Zero-sums of length \( kq \) in \( \mathbb{Z}_q^d \)

by

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1. Introduction. Let \( n \) and \( d \) be positive integers. A sequence \( A \) in \( \mathbb{Z}_n^d \) is called a zero-sum if the sum of all elements of \( A \) is zero in \( \mathbb{Z}_n^d \). By \( s_k(\mathbb{Z}_n^d) \) we denote the smallest integer \( t \) such that any sequence of length \( t \) in \( \mathbb{Z}_n^d \) contains a zero-sum of length \( kn \). The case \( k = 1 \), \( s_1(\mathbb{Z}_n^d) \) then denoted by \( f(n, d) \), was first studied by Harborth ([7]) and generated a lot of research. Already in 1961 the one-dimensional case had been solved by Erdős, Ginzburg and Ziv, which initiated a whole new branch in combinatorial number theory.

**Theorem** (P. Erdős, A. Ginzburg, A. Ziv, 1961 [3]). *For any positive integer \( n \) we have \( f(n, 1) = s_1(\mathbb{Z}_n) = 2n - 1 \).*

Kemnitz’ Conjecture \( f(n, 2) = s_1(\mathbb{Z}_n^2) = 4n - 3 \) (see [8]) was open for about twenty years and was recently proved by Reiher in [10]. The best result until then was the following:

**Theorem** (W. D. Gao, 2001 [4]). *Let \( q \) be a prime power. Then we have \( f(q, 2) = s_1(\mathbb{Z}_q^2) \leq 4q - 2 \) and \( s_2(\mathbb{Z}_q^2) \leq 4q - 2 \).*

This improves a result of Rónyai ([11]) who showed this only a little earlier for primes \( p \) instead of prime powers \( q \). Up to now the best general bounds for odd primes \( p \) and higher dimensions \( d \) are

\[
f(p, d) \geq 1.125^{[d/3]}2^d(p - 1) + 1,
\]

by Elsholtz ([2]), where \( 2^d(p - 1) + 1 \) is the trivial lower bound, and

\[
f(p, d) \leq (cd\log d)^d p
\]

by Alon and Dubiner ([1]). They conjectured that \( f(p, d) \leq c^dp \).

For \( k \neq 1 \) the constant \( s_k(\mathbb{Z}_n^d) \) was first studied by Gao and Thangadurai. They verified that \( s_k(\mathbb{Z}_p^3) = (k + 3)p - 3 \) for \( k \geq 4 \) (see [6]) and in higher dimensions \( s_k(\mathbb{Z}_q^d) = (k + d)q - d \) for \( k \geq q^{d-1} \) (see [5]).
The sequence consisting of \( kn - 1 \) copies of the zero-vector and \( n - 1 \) copies of each of the \( d \) basis vectors obviously does not contain a zero-sum of length \( kn \). Therefore we have

\[
s_k(\mathbb{Z}_n^d) \geq kn - 1 + (n - 1)d + 1 = (k + d)n - d.
\]

For \( k < d \) the above example can be extended by \( \left\lfloor \frac{d-k}{d-1} n \right\rfloor - 1 \) copies of the vector \( (1, \ldots, 1) \). So we get

\[
s_k(\mathbb{Z}_n^d) \geq (k + d)n - d + \left\lfloor \frac{d-k}{d-1} n - 1 \right\rfloor.
\]

Again this example can be improved by using vectors with exactly \( l \) (\( > k \)) entries 1 and the other entries 0 instead of the all-one vector. But as opposed to the case \( k = 1 \), where a simple example shows that \( s_1(\mathbb{Z}_n^d) > 2^d(n - 1) \), it is not obvious that for \( 2 \leq k < d \) the growth of \( s_k(\mathbb{Z}_n^d) \) is not linear in \( d \).

In this paper we suggest the following conjecture:

\textbf{Conjecture.} For positive integers \( k \geq d \) and \( n \) we have

\[
s_k(\mathbb{Z}_n^d) = (k + d)n - d.
\]

This has been proved by Gao ([5]) for prime powers \( n = q \) and \( k \geq q^{d-1} \) using Olson’s result about Davenport’s Constant ([9]). Here the Conjecture will be verified for a large class of smaller values of \( k \) in the case of general \( d \) (Theorems 2 and 4) as well as for \( d \leq 4 \) (Theorem 1).

These are our main results:

\textbf{Theorem 1.} Let \( p \) be a prime and \( q \) be a power of \( p \). For any positive integer \( k \) we have

1. \( s_k(\mathbb{Z}_q) = (k + 1)q - 1 \) (Gao, Thangadurai, 2003 [6]),
2. \( s_k(\mathbb{Z}_q^2) = (k + 2)q - 2 \) for \( k \geq 2 \) (Gao, Thangadurai, 2003 [6]),
3. \( s_k(\mathbb{Z}_q^3) = (k + 3)q - 3 \) for \( k \geq 3 \) and \( s_2(\mathbb{Z}_q^3) \leq 6q - 3 \), both for \( p > 3 \),
4. \( s_k(\mathbb{Z}_q^4) = (k + 4)q - 4 \) for \( k \geq 4 \) and \( p \geq 7 \) (actually for \( p \geq 5 \), if \( k \) is even), and \( s_2(\mathbb{Z}_q^4) \leq 8q - 4 \) and \( s_3(\mathbb{Z}_q^4) \leq 8q - 4 \), both for \( p \geq 5 \).

\textbf{Theorem 2.} Let \( p \) be a prime and \( q \) be a power of \( p \). Then the Conjecture holds for \( s_{np}(\mathbb{Z}_{q}^{d}) \), where \( n \) and \( d \) are any positive integers:

\[
s_{np}(\mathbb{Z}_{q}^{d}) = (np + d)q - d.
\]

Our next result is a general upper bound for \( s_k(\mathbb{Z}_q^d) \) with \( k \geq d \).

\textbf{Theorem 3.} Let \( d \) and \( k \geq d \) be positive integers, \( p > \min(2k, 2d) \) be a prime and \( q \) be a power of \( p \). Then

\[
s_k(\mathbb{Z}_q^d) \leq \left( \frac{3}{8} d^2 + \frac{3}{2} d - \frac{3}{8} + k \right) q - d.
\]

Certainly this could be improved with some additional effort but we do not see how to obtain an upper bound for \( s_k(\mathbb{Z}_q^d) \) with linear growth in \( d \).
As a corollary of Theorems 2 and 3 we prove the Conjecture for sufficiently large \( k \).

**Theorem 4.** Let \( d \) and \( k \) be positive integers, \( p > 2d \) be a prime and \( q \) be a power of \( p \). If \( \left[ \frac{k-d}{p} \right] p \geq \frac{3}{8}d^2 + \frac{d}{2} - \frac{3}{8} \), then the Conjecture holds for \( s_k(\mathbb{Z}_q^d) \).

The cases \( d = 1 \) and \( d = 2 \) are simple consequences of the Erdős–Ginzburg–Ziv Theorem and of the above theorem of Gao \( (s_2(\mathbb{Z}_q^2) \leq 4q - 2) \). To handle the other cases in the following sections we will generalize the method that Rónyai \( ([11]) \) developed to prove \( f(p, 2) \leq 4p - 2 \).

### 2. Rónyai’s method.

In order to prove \( f(p, 2) \leq 4p - 2 \), Rónyai \( ([11]) \) used special polynomial functions \( P : \{0, 1\}^m \to \mathbb{F}_p \), depending on the given sequence \( A \). For sufficiently large \( m = |A| \) there is an \( x \in \{0, 1\}^m \), \( x \neq 0 \), such that \( P(x) \neq 0 \). This \( x \) is related to a zero-sum with length \( p \) within \( A \).

In order to adapt these polynomials to prime powers \( q \) instead of primes \( p \) we had to change them a bit. Furthermore, in higher dimensions \( d > 2 \) this method can be generalized to prove that, for a given set \( \mathcal{L} = \{l_1, \ldots, l_{[d/2]}\} \), any sufficiently long sequence in \( \mathbb{Z}_q^d \) contains a zero-sum of length \( lq \) for at least one \( l \in \mathcal{L} \) and, iterating this, the existence of zero-sums of length \( kq \) for any given \( k \).

We use the following easy fact, proved e.g. by Rónyai \( ([11]) \).

**Lemma 2.1.** Let \( \mathbb{F} \) be a field and \( m \) be a positive integer. Then the monomials \( \prod_{i \in I} x_i, I \subseteq \{1, \ldots, m\} \), constitute a base of the \( \mathbb{F} \)-linear space of all functions \( f : \{0, 1\}^m \to \mathbb{F} \). (Here 0 and 1 are viewed as elements of \( \mathbb{F} \).)

Therefore any polynomial function \( P : \{0, 1\}^m \to \mathbb{F}_p, p > 2 \), has a unique representation of the form \( \sum_{I \subseteq \{1, \ldots, m\}} a_I \prod_{i \in I} x_i \). With respect to this representation we define the degree of \( P \) as

\[
\deg P = \max_{I \subseteq \{1, \ldots, m\}} \deg \left( \prod_{i \in I} x_i \right) = \max_{I \subseteq \{1, \ldots, m\}, a_I \neq 0} |I|.
\]

**Definition 1.** Let \( d > 1 \) be an integer and \( \mathcal{L} \subseteq \mathbb{N} \) be a set at least of cardinality \( \left[ \frac{d}{2} \right] \). An integer \( K \) is said to have Property (1) if

\[
|\mathcal{L} \cup (K - \mathcal{L})| \geq d.
\]

Here \( K - \mathcal{L} \) denotes the set \( \{K - l \mid l \in \mathcal{L}\} \).

Note that all \( K > 2 \max_{l \in \mathcal{L}} l \) have Property (1).

Now we can prove the following theorem.

**Theorem 2.1.** Let \( d > 1 \) be an integer, \( p \) be a prime and \( q \) be a power of \( p \). Let \( \mathcal{L} \) be a set of at least \( \left[ \frac{d}{2} \right] \) positive integers and \( K \) be an integer with
Property (1). If $p > \max(\{1, \ldots, K-1\} \setminus (L \cup (K-L)))$, then any sequence $(a_1, \ldots, a_{Kq})$ in $\mathbb{Z}_q^d$ with $\sum_{i=1}^{Kq} a_i = 0$ (in $\mathbb{Z}_q^d$) contains a zero-sum of length $lq$ for at least one $l \in L$.

Proof. Assume to the contrary that for no $l \in L$ there is a zero-sum of length $lq$ within $(a_1, \ldots, a_{Kq})$. Then there is no zero-sum of length $(K-l)q$, $l \in L$, either. So if there are zero-sums of length $kq > 0$ apart from the whole sequence, $k$ has to be in $J = \{1, \ldots, K-1\} \setminus (L \cup (K-L))$, $|J| \leq K-1-d$.

We define the polynomial function $P : \{0,1\}^{Kq} \to \mathbb{F}_p$ as

$$P(x) = Q(x) \prod_{\delta=1}^{d} R_{\delta}(x) \prod_{j \in J} S_j(x),$$

where

$$Q(x) = \left( g(x) - 1 \over q - 1 \right), \quad R_{\delta}(x) = \left( \sum_{i=1}^{Kq} a_i, \delta x_i - 1 \over q - 1 \right), \quad S_j(x) = \left( g(x) \over q \right) - j.$$  

Here $g(x)$ is the Hamming weight of $x \in \{0,1\}^{Kq}$,

$$g(x) = \sum_{i=1}^{Kq} x_i.$$  

Any vector $x \in \{0,1\}^{Kq}$ corresponds to a subsequence $B_x = (a_i)_{i=1}^{Kq}$ of length $g(x)$. Note that $P(x)$ vanishes in each of the following three cases:

1. $g(x)$ is not divisible by $q$ (because of $Q$),
2. the corresponding subsequence $B_x$ is not a zero-sum (because of $R_{\delta}$),
3. $B_x$ is of length $jq$ with $j \in J + p\mathbb{N}$ (because of $S_j$).

Therefore we get

$$P(x) = P(0)\chi_0(x) + P(1)\chi_1(x)$$

where $\chi_0(x) = \prod_{i=1}^{Kq} (1 - x_i)$ and $\chi_1(x) = \prod_{i=1}^{Kq} x_i$ are the characteristic functions of the all-zero resp. the all-one vector and

$$P(0) = \prod_{j \in J} (-j) = (-1)^{|J|}P(1).$$

So the degree of $P$ is at least $\deg P \geq Kq - 1$.

On the other hand the degree of $P$ can be determined via the representation as a linear combination of monomials one gets using the relations $x_i^2 = x_i$ ($x_i \in \{0,1\}$). Since this reduction cannot increase the degree, we have

$$\deg P \leq \deg Q + \sum_{\delta=1}^{d} \deg R_{\delta} + \sum_{j \in J} \deg S_j \leq (d+1)(q-1) + |J|q \leq Kq - d,$$

a contradiction to $\deg P \geq Kq - 1$. □
In a second step we will start with sequences which are not necessarily zero-sums of length $Kq$.

**Theorem 2.2.** Let $p$ be a prime, $q$ be a power of $p$ and $d > 1$ be an integer. Given a set $\mathcal{L} = \{l_1, \ldots, l_{[d/2]}\} \subset \mathbb{N}$ let $K_1 < \cdots < K_{[d/2]}$ be the $\left\lceil \frac{d}{2} \right\rceil$ smallest positive integers with Property (1). Define the set $J := \{1, \ldots, K_{[d/2]}\} \setminus (\mathcal{L} \cup \{K_1, \ldots, K_{[d/2]}\})$. Then for $m \geq (K_{[d/2]} + 1)q - d$ and $p > \max_{j \in J} j$ any sequence $(a_i)_{i=1}^m$ in $\mathbb{Z}_q^d$ has a zero-sum of length $lq$ for at least one $l \in \mathcal{L}$.

**Proof.** Assume to the contrary that a sequence $(a_i)_{i=1}^m$ contains no zero-sums of length $lq$ for any $l \in \mathcal{L}$. Then by Theorem 2.1 for any $K$ with Property (1) there are no zero-sums of length $Kq$ either. Now look at $P : \{0, 1\}^m \to \mathbb{F}_p$,

$$ P(x) = Q(x) \prod_{\delta=1}^d R_{\delta}(x) \prod_{j \in J} S_j(x) , $$

and proceed as above.

Theorem 2.2 has the following immediate consequences:

**Corollary 2.1.** For $q$ a power of the prime $p$ and an integer $d \geq 2$ let $(a_i)_{i=1}^m$ be a sequence in $\mathbb{Z}_q^d$.

1. If $m \geq (2d - \left\lceil \frac{d}{2} \right\rceil + 1)q - d$ and $p > d$, then $(a_i)$ contains a zero-sum of length $lq$ for at least one $l \in \{1, \ldots, \left\lceil \frac{d}{2} \right\rceil\}$.
2. If $m \geq (2d - \left\lceil \frac{d}{2} \right\rceil)q - d$ and $p \geq d$, then $(a_i)$ contains a zero-sum of length $lq$ for at least one $l \in \{1, \ldots, \left\lceil \frac{d}{2} \right\rceil\}$.
3. If $m \geq 2dq - d$ and $p \geq d + \left\lceil \frac{d}{2} \right\rceil$, then $(a_i)$ contains a zero-sum of length $lq$ for at least one $l \in \{1, \ldots, \left\lceil \frac{d}{2} \right\rceil - 1, d\}$.
4. If $m \geq (2d - \left\lceil \frac{d}{2} \right\rceil)q - d$ and $p > d$, then $(a_i)$ contains a zero-sum of length $lq$ for at least one $l \in \{1, \ldots, \left\lceil \frac{d}{2} \right\rceil + 1\}$.
5. If $m \geq 2dq - d$ and $p \geq 2d - 1$, then $(a_i)$ contains a zero-sum of length $lq$ for at least one odd $l \leq d$.
6. If $m \geq 2dq - d$ and $p \geq 2d - 1$, then $(a_i)$ contains a zero-sum of length $lq$ for at least one even $l \leq d + 1$.

**Proof.** This directly follows by an application of Theorem 2.2 with the following choice of the sets $J$:

1. $J = \{\left\lceil \frac{d}{2} \right\rceil + 1, \ldots, d\}$,
2. $J = \{\left\lceil \frac{d}{2} \right\rceil + 1, \ldots, d - 1\}$,
3. $J = \{\left\lceil \frac{d}{2} \right\rceil, \ldots, d + \left\lceil \frac{d}{2} \right\rceil - 1\} \setminus \{d\}$,
4. $J = \{\left\lceil \frac{d}{2} \right\rceil + 2, \ldots, d\}$,
5. $J = \{2, 4, \ldots, 2(d - 1)\}$,
6. $J = \{1, 3, \ldots, 2\left\lceil \frac{d}{2} \right\rceil - 1\} \cup \{2\left\lceil \frac{d}{2} \right\rceil + 2, 2\left\lceil \frac{d}{2} \right\rceil + 4, \ldots, 2(d - 1)\}$. **
A slightly weaker result than item (5) in the above corollary is due to Gao and Thangadurai ([6]) who showed that for primes \( p > 2 \) any sequence of length \( 2(d + 1)(p - 1) + 1 \) in \( \mathbb{Z}_p^d \) has a zero subsequence of length \( lp \) for some odd \( l \).

3. Proofs of our main results. To handle the higher-dimensional problem we combine the parts of Corollary 2.1 in order to ensure the existence of a zero-sum of length \( kq \) for a fixed \( k \geq d \) within a sufficiently large sequence in \( \mathbb{Z}_q^d \). We point out that a slightly weaker version of part (3) of Theorem 1 (for primes \( p > 3 \) and \( k \geq 4 \)) has been proved by Gao and Thangadurai in [6], using different methods.

Proof of Theorem 1. Since \( s_k(\mathbb{Z}_q^d) \geq (k + d)q - d \) (see introduction) we only have to show that the claimed constants are upper bounds.

(3) Let \( p > 3 \) and \( A \) be a sequence in \( \mathbb{Z}_q^3 \) of length \( 6q - 3 \). First we search for zero-sums of length \( 2q \) and \( 3q \). By Corollary 2.1(1), (3), \( A \) contains a zero-sum of length \( q \) or two zero-sums of length \( 2q \) and of length \( 3q \). In the first case there are \( 5q - 3 \) elements left, which provide by Corollary 2.1(1) another zero-sum of length \( 2q \) (then we are done) or of length \( q \). In this last case, to find a zero-sum of length \( 3q \), we apply Corollary 2.1(2) to the remaining \( 4q - 3 \) elements. So we have \( s_2(\mathbb{Z}_q^3), s_3(\mathbb{Z}_q^3) \leq 6q - 3 \). Therefore any sequence in \( \mathbb{Z}_q^3 \) of cardinality \((k + 3)q - 3 \) \((k \equiv 2, 3 \mod 3, k \geq 3)\) contains disjoint zero-sums, one of length \( 2q \) resp. \( 3q \) and \( \left\lfloor \frac{k-2}{3} \right\rfloor \) of length \( 3q \). We get \( s_k(\mathbb{Z}_q^d) \leq (k + 3)q - 3 \) for all \( k \equiv 2, 3 \mod 3, k \geq 3 \).

To show the upper bound for \( k = 4 \) take a sequence of length \( 7q - 3 \). Repeated application of Corollary 2.1(1) proves the existence either of a zero-sum of length \( 4q \) or of two zero-sums of lengths \( q \) and \( 2q \). In this second case \( 4q - 3 \) elements are left which by Corollary 2.1(2) contain a zero-sum of length \( q \), \( 2q \) or \( 3q \). Therefore we have \( s_k(\mathbb{Z}_q^d) \leq (k + 3)q - 3 \) for all \( k \equiv 1 \mod 3, k \geq 4 \).

(4) We get \( s_2(\mathbb{Z}_q^4) \leq 8q - 4 \) from Corollary 2.1(1), and Corollary 2.1(2) tells us \( s_4(\mathbb{Z}_q^4) \leq 8q - 4 \), both for \( p \geq 5 \).

Let now \( p \geq 7 \). To show \( s_3(\mathbb{Z}_q^4) \leq 8q - 4 \) let \( A \) be a sequence in \( \mathbb{Z}_q^4 \) of length \( 8q - 4 \). By Corollary 2.1(5) it contains a zero-sum of length \( q \) or \( 3q \). In the first case within the \( 7q - 4 \) remaining elements we find by Corollary 2.1(1) a zero-sum of length \( q \) or \( 2q \). If this again is a zero-sum of length \( q \), then the last \( 6q - 4 \) elements contain a zero-sum of length \( q \), \( 2q \) or \( 3q \) and so we are done.

Now we search for a zero-sum of length \( 5q \) within an arbitrary sequence in \( \mathbb{Z}_q^4 \) of length \( 9q - 4 \). We already know that there must be a zero-sum of length \( 3q \). By Corollary 2.1(2) the \( 6q - 4 \) remaining elements contain a zero-sum of length \( q \), \( 2q \) or \( 4q \). In the case of length \( 2q \) we are done. If
there is a zero-sum of length \( q \), we delete these \( q \) elements from the original sequence and because of \( s_4(\mathbb{Z}_q^4) \leq 8q - 4 \) we find a zero-sum of length \( 4q \) and so have a zero-sum of length \( 5q \). In the last case (i.e. of disjoint zero-sums of lengths \( 3q \) and \( 4q \)) we apply Theorem 2.1 to the zero-sum of length \( 7q \) and \( \mathcal{L} = \{1, 5\} \). So either we directly get a zero-sum of length \( 5q \) or in the case of length \( q \) we proceed as above. So we have shown \( s_5(\mathbb{Z}_q^4) \leq 9q - 4 \). Combining the results in this part we get \( s_k(\mathbb{Z}_q^4) = (k + 4)q - 4 \) for all \( k \geq 4 \).

Proof of Theorem 2. The proof of \( s_p(\mathbb{Z}_q^d) = (p + d)q - d \) is analogous to that of Theorem 2.2 with \( m = (p + d)q - d \) and \( P : \{0, 1\}^m \to \mathbb{F}_p \) defined as

\[
P = Q \prod_{\delta = 1}^{d} R_{\delta} \prod_{j = 1}^{p - 1} S_j
\]

where \( Q, R_{\delta} \) and \( S_j \) are as above. So, within a sequence of length \( (np + d)q - d \) there are \( n \) disjoint zero-sums of length \( pq \).

Proof of Theorem 3. First let \( k \) be in \( \{d, \ldots, 2d - 1\} \). The idea is to use Theorem 2.2 in order to extract \( \left\lceil \frac{d}{2} \right\rceil - 1 \) pairwise disjoint zero-sums of different lengths \( l_jq \) (\( \neq kq \)) first and then to find a zero-sum of length \( (k - l_j)q \) or \( kq \).

So let \( \mathcal{A} \) be a sequence in \( \mathbb{Z}_q^d \) of length

\[
(3 \dfrac{d^2}{8} + \dfrac{3}{2}d - \dfrac{3}{8} + k)q - d \geq \left(3 \dfrac{d^2}{8} - \dfrac{d}{2} + \dfrac{5}{8} + 2k\right)q - d.
\]

By Theorem 2.2 the sequence \( \mathcal{A}_1 := \mathcal{A} \) contains a zero-sum of length \( l_1q \) for at least one \( l_1 \in \mathcal{L}_1 := \{1, 2, \ldots, \left\lfloor \frac{d}{2} \right\rfloor - 1, k\} \). Let \( \mathcal{A}_2 \) be the sequence \( \mathcal{A}_1 \) without this zero-sum. So either \( \mathcal{A}_2 \) has length at least \( |\mathcal{A}_1| - (\left\lfloor \frac{d}{2} \right\rfloor - 1)q \) or we have already obtained a zero-sum of length \( kq \).

We use Theorem 2.2 with \( \mathcal{L}_j := \{1, 2, \ldots, \left\lfloor \frac{d}{2} \right\rfloor - 2 + j, k\} \) \( \setminus \{l_1, \ldots, l_{j-1}\} \) iteratively where \( \mathcal{A}_j \) is the sequence \( \mathcal{A}_{j-1} \) without the zero-sum of length \( l_{j-1}q \) until we have found a zero-sum of length \( kq \) or \( \left\lfloor \frac{d}{2} \right\rfloor - 1 \) pairwise disjoint zero-sums of lengths \( l_jq \). In both cases \( \mathcal{A}' := \mathcal{A}_{[d/2]} \) has length at least

\[
|\mathcal{A}| - \sum_{j = 1}^{\left\lceil \frac{d}{2} \right\rceil - 1} \left(\left\lceil \frac{d}{2} \right\rceil - 2 + j\right)q \geq \left(\left\lceil \frac{d}{2} \right\rceil - 2 + 2k\right)q - d.
\]

Therefore by Theorem 2.2 the sequence \( \mathcal{A}' \) contains a zero-sum of length \( l'q \) for at least one \( l' \in \mathcal{L}' := \{k - l_1, \ldots, k - \left\lceil \frac{d}{2} \right\rceil - 1, k\} \).

Note that in all these steps \( \max J \) does not exceed \( 2k \), so \( p > 2k \) guarantees \( p > \max J \).
Now if $k \geq 2d$, then $A$ contains $\lfloor \frac{k}{d} \rfloor - 1$ disjoint zero-sums of length $dq$ and because of
\[
\frac{3}{8} d^2 + \frac{3}{2} d - \frac{3}{8} + k - \left( \left\lfloor \frac{k}{d} \right\rfloor - 1 \right) d \geq \frac{3}{8} d^2 - \frac{d}{2} + \frac{5}{8} + 2 \left( k - \left( \left\lfloor \frac{k}{d} \right\rfloor - 1 \right) d \right) \leq 2d-1
\]
there is a zero-sum of length $(k - (\lfloor \frac{k}{d} \rfloor - 1)d)q$ within the remaining sequence.

**Proof of Theorem 4.** Let $A$ be a sequence in $\mathbb{Z}_q^d$ of length $(d + k)q - d$ where $k$ is of the form $np + r$ with $d \leq r \leq p + d - 1$ and $np \geq \frac{3}{8} d^2 + \frac{d}{2} - \frac{3}{8}$. Since $d + k = d + np + r \geq \frac{3}{8} d^2 + \frac{3}{2} d - \frac{3}{8} + r$ the given sequence $A$ contains by Theorem 3 a zero-sum of length $rq$. Within the remaining $(d + np)q - d$ elements there is by Theorem 2 a zero-sum of length $npq$.

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