/2}, \qquad j = 1, \ldots, p-1,$$

and $f_p(1) = 0$, *where*

$$z_p := \exp\left(\frac{2\pi i}{p}\right)$$

is the first pth root of unity, and $\varepsilon_p \in \{-1, 1, -i, i\}$.

The choice of ε_p is more subtle. This is also a result of Gauss (see [8]).

LEMMA 1.2 (Gauss). *In Lemma* 1.1 *we have*

$$\varepsilon_p = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 4, \\ i & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

In [6] the author extended (1.1) to subarcs of the unit circle. Namely it is proved that there is an absolute constant $c_1 > 0$ such that

$$M_0(f_p, [\alpha, \beta]) \geq c_1 p^{1/2}$$

for all primes p and for all $\alpha, \beta \in \mathbb{R}$ such that $(\log p)^{3/2} p^{-1/2} \leq \beta - \alpha \leq 2\pi$.

2. New results. We give an upper bound for the average value of $|f_p(z)|^q$ over any subarc I of the unit circle, valid for all sufficiently large primes p and exponents $q > 0$.

THEOREM 2.1. *There is a constant* $c_2(q, \varepsilon)$ *depending only on* $q > 0$ *and* $\varepsilon > 0$ *such that*

$$\left(\frac{1}{|I|} \int_I |f_p(z)|^q \, |dz|\right)^{1/q} \leq c_2(q, \varepsilon) p^{1/2}$$

for every subarc I *of the unit circle with length* $|I| \geq 2p^{-1/2+\varepsilon}$.

We remark that together with the result from [6] mentioned at the end of the Introduction, Theorem 2.1 shows that there is an absolute constant $c_1 > 0$ and a constant $c_2(q, \varepsilon) > 0$ depending only on $q > 0$ and $\varepsilon > 0$ such that

$$c_1 p^{1/2} \leq \left(\frac{1}{|I|} \int_I |f_p(z)|^q \, |dz|\right)^{1/q} \leq c_2(q, \varepsilon) p^{1/2}$$

for every subarc I of the unit circle with $|I| \geq 2p^{-1/2+\varepsilon} \geq (\log p)^{3/2} p^{-1/2}$.

THEOREM 2.2. *For every sufficiently large prime* p *and for every* $8\pi p^{-1/8} \leq s \leq 2\pi$ *there is a closed subset* $E := E_{p,s}$ *of the unit circle with linear measure* $|E| = s$ *such that*

$$\frac{1}{|E|} \int_E |f_p(z)| \, |dz| \geq c_3 \, p^{1/2} \log\log(1/s)$$

with an absolute constant $c_3 > 0$.

3. Proofs. Our proof of Theorem 2.1 is a combination of Lemma 1.1 due to Gauss, a well-known direct approximation theorem due to Jackson, and the Marcinkiewicz–Zygmund inequality [10], [16, Theorem 7.5, Chapter X]. The Marcinkiewicz–Zygmund inequality asserts that there is a constant $c_4(q)$ depending only on q such that

$$c_4(q)^{-1}\,\frac{1}{n}\sum_{j=1}^{n}|P(z_n^j)|^q \le \int_0^{2\pi}|P(e^{it})|^q\,dt \le c_4(q)\,\frac{1}{n}\sum_{j=1}^{n}|P(z_n^j)|^q$$

for any polynomial P of degree at most $n-1$ and for any $1<q<\infty$, where

$$z_n := \exp(2\pi i/n)$$

is the first nth root of unity.

Proof of Theorem 2.1. It is well known that

$$\left(\frac{1}{|I|}\int_I |f_p(z)|^q\,|dz|\right)^{1/q}$$

is an increasing function of q on $(0,\infty)$. So it is sufficient to prove the theorem only for $q>\varepsilon^{-1}>2$. Let $q>1$; we will use $q\ge\varepsilon^{-1}>2$ only at the end of the proof. Without loss of generality we may assume that $|I|\le 2\pi/3$. We introduce the *truncated Fekete polynomials* $f_{p,m}$ by

$$f_{p,m}(z) := \sum_{k=1}^{p-(m+1)}\left(\frac{k}{p}\right)z^k,$$

with $m:=\lfloor p^{1/2}\rfloor$. Then $f_{p,m}$ is a polynomial of degree $p-(m+1)$.

Let $I=\{e^{it}: t\in[a,b]\}$ and let $3I:=\{e^{it}: t\in[2a-b,2b-a]\}$ be the arc centered at the midpoint of I with arclength $3|I|$. We first define the piecewise linear function L_I on $[2a-b,2a-b+2\pi]$ by

$$L_I(t) := \begin{cases} 1 & \text{if } t\in[a,b],\\ \dfrac{t-(2a-b)}{b-a} & \text{if } t\in[2a-b,a],\\ \dfrac{(2b-a)-t}{b-a} & \text{if } t\in[b,2b-a],\\ 0 & \text{if } t\in[2b-a,2a-b+2\pi],\end{cases}$$

and then we extend it as a periodic function with period 2π defined on $\mathbb{R}$. By a well-known direct approximation theorem (see [5, p. 205], for example) there is a real trigonometric polynomial T_m of degree at most $m/2$ such that

$$\max_{t\in\mathbb{R}}|L_I(t)-T_m(t)| \le \frac{c_5}{m|I|} \le \frac{1}{2} \tag{3.1}$$

with an absolute constant $c_5 > 0$. Without loss of generality we may assume that $T_m(t) \geq 0$ for every $t \in \mathbb{R}$, hence $T_m(t) = |Q_m(e^{it})|$ with an appropriate algebraic polynomial Q_m of degree at most m. Note that $1/2 \leq |Q_m(z)| \leq 3/2$ for every $z = e^{it} \in I$.

Observe that

$$|f_p(z) - f_{p,m}(z)| \leq m, \qquad z = e^{it},\ t \in \mathbb{R}. \tag{3.2}$$

Using Lemma 1.1 and (3.2) we can deduce that

$$|f_{p,m}(z_p^j)| \leq |f_p(z_p^j)| + |f_{p,m}(z_p^j) - f_p(z_p^j)| \leq p^{1/2} + m, \quad j = 1, \ldots, p. \tag{3.3}$$

Combining the inequality

$$|a+b|^q \leq 2^{q-1}(|a|^q + |b|^q), \qquad a, b \in \mathbb{C},\ q \in [1, \infty),$$

with (3.2), and then recalling that $1/2 \leq |Q_m(z)|$ for all $z = e^{it} \in I$, we obtain

$$\begin{aligned} \int_I |f_p(z)|^q\,|dz| &\leq \int_I 2^{q-1}(|f_{p,m}(z)|^q + |f_p(z) - f_{p,m}(z)|^q)\,|dz| \\ &= 2^{q-1}\int_I |f_{p,m}(z)|^q\,|dz| + 2^{q-1}\int_I |f_p(z) - f_{p,m}(z)|^q\,|dz| \\ &\leq 2^{q-1}\int_I |f_{p,m}(z)|^q\,|dz| + 2^{q-1}m^q|I| \\ &\leq 2^{q-1}2^q\int_I |(f_{p,m}Q_m)(z)|^q\,|dz| + 2^{q-1}m^q|I|. \end{aligned} \tag{3.4}$$

Applying the Marcinkiewicz–Zygmund inequality to the polynomial

$$P := f_{p,m}Q_m$$

of degree at most $p-1$, then using (3.3), we obtain

$$\begin{aligned} \int_I |(f_{p,m}Q_m)(z)|^q\,|dz| &\leq c_4(q)\frac{1}{p}\sum_{j=1}^{p} |(f_{p,m}Q_m)(z_p^j)|^q \\ &\leq c_4(q)(p^{1/2}+m)^q\,\frac{1}{p}\sum_{j=1}^{p} |Q_m(z_p^j)|^q. \end{aligned} \tag{3.5}$$

Observe that (3.1) implies that

$$|Q_m(z_p^j)|^q \leq \begin{cases} 2^q, & z_p^j \in 3I, \\ \left(\dfrac{c_5}{m|I|}\right)^q, & z_p^j \notin 3I, \end{cases}$$

and there are at most $3p|I|/(2\pi) + 1$ values of $j = 1, \ldots, p$ for which $z_p^j \in 3I$.

Hence

$$\text{(3.6)}\qquad \frac{1}{p}\sum_{j=1}^{p}|Q_m(z_p^j)|^q \le \frac{1}{p}\left(2^q\left(\frac{3p|I|}{2\pi}+1\right)+\left(\frac{c_5}{m|I|}\right)^q p\right)$$
$$\le \left(2^q\left(\frac{3|I|}{2\pi}+\frac{1}{p}\right)+(2c_5)^q|I|\right) \le c_6(q)|I|$$

with a constant $c_6(q)$ depending only on q, whenever

$$\left(\frac{c_5}{m|I|}\right)^q \le (2c_5)^q|I|,$$

that is, whenever

$$1/m \le 2p^{-1/2} \le 2\,|I|^{1+1/q}.$$

Combining (3.4)–(3.6), and recalling that $m \le p^{1/2}$, we conclude

$$\frac{1}{|I|}\int_I |f_p(z)|^q\,|dz| \le \frac{4^q}{|I|}\Big(\int_I |(f_{p,m}Q_m)(z)|^q\,|dz|\Big)+2^q m^q$$
$$\le \frac{4^q}{|I|}c_4(q)(p^{1/2}+m)^q\,\frac{1}{p}\,\Big(\sum_{j=1}^{p}|Q_m(z_p^j)|^q\Big)+2^q m^q$$
$$\le 4^q c_4(q)2^q p^{q/2}c_6(q)+2^q m^q \le c_7(q)p^{q/2}$$

with a constant $c_7(q)$ depending only on q, whenever

$$1/m \le 2p^{-1/2} \le 2\,|I|^{1+1/q}.$$

So the theorem is proved for all $q>0$ satisfying

$$\frac{-1/2}{1+1/q} \le -\frac{1}{2}+\varepsilon,$$

hence for all $q > \varepsilon^{-1} > 2$, with a constant $c_2(q,\varepsilon)$ depending only on q and ε. ■

To prove Theorem 2.2 we follow [11]. Let $e(t) = \exp(2\pi i t)$. Our first lemma is Lemma 1 of [11].

Lemma 3.1. *Let χ be a primitive character (mod q), $q>1$. Then*

$$\sum_{m=0}^{q-1}\chi(m)e(m\alpha) = \tau(\chi)\,q^{-1}e\big(\tfrac{1}{2}q\alpha\big)(\sin(\pi q\alpha))T(\alpha,\overline{\chi}),$$

where $\tau(\chi)$ is the Gauss sum

$$\tau(\chi) = \sum_{n=1}^{q}\chi(n)e\Big(\frac{n}{q}\Big),$$

and

$$T(\alpha,\chi) = \sum_{a=1}^{q}\chi(a)\cot\Big(\pi\Big(\alpha-\frac{a}{q}\Big)\Big).$$

Note that if
$$\chi(n) = \left(\frac{n}{p}\right)$$
is the quadratic character, then Lemma 1.1 implies $\tau(\chi) = \varepsilon_p p^{1/2}$, and the content of Lemma 3.1 is just the identity obtained by expressing the Fekete polynomial f_p by the Lagrange interpolation formula associated with the pth root of unity. In fact, in the proof of Theorem 2.2 we will need Lemma 3.1 above only in the case when χ is the quadratic character.

Our second lemma is Lemma 2 of [11].

LEMMA 3.2. *Let p be a prime. For $k \geq 1$ let $a_1, \ldots, a_k$ be integers, distinct modulo p, and put $f(x) = \prod_{j=1}^{k}(x - a_j)$. Then*
$$\left|\sum_{n=1}^{p}\left(\frac{f(n)}{p}\right)\right| \leq (k-1)p^{1/2}.$$

Montgomery writes "This is a consequence of Weil's Riemann Hypothesis for the zeta function of a curve over a finite field; see Weil [14], [15]. The derivation of the particular bound above is given by Burgess ([2]; §2)."

Proof of Theorem 2.2. We rely heavily on Montgomery's beautiful line of proof in [11] to connect the two lemmas above to the proof of the theorem. Let $T(\alpha) := T(\alpha, \chi)$ with
$$\chi(h) = \left(\frac{h}{p}\right).$$
It follows from Lemma 1.1 that $|\tau(\chi)| = p^{1/2}$ and hence Lemma 3.1 implies
$$\left|f_p\left(e\left(\frac{2n+\delta}{2p}\right)\right)\right| \geq \frac{1}{\sqrt{2}}\,p^{-1/2}\left|T\left(\frac{2n+\delta}{2p}\right)\right| \tag{3.7}$$
for every $n = 1, \ldots, p$ and $\delta \in [1/2, 3/2]$. We define
$$W(n) := W_H(n) := \prod_{h=1}^{H}\left(1 - \left(\frac{n+h}{p}\right)\right)\prod_{h=0}^{H}\left(1 + \left(\frac{n-h}{p}\right)\right), \tag{3.8}$$
and compute the size of the weighted sum
$$\sum_{n=1}^{p} T\left(\frac{2n+\delta}{2p}\right)W(n)$$
for $\delta \in [1/2, 3/2]$. By multiplying the product (3.8) out, we have
$$W(n) = 1 + \sum_{f}\varepsilon_f\left(\frac{f(n)}{p}\right), \qquad \varepsilon_f \in \{-1, 1\},$$
where f runs through $2^{2H+1} - 1$ polynomials of the sort considered in

Lemma 3.2. Hence, using Lemma 3.2 we can deduce that

$$\sum_{n=1}^{p} W(n) = p + O(H2^{2H}p^{1/2}). \tag{3.9}$$

Similarly,

$$\sum_{n=1}^{p} W(n)\left(\frac{n-a}{p}\right) = c(a)p + O(H2^{2H}p^{1/2}), \tag{3.10}$$

where $c(a) = 1$ if $0 \le a \le H$, $c(a) = 0$ if $H < a < p-H$, and $c(a) = -1$ if $p - H \le a < p$. We have

$$\begin{aligned}(3.11)\qquad &\sum_{n=1}^{p} T\left(\frac{2n+\delta}{2p}\right)W(n) \\ &= \sum_{n=1}^{p}\sum_{a=1}^{p}\left(\frac{a}{p}\right)\cot\left(\pi\left(\frac{2n+\delta}{2p} - \frac{a}{p}\right)\right)W(n) \\ &= \sum_{a=1}^{p}\sum_{n=1}^{p}\left(\frac{n-a}{p}\right)W(n)\cot\left(\pi\left(\frac{2a+\delta}{2p}\right)\right) = \sum_{a=1}^{H} + \sum_{a=p-H}^{p} + \sum_{a=H+1}^{p-H-1}\end{aligned}$$

for every $\delta \in [1/2, 3/2]$. Using (3.10) and the facts that

$$\cot x = -\cot(\pi - x) = \begin{cases} x^{-1} + O(x) & \text{if } x \in (0, \pi/2], \\ -(\pi - x)^{-1} + O(\pi - x) & \text{if } x \in [\pi/2, \pi), \end{cases}$$

and

$$\sum_{a=H+1}^{p-H-1}\cot\left(\pi\left(\frac{2a+\delta}{2p}\right)\right) = O\left(\sum_{a=H+1}^{p-H-1}\frac{p}{a}\right) = O(p\log p),$$

we obtain

$$\begin{aligned}(3.12)\qquad &\sum_{a=1}^{H} + \sum_{a=p-H}^{p} + \sum_{a=H+1}^{p-H-1} \\ &= \frac{4p^2}{\pi}\sum_{a=1}^{H}\frac{1}{2a-1} + O(p^2) + O(H2^{2H}p^{1/2}p\log p) \\ &= \frac{2}{\pi}p^2\log H + O(p^2) + O(H2^{2H}p^{1/2}p\log p) = \frac{2}{\pi}p^2\log H + O(p^2)\end{aligned}$$

whenever $\delta \in [1/2, 3/2]$ and $2 \le H \le \frac{1}{8}\log p$. Combining (3.11) and (3.12), we conclude

$$\sum_{n=1}^{p} T\left(\frac{2n+\delta}{2p}\right)W(n) = \frac{2}{\pi}p^2\log H + O(p^2) \tag{3.13}$$

whenever $\delta \in [1/2, 3/2]$ and $2 \le H \le \frac{1}{8}\log p$.

Now let $A := A_{p,H}$ be the union of all intervals

$$\left[\frac{2n+1/2}{2p}, \frac{2n+3/2}{2p}\right]$$

with $W(n) := W_H(n) \neq 0$, $n = 1, \dots, p$. Let $B = B_{p,H} := \{e(t) : t \in A\}$. Note that

$$W(n) \in \{2^{2H}, 2^{2H+1}, 0\}, \qquad n = 1, \dots, p. \tag{3.14}$$

This, together with (3.9), implies that the linear measure of B can be estimated as

$$|B| \le \frac{p}{2^{2H}} \frac{2\pi}{2p} + O(Hp^{-1/2}) = (\pi + O(p^{-1/4}\log p))2^{-2H} \tag{3.15}$$

whenever $2 \le H \le \frac{1}{8}\log p$. Also $|B| \le 2\pi 2^{-2H}$ for all sufficiently large primes p and for all integers $2 \le H \le \frac{1}{8}\log p$. Using (3.7) we obtain

$$\begin{aligned}\int_B |f_p(z)|\,|dz| &= 2\pi \int_A |f_p(e(t))|\,dt \\ &= \frac{\pi}{p} \sum_{\substack{n=1\\W(n)\neq 0}}^{p} \int_{1/2}^{3/2} \left| f_p\left(e\left(\frac{2n+\delta}{2p}\right)\right)\right| d\delta \\ &\ge \frac{\pi}{p}\frac{1}{\sqrt{2}}p^{-1/2} \sum_{\substack{n=0\\W(n)\neq 0}}^{p} \int_{1/2}^{3/2} \left| T\left(\frac{2n+\delta}{2p}\right)\right| d\delta \\ &\ge \frac{\pi}{\sqrt{2}} p^{-3/2} \int_{1/2}^{3/2} \left(\sum_{\substack{n=1\\W(n)\neq 0}}^{p} T\left(\frac{2n+\delta}{2p}\right)\right) d\delta.\end{aligned} \tag{3.16}$$

Using (3.14) and (3.13) we can continue as follows:

$$\begin{aligned}&\frac{\pi}{\sqrt{2}} p^{-3/2} \int_{1/2}^{3/2} \left(\sum_{\substack{n=1\\W(n)\neq 0}}^{p} T\left(\frac{2n+\delta}{2p}\right)\right) d\delta \\ &\ge \frac{\pi}{\sqrt{2}} p^{-3/2} 2^{-(2H+1)} \int_{1/2}^{3/2} \left(\sum_{\substack{n=1\\W(n)\neq 0}}^{p} T\left(\frac{2n+\delta}{2p}\right) W(n)\right) d\delta \\ &\ge \frac{\pi}{\sqrt{2}} p^{-3/2} 2^{-(2H+1)} \left(\frac{2}{\pi} p^2 \log H + O(p^2)\right) \\ &\ge \frac{\pi}{\sqrt{2}} 2^{-(2H+1)} \left(\frac{2}{\pi} p^{1/2} \log H + O(p^{1/2})\right).\end{aligned} \tag{3.17}$$

Thus (3.16) and (3.17) imply

$$\int_B |f_p(z)|\,|dz| \geq \frac{\pi}{\sqrt{2}}\, 2^{-(2H+1)} \left(\frac{2}{\pi} p^{1/2} \log H + O(p^{1/2})\right). \tag{3.18}$$

Now let $8\pi p^{-1/8} \leq s \leq 2\pi$ be fixed. Without loss of generality we may assume that $s \leq 1$. Let $H \geq 2$ be the (only) integer such that

$$s/4 < 2\pi\, 2^{-2H} \leq s. \tag{3.19}$$

Then

$$H \leq \frac{\log p}{16 \log 2} \leq \frac{1}{8} \log p.$$

As $|B_{p,H}| \leq 2\pi\, 2^{-2H}$ for all sufficiently large primes p and for all integers $2 \leq H \leq \frac{1}{8} \log p$, there is a closed subset $E := E_{p,s}$ of the unit circle with linear measure s containing $B := B_{p,H}$. Then (3.18) and (3.19) imply that

$$\frac{1}{s} \int_E |f_p(z)|\,|dz| \geq c(p^{1/2} \log\log(1/s) + O(p^{1/2}))$$

with an absolute constant $c > 0$. ■

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Tamás Erdélyi
Department of Mathematics
Texas A&M University
College Station, TX 77843-3368, U.S.A.
E-mail: terdelyi@math.tamu.edu

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