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On the distribution of algebraic numbers with prescribed factorization properties

by

Maciej Radziejewski (Poznań)

1. Introduction. Our objective is to study oscillatory behaviour of the counting functions of various sets of algebraic numbers with prescribed factorization properties.

Let K be an algebraic number field of finite degree, $\mathcal{O}_K$ its ring of algebraic integers, and Γ a subgroup of $H^*(K)$, the class group of K in the narrow sense. We denote by S the semigroup of non-zero ideals of $\mathcal{O}_K$ whose classes belong to Γ. Such a semigroup is a special case of the generalized Hilbert semigroup defined by F. Halter-Koch [8, Beispiel 4] (cf. also [5]). In particular, for appropriate choices of Γ, we can have S isomorphic to the reduced multiplicative semigroup of $\mathcal{O}_K$ (the case studied most extensively) or the reduced semigroup of totally positive algebraic integers in K, with multiplication. S is a subset of the semigroup of non-zero ideals $\mathcal{I}(\mathcal{O}_K)$ and a Krull monoid (cf. [8]).

We denote the class group of S by $\mathrm{Cl}(S)$ and its class number by h. The characters of $\mathrm{Cl}(S)$ are numbered $\chi_0, \ldots, \chi_{h-1}$ with χ_0 denoting the principal character. We tacitly identify characters of $\mathrm{Cl}(S) \cong H^*(K)/\Gamma$ with the corresponding characters of $H^*(K)$ and $\mathcal{I}(\mathcal{O}_K)$. As usual, $s = \sigma + it$ denotes a complex variable. We write

$$\zeta(s,\chi) = \sum_{\mathfrak{a} \in \mathcal{I}(\mathcal{O}_K)} \frac{\chi(\mathfrak{a})}{\mathrm{N}(\mathfrak{a})^s}, \qquad \sigma > 1,$$

to denote the Hecke zeta function corresponding to $\chi \in \widehat{\mathrm{Cl}(S)}$. All such functions are in the Selberg class $\mathcal{S}$ (see, e.g., [13] or [12]) as $\chi \in \widehat{\mathrm{Cl}(S)}$ induces a primitive Hecke character on $\mathcal{I}(\mathcal{O}_K)$.

For any complex function $F(s)$ regular in a certain half-plane $\sigma > \sigma_0$ and non-vanishing in a half-plane $\sigma \geq \sigma_1 > \sigma_0$, and such that $\arg F(\sigma)$

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is close to 0 when σ is large, we choose the branch of $\log F(s)$ such that $\operatorname{Im}\log F(\sigma)$ is close to 0 when σ is large and extend it to the half-plane $\sigma > \sigma_0$ with cuts from the edge of the half-plane to the zeros of $F(s)$ in the unique way. In particular, $\log s$ will denote the principal branch of the logarithm. We let $\log \mathcal{S}$ denote the set of logarithms of functions from $\mathcal{S}$ (cf. [12]). The multiplicity of a zero of a complex function $F(s)$ at $s = \varrho$, $\varrho \in \mathbb{C}$, is written as $m(\varrho, F)$, or, in case $F(s) = \zeta(s,\chi)$, as $m(\varrho,\chi)$. The characteristic function of a set A is written as char_A.

For $\alpha \in S$ let $L(\alpha)$ denote the set of lengths of factorizations of α into irreducibles in S. Let M denote the set of irreducibles in S, M_k the set of products of k or less irreducibles (i.e. α such that $\min L(\alpha) \le k$), M_k' the set of products of k irreducibles (i.e. $k \in L(\alpha)$), and $M_{a,b}$, for $a, b \in \mathbb{N}$, $a \le b$, the set of $\alpha \in S$ with $L(\alpha) \subseteq [a,b]$. Let $G_{a,b}$ $(a,b \in \mathbb{N},\ a \le b)$ denote the set of $\alpha \in S$ with $|L(\alpha)| \in [a,b]$. The set $G_{1,m}$ is usually denoted as G_m, and $G_{m,m}$ as $\overline{G}_m$. We use the notation $G_{a,b}$ to treat both of these together.

For a set $A \subseteq S$ let $A(x)$ be the number of elements $\alpha \in A$ with $\mathrm{N}(\alpha) \le x$, and let

$$\zeta(s,A) = \sum_{\mathfrak{a}\in A} \frac{1}{\mathrm{N}(\mathfrak{a})^s}, \qquad \sigma > 1.$$

If the function $\zeta(s,A)$ is regular around $[1/2,1]$ except for the real points to the left of $1/2$, and $\mathcal{C}$ is a contour starting at $1/2-\delta$, for a small $\delta > 0$, going closely around $[1/2,1]$, counterclockwise, and back to $1/2-\delta$, we call

$$\mathcal{A}(x) = \frac{1}{2\pi i}\int_{\mathcal{C}} \zeta(s,A)\,\frac{x^s}{s}\,ds, \qquad x \ge 1,$$

the *main term* of $A(x)$, similarly to [11] and [12, Theorem 3]. For $x < 1$ we put $\mathcal{A}(x) = 0$. The asymptotic expansion of $\mathcal{A}(x)$ as x tends to infinity is usually quite complicated. We refer the reader to [11] for a detailed treatment of this problem. We show that the main terms corresponding to the sets M, M_k, M_k', $M_{a,b}$, and $G_{a,b}$, are well defined and denote them by $\mathcal{M}(x)$, $\mathcal{M}_k(x)$, $\mathcal{M}_k'(x)$, $\mathcal{M}_{a,b}(x)$, and $\mathcal{G}_{a,b}(x)$, respectively.

We say that a real, piecewise continuous function $f(x)$ *is subject to oscillations of lower logarithmic frequency* γ *and size* $x^{\theta-\varepsilon}$ (for $\gamma > 0$, $\theta \in \mathbb{R}$) if there exists an increasing sequence of positive real numbers $(x_n)_{n=1}^{\infty}$, $\lim_{n\to\infty} x_n = \infty$, such that:

(1) We have $f(x_n) \ne 0$ for each n and the signs of $f(x_n)$ alternate.

(2) If $V(Y)$ denotes the number of terms of (x_n) not exceeding Y, then

$$\liminf_{Y\to\infty} \frac{V(Y)}{\log Y} = \gamma.$$

(3) If $\varepsilon > 0$, then for any Y sufficiently large the segment $[Y^{1-\varepsilon}, Y]$ contains at least one element of (x_n).

(4) We have

$$\liminf_{n\to\infty} \frac{|f(x_n)|}{x_n^{\theta-\varepsilon}} = \infty$$

for every $\varepsilon > 0$.

The main arithmetic results of this paper are:

THEOREM 1. *The error terms* $M(x) - \mathcal{M}(x)$, $M_k(x) - \mathcal{M}_k(x)$ $(k \in \mathbb{N})$, $M'_k(x) - \mathcal{M}'_k(x)$ $(k \in \mathbb{N})$, *and* $M_{a,b}(x) - \mathcal{M}_{a,b}(x)$ $(a, b \in \mathbb{N},\ a \le b)$ *are subject to oscillations of positive lower logarithmic frequency and size* $x^{1/2-\varepsilon}$.

THEOREM 2. *Suppose* $h \ge 3$ *and let* $a, b \in \mathbb{N}$, $a \le b$. *If* $a \ge 2$, *or* $a = 1$ *and* b *is sufficiently large*, *then the error term* $G_{a,b}(x) - \mathcal{G}_{a,b}(x)$ *is subject to oscillations of positive lower logarithmic frequency and size* $x^{1/2-\varepsilon}$.

For a subset U of an additively written finite abelian group G let $\mathcal{F}(U)$ denote the free abelian monoid over U. Elements of $\mathcal{F}(U)$ are denoted formally $\prod_{g\in U} g^{\alpha_g}$ and called *sequences*. The block monoid over U consists of sequences $\prod_{g\in U} g^{\alpha_g}$ whose sum $\sum_{g\in U} \alpha_g g$ is zero, and is denoted $\mathcal{B}(U)$ (cf. [19] and [21, Chapter 9]). The set U is called *half-factorial* if the monoid $\mathcal{B}(U)$ is *half-factorial*, i.e., each element of $\mathcal{B}(U)$ has a unique length of factorization into irreducibles. A set U is half-factorial if and only if we have

$$\sum_{g\in U} \frac{\alpha_g}{\operatorname{ord} g} = 1$$

for each irreducible element $\prod_{g\in U} g^{\alpha_g}$ of $\mathcal{B}(U)$; cf. e.g. [28, 32] for some early results and [3] for a more recent treatment of half-factorial sets.

Let $\mu(G)$ be the maximum cardinality of a half-factorial subset of G. It is well known (cf. [1]) that $\mu(G) = |G|$ if and only if $h \le 2$. In the case $h \le 2$ the sets $G_{a,b}$ reduce either to $\emptyset$ or to S, otherwise they are non-empty proper subsets of S (cf. [29]). The remaining case of $G_{1,b}(x)$ for $h \ge 3$ and small b, not covered by Theorem 2, appears to be more difficult as we have neither sufficient knowledge about the structure of the set $G_{1,b}$ nor about the multiplicities of the zeros of $\zeta(s,\chi)$, $\chi \in \widehat{\mathrm{Cl}(S)}$.

Let $m(S)$ denote the smallest positive integer m such that for some complex non-real zeros $\varrho_1, \ldots, \varrho_q$ of $\prod_{\chi\in\mathrm{Cl}(S)} \zeta(s,\chi)$, and some $k_1, \ldots, k_q \in \mathbb{Z}$, we have

$$\sum_{j=1}^{q} k_j m(\varrho_j, \chi) = \begin{cases} m, & \chi = \chi_0, \\ 0, & \chi \in \widehat{\mathrm{Cl}(S)},\ \chi \ne \chi_0. \end{cases}$$

We also use the notation $m(K)$ if S is the semigroup of non-zero principal ideals of $\mathcal{O}_K$. Results of [12] imply $m(S) < \infty$. We show the existence of oscillations of $G_{1,b}(x)$ under additional assumptions on $m(S)$:

Theorem 3. *Suppose $h \geq 3$ and $b \in \mathbb{N}$. If $m(S)$ is not a multiple of $h/(h, \mu(\mathrm{Cl}(S)))$, then the error term $G_{1,b}(x) - \mathcal{G}_{1,b}(x)$ is subject to oscillations of positive lower logarithmic frequency and size $x^{1/2-\varepsilon}$.*

In particular, we get the required oscillations for all S such that $(m(S), h) = 1$ and $h \geq 3$. Using numerical computations we show

Theorem 4. *We have $m(K) = 1$ for K equal to $\mathbb{Q}(\alpha)$, $\mathbb{Q}(\beta)$, $\mathbb{Q}(\gamma)$, $\mathbb{Q}(\delta)$, and $\mathbb{Q}(\omega)$, where $\alpha^2 = -65$, $\beta^2 = -9982$, $\gamma^3 - \gamma^2 + 7\gamma + 8 = 0$, $\delta^3 - \delta^2 - 97\delta - 384 = 0$, and $\omega^2 = 26$.*

Corollary 1. *The error term $G_{1,b}(x) - \mathcal{G}_{1,b}(x)$, $b \in \mathbb{N}$, is subject to oscillations of positive lower logarithmic frequency and size $x^{1/2-\varepsilon}$ for the semigroups of non-zero principal integral ideals of $\mathbb{Q}(\alpha)$, $\mathbb{Q}(\beta)$, $\mathbb{Q}(\gamma)$, and $\mathbb{Q}(\delta)$, where $\alpha^2 = -65$, $\beta^2 = -9982$, $\gamma^3 - \gamma^2 + 7\gamma + 8 = 0$, and $\delta^3 - \delta^2 - 97\delta - 384 = 0$.*

Another approach to the problem of oscillations of $G_{1,b}(x) - \mathcal{G}_{1,b}(x)$ for small b is related to combinatorial properties of the class group $\mathrm{Cl}(S)$. Let G be a finite abelian group, $b \in \mathbb{N}$. Consider all half-factorial $U \subseteq G$ with $|U| = \mu(G)$ and sequences $F = \prod_{g \in G \setminus U} g^{\alpha_g} \in \mathcal{F}(G \setminus U)$ such that all blocks of the form $F \prod_{g \in U} g^{\beta_g}$ have at most b distinct factorization lengths in the block monoid $\mathcal{B}(G)$. The maximum of $\sum_{g \in G \setminus U} \alpha_g$ over all such U and F is denoted by $\psi(G, b)$, as in [4]. Obviously $0 \leq \psi(G, 1) \leq \psi(G, 2) \leq \cdots$. The value of $\psi(\mathrm{Cl}(S), b)$ is related to the first term in the asymptotic expansion of $G_{1,b}(x)$:

$$G_{1,b}(x) \sim Cx(\log x)^{-1+\mu(\mathrm{Cl}(S))/h}(\log\log x)^{\psi(\mathrm{Cl}(S),b)}$$

for a $C > 0$, provided $h \geq 3$ (cf. [4]).

Theorem 5. *Suppose $h \geq 3$ and $b \in \mathbb{N}$. If $\psi(\mathrm{Cl}(S), b) > 0$, then the error term $G_{1,b}(x) - \mathcal{G}_{1,b}(x)$ is subject to oscillations of positive lower logarithmic frequency and size $x^{1/2-\varepsilon}$.*

In [24] W. A. Schmid and the author prove that $\psi(G, 2) > 0$ for every finite abelian group G with at least three elements and that $\psi(G, 1) > 0$ for several classes of groups. We state

Conjecture. *The inequality $\psi(G, 1) > 0$ holds for every finite abelian group G with at least three elements.*

Our main technical result is Theorem 6 of Section 2 which allows us to establish the existence of non-real singularities of the zeta functions of the sets we study.

The asymptotics of $M(x)$ in the case of the multiplicative semigroup of $\mathcal{O}_K$ was found by P. Rémond [25, 26] and refined by J. Kaczorowski [11]. The counting functions of $G_{m,m}$ and $G_{1,m}$ (and of the corresponding subsets of $\mathbb{N}$) were investigated by W. Narkiewicz [16, 17, 18, 20] (cf. also [21]), R. Odoni [22], J. Śliwa [28, 29], J. Kaczorowski [11], A. Geroldinger [4], and, in more generality, by F. Halter-Koch [9], who also considered $M_k(x)$ and $M'_k(x)$ (see [10]). A general, axiomatic treatment of those and related sets is due to A. Geroldinger, F. Halter-Koch, and J. Kaczorowski [7, 6].

The first result on oscillations of counting functions of sets mentioned here was due to J. Kaczorowski and J. Pintz [14] who showed that $M(x)$ oscillates around its main term under additional assumptions implying the existence of singularities of $\zeta(s, M)$. J. Kaczorowski and A. Perelli [12] proved the same unconditionally. Their method is also sufficient to treat the sets M_k, M'_k, and $M_{a,b}$, whose zeta functions are essentially polynomials in $\log \mathcal{S}$. Zeta functions of G_m and related sets are combinations of such polynomials with complex powers of Hecke zeta functions corresponding to characters of $\mathrm{Cl}(S)$. A theorem that relates singularities of such functions to oscillations of the corresponding counting functions was demonstrated in [23] where the oscillations of $G_1(x)$ in the special case of the Hilbert semigroup modulo 5 were also treated.

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2. Existence of singularities. We need some further notation. Let $\Omega_X(\mathfrak{a})$ denote the number of prime divisors of $\mathfrak{a} \in S$ in the class $X \in \mathrm{Cl}(S)$, counted with multiplicities, $\Omega(\mathfrak{a})$ the number of all prime divisors. For $U \subseteq \mathrm{Cl}(S)$ and $A\colon \mathrm{Cl}(S) \setminus U \to \mathbb{N} \cup \{0\}$ we call the pair (U, A) a *system* (cf. [28]) and put

$$N_{U,A} = \{\mathfrak{a} \in S : \Omega_X(\mathfrak{a}) = A(X),\ X \in \mathrm{Cl}(S) \setminus U\}.$$

While $\langle U \rangle$ denotes the subgroup of $\mathrm{Cl}(S)$ generated by U, we use $\langle \chi | U \rangle$ for the scalar product of $\chi \in \widehat{\mathrm{Cl}(S)}$ and the characteristic function of $U \subseteq \mathrm{Cl}(S)$:

$$\langle \chi | U \rangle = \frac{1}{h} \sum_{X \in U} \chi(X).$$

We replace "$\chi \in \widehat{\mathrm{Cl}(S)}$" by "$\chi$" (and "$\psi \in \widehat{\mathrm{Cl}(S)}$" by "$\psi$") in the subscripts of sums or products. Likewise, we write $\sum_{X \notin U}$ instead of $\sum_{X \in \mathrm{Cl}(S) \setminus U}$ if U is a subset of $\mathrm{Cl}(S)$. The letter $\mathfrak{p}$ denotes prime ideals of $\mathcal{O}_K$ and $[\mathfrak{a}]$ is the class of an ideal $\mathfrak{a}$ in $\mathrm{Cl}(S)$. Since $\zeta(s, \chi_0)$ is the Dedekind zeta function, we also write it as $\zeta_K(s)$. Let D denote a region containing the set

$$\{s \in \mathbb{C} : \sigma \geq 1/2,\ t \neq 0\} \cup \{s \in \mathbb{C} : \sigma > 1/2,\ t = 0\}$$

such that each $\zeta(2s,\chi)$, $\chi \in \widehat{\mathrm{Cl}(S)}$, is regular and non-vanishing in D (in particular $1/2 \notin D$). See [21] for a specific zero-free region.

In this section we prove the following theorem:

THEOREM 6. *Let (U_i, A_i), $i = 1, \dots, n$, be systems such that all N_{U_i,A_i} are non-empty. Let $M = \max_{|U_i| \neq h} |U_i|$ and*

$$Z(s) = \sum_{i=1}^{n} \alpha_i \zeta(s, N_{U_i,A_i}), \qquad \sigma > 1, \tag{1}$$

where $\alpha_i \in \mathbb{C}$, with $\alpha_i > 0$ whenever $|U_i| = M$. If $\max_{|U_i|=M} \sum_{X \notin U_i} A_i(X) > 0$, then $Z(s)$ has infinitely many singularities in the strip $1/2 \le \sigma < 1$. If $M > 0$ and $m(S)$ is not a multiple of $h/(h, M)$, then $Z(s)$ has at least one singularity in $\{s \in \mathbb{C} : 1/2 \le \sigma < 1,\, t \neq 0\}$.

We make use of the following:

THEOREM 7 (Kaczorowski, Perelli [12]). *Let $\log F_1, \dots, \log F_N \in \log \mathcal{S}$ be linearly independent over $\mathbb{Q}$ and let P be a polynomial in N variables of positive degree with coefficients regular in a region Ω containing the set*

$$\{s \in \mathbb{C} : \sigma \ge 1/2,\, |t| \ge T_1\} \cup \{s \in \mathbb{C} : \sigma > 1,\, |t| < T_1\}$$

for some $T_1 > 0$. Then the function

$$p(s) = P(\log F_1(s), \dots, \log F_N(s), s)$$

has infinitely many singularities in the half-plane $\sigma \ge 1/2$.

LEMMA 1 ([23]). *Suppose $\varrho \in \mathbb{C}$ and $\eta > 0$. Every function F defined in the neighbourhood $|s - \varrho| \le \eta$ with the exclusion of the segment $[\varrho - \eta, \varrho]$ by*

$$F(s) = \sum_{j=1}^{m} (s - \varrho)^{w_j} P_j(\log(s - \varrho)),$$

where $m \ge 0$, $w_j \in \mathbb{C}$, and P_j are polynomials with coefficients regular in $|s - \varrho| \le \eta$, $j = 1, \dots, m$, can be uniquely represented in the form

$$F(s) = \sum_{j=1}^{m'} (s - \varrho)^{w'_j} Q_j(\log(s - \varrho))$$

with m', w'_j, and Q_j as m, w_j and P_j above, but w'_j $(j = 1, \dots, m')$ pairwise non-congruent mod $\mathbb{Z}$ *and the coefficients of Q_j $(j = 1, \dots, m')$ not all attaining the value 0 at ϱ. Each w'_j $(j = 1, \dots, m')$ is congruent* mod $\mathbb{Z}$ *to one of the w_j's. F can be analytically continued to a neighbourhood of ϱ if and only if either $m' = 0$ or $m' = 1$, w'_1 is a non-negative integer and Q_1 is of degree 0.*

We also need some other lemmas.

LEMMA 2. *Let Ω be the interior of $\{\sigma+it\in\mathbb{C}:\sigma>f(t)\}$ for a real, piecewise continuous function f. Suppose $F_1,\dots,F_k\in\mathcal{S}$ are regular in Ω. Let $G_1,\dots,G_m$ be regular in Ω and non-vanishing in a certain half-plane $\sigma>\sigma_0\ge 1$, $\lim_{\sigma\to\infty}\arg G_j(\sigma)=0$, $j=1,\dots,m$, $P_1,\dots,P_n$ polynomials with coefficients regular in Ω, and $\alpha_{i,j}$ $(i=1,\dots,n,\ j=1,\dots,m)$ complex numbers. If the function*

$$Z(s)=\sum_{i=1}^{n}\Big(\prod_{j=1}^{m}G_j(s)^{\alpha_{i,j}}\Big)P_i(\log F_1(s),\dots,\log F_k(s),s),\qquad \sigma>\sigma_0,$$

has a regular continuation in Ω, then

$$Z(s)=\sum_{i\in I}\Big(\prod_{j=1}^{m}G_j(s)^{\alpha_{i,j}}\Big)P_i(\log F_1(s),\dots,\log F_k(s),s),\qquad \sigma>\sigma_0,\tag{2}$$

where $I=\{i\in\{1,\dots,n\}:\sum_{j=1}^{m}\alpha_{i,j}m(\varrho,G_j)\in\mathbb{Z},\ \varrho\in\Omega\}$. Furthermore, if $I'\ne I$ is an equivalence class of the relation $\sim$ defined by

$$i\sim i'\Leftrightarrow\bigwedge_{\varrho\in\Omega}\sum_{j=1}^{m}\alpha_{i,j}m(\varrho,G_j)\equiv\sum_{j=1}^{m}\alpha_{i',j}m(\varrho,G_j)\ (\operatorname{mod}\mathbb{Z}),\qquad i,i'=1,\dots,n,$$

then

$$\sum_{i\in I'}\Big(\prod_{j=1}^{m}G_j(s)^{\alpha_{i,j}}\Big)P_i(\log F_1(s),\dots,\log F_k(s),s)=0,\qquad \sigma>\sigma_0.\tag{3}$$

If, moreover, Ω contains the set

$$\{s\in\mathbb{C}:\sigma\ge 1/2,\ |t|\ge T_1\}\cup\{s\in\mathbb{C}:\sigma>1,\ |t|<T_1\}$$

for a $T_1>0$, then

$$Z(s)=\sum_{i\in I}\Big(\prod_{j=1}^{m}G_j(s)^{\alpha_{i,j}}\Big)H_i(s),\qquad \sigma>\sigma_0,\tag{4}$$

where $H_i(s)$ denotes the constant term of the polynomial P_i.

Proof. Let Ω' denote the region obtained from Ω by making cuts from each zero of $\prod_{i=1}^{k}F_i(s)\prod_{j=1}^{m}G_j(s)$ in Ω towards the left, to the edge of Ω. Let $\varrho\in\Omega$. For s sufficiently close to ϱ, $\operatorname{Im}s<\operatorname{Im}\varrho$, we have $s\in\Omega'$ and

$$\Big(\prod_{j=1}^{m}G_j(s)^{\alpha_{i,j}}\Big)P_i(\log F_1(s),\dots,\log F_k(s),s)$$
$$=(s-\varrho)^{\sum_j\alpha_{i,j}m(\varrho,G_j)}P_{i,\varrho}(\log(s-\varrho),s),\qquad i=1,\dots,n,$$

where $P_{i,\varrho}$ are polynomials in $\log(s-\varrho)$ with coefficients regular in a neighbourhood of ϱ.

Consider sets $J \subseteq \{1, \ldots, n\}$ such that $I \subseteq J$ and

$$\sum_{i\in J}\Big(\prod_{j=1}^{m} G_j(s)^{\alpha_{i,j}}\Big) P_i(\log F_1(s), \ldots, \log F_k(s), s) = Z(s), \qquad s \in \Omega',$$

and choose any J_0 minimal among them. If $J_0 \neq I$, we pick $i_0 \in J_0 \setminus I$ and $\varrho \in \Omega$ such that

$$\sum_{j=1}^{m} \alpha_{i_0,j} m(\varrho, G_j) \notin \mathbb{Z}.$$

By Lemma 1 and the regularity at ϱ of

$$Z(s) = \sum_{i\in J_0} (s-\varrho)^{\sum_j \alpha_{i,j} m(\varrho,G_j)} P_{i,\varrho}(\log(s-\varrho), s)$$

we get

$$\begin{aligned} Z(s) &= \sum_{\substack{i\in J_0\\ \sum_j \alpha_{i,j} m(\varrho,G_j)\in\mathbb{Z}}} (s-\varrho)^{\sum_j \alpha_{i,j} m(\varrho,G_j)} P_{i,\varrho}(\log(s-\varrho), s) \\ &= \sum_{\substack{i\in J_0\\ \sum_j \alpha_{i,j} m(\varrho,G_j)\in\mathbb{Z}}} \Big(\prod_{j=1}^{m} G_j(s)^{\alpha_{i,j}}\Big) P_i(\log F_1(s), \ldots, \log F_k(s), s) \end{aligned}$$

in the neighbourhood of ϱ. The equality can be extended to Ω', contradicting the minimality of J_0. Hence $J_0 = I$ and (2) is proved.

If we consider I' of the second assertion, we may choose a minimal subset $J_1 \subseteq \{1, \ldots, n\}$ among those containing I' and such that

$$\sum_{i\in J_1}\Big(\prod_{j=1}^{m} G_j(s)^{\alpha_{i,j}}\Big) P_i(\log F_1(s), \ldots, \log F_k(s), s) = 0, \qquad s \in \Omega'.$$

We know that the set $\{1, \ldots, n\} \setminus I$ satisfies the above conditions (since I and I' are disjoint), so the family of sets to choose from is indeed non-empty. If assertion (3) were not satisfied, we could choose $i' \in I'$, $i'' \in J_1 \setminus I'$, and $\varrho \in \Omega$ such that

$$\sum_{j=1}^{m} \alpha_{i',j} m(\varrho, G_j) \not\equiv \sum_{j=1}^{m} \alpha_{i'',j} m(\varrho, G_j) \pmod{\mathbb{Z}}.$$

Then the sum

$$\sum_{i\in J_1} (s-\varrho)^{\sum_j \alpha_{i,j} m(\varrho,G_j)} P_{i,\varrho}(\log(s-\varrho), s) = 0$$

would contain powers of $s - \varrho$ with exponents in at least two classes $\bmod\ \mathbb{Z}$. By Lemma 1 the sum over each of these classes must vanish identically, contradicting the minimality of J_1, so (3) must hold.

Suppose now that Ω satisfies also the assumptions of the last assertion. The polynomial

$$P(z_1,\dots,z_k,s)=\sum_{i\in I}\Big(\prod_{j=1}^{m}G_j(s)^{\alpha_{i,j}+M}\Big)P_i(z_1,\dots,z_k,s),\qquad s\in\Omega',$$

has coefficients regular in Ω, provided M is a sufficiently large natural number. Without loss of generality we may assume that $\log F_1,\dots,\log F_r$ are linearly independent over $\mathbb{Q}$ and

$$\log F_{r+i}=L_i(\log F_1,\dots,\log F_r),\qquad i=1,\dots,k-r,$$

for some rational linear forms $L_1,\dots,L_{k-r}$. The regularity of

$$\begin{aligned}\Big(\prod_{j=1}^{m}G_j(s)^{M}\Big)Z(s)=P(\log F_1(s),\dots,\log F_r(s),L_1(\log F_1(s),\dots,\log F_r(s)),\\ \dots,L_{k-r}(\log F_1(s),\dots,\log F_r(s)),s),\qquad s\in\Omega',\end{aligned}$$

in Ω implies, in view of Theorem 7, that $P(z_1,\dots,z_r,L_1(z_1,\dots,z_r),\dots,L_{k-r}(z_1,\dots,z_r))$ is of degree 0, hence

$$\begin{aligned}\Big(\prod_{j=1}^{m}G_j(s)^{M}\Big)Z(s)&=P(0,\dots,0,L_1(0,\dots,0),\dots,L_{k-r}(0,\dots,0),s)\\ &=P(0,\dots,0,s),\qquad s\in\Omega',\end{aligned}$$

and (4) follows. ∎

LEMMA 3. *For $X\in\mathrm{Cl}(S)$, $z\in\mathbb{C}$, we have*

$$\sum_{\substack{\mathfrak{a}\in\mathcal{I}(\mathcal{O}_K)\\ \mathfrak{p}\mid\mathfrak{a}\Rightarrow\mathfrak{p}\in X}}\frac{z^{\Omega(\mathfrak{a})}}{\mathrm{N}(\mathfrak{a})^s}=\Big(\prod_{\psi}\zeta(s,\psi)^{z\overline{\psi(X)}}\Big)F_{X,z}(s),\qquad \sigma>1,$$

where $F_{X,z}(s)$ is regular and non-vanishing in $s\in D$.

Proof. Let

$$Z_X(s,z)=\sum_{\substack{\mathfrak{a}\in\mathcal{I}(\mathcal{O}_K)\\ \mathfrak{p}\mid\mathfrak{a}\Rightarrow\mathfrak{p}\in X}}\frac{z^{\Omega(\mathfrak{a})}}{\mathrm{N}(\mathfrak{a})^s},\qquad \sigma>1,\ z\in\mathbb{C},$$

and

$$P_X(s)=\sum_{\mathfrak{p}\in X}\frac{1}{\mathrm{N}(\mathfrak{p})^s},\qquad \sigma>1.$$

We have

$$\log Z_X(s,z)=zP_X(s)+\frac{z^2}{2}P_X(2s)+g_{X,z}(s),\qquad \sigma>1, \tag{5}$$

for $g_{X,z}(s)$ regular in $\sigma>1/3$. Substituting

$$P_X(s)=\frac{1}{h}\sum_{\chi}\overline{\chi(X)}\Big(\log\zeta(s,\chi)-\frac{1}{2}\,\zeta(2s,\chi^2)\Big)+g_X(s),\qquad \sigma>1,$$

in (5), $g_X(s)$ being regular in $\sigma>1/3$, we arrive at the desired conclusion. ∎

LEMMA 4. *For every system* (U,A) *we have*

$$\zeta(s,N_{U,A})=\Big(\frac{1}{h}\sum_{\chi}\chi(Y)\prod_{\psi}\zeta(s,\psi)^{\langle\chi\overline{\psi}|U\rangle}\prod_{X\in U}F_{X,\chi(X)}(s)\Big)$$
$$\times\prod_{X\notin U}P_{X,A(X)}(\log\zeta(s,\chi_0),\ldots,\log\zeta(s,\chi_{h-1}),s),\qquad \sigma>1,$$

where $Y=\prod_{X\notin U}X^{A(X)}$, $F_{X,z}(s)$ *is as in Lemma* 3, *and* $P_{X,m}$ $(m\ge 0)$ *is a polynomial of degree* m *in the first* h *variables*, *with coefficients regular in* $s\in D$ *and the coefficient at* $\log^m\zeta(s,\chi_0)$ *constant and equal to* $1/h^m m!$.

Proof. We have

$$\zeta(s,N_{U,A})=\Big(\frac{1}{h}\sum_{\chi}\chi(Y)\prod_{X\in U}Z_X(s,\chi(X))\Big)\prod_{X\notin U}Z_{X,A(X)}(s),\qquad \sigma>1,$$

where $Z_X(s,z)$ is as in the proof of Lemma 3 and

$$Z_{X,m}(s)=\sum_{\substack{\mathfrak{a}\in\mathcal{I}(\mathcal{O}_K)\\ \mathfrak{p}|\mathfrak{a}\Rightarrow\mathfrak{p}\in X\\ \Omega(\mathfrak{a})=m}}\frac{1}{\mathrm{N}(\mathfrak{a})^s},\qquad \sigma>1,\ m\in\mathbb{N}\cup\{0\}.$$

We have (cf. [11])

$$Z_{X,m}(s)=\sum_{k=0}^{m}\frac{1}{k!}\sum_{\substack{m_1=1\\ m_1+\cdots+m_k=m}}^{\infty}\cdots\sum_{m_k=1}^{\infty}\frac{1}{m_1\cdots m_k}P_X(m_1s)\cdots P_X(m_ks),\qquad \sigma>1,$$

with $P_X(s)$ as before. Substituting $P_X(s)$ again we get the assertion. ∎

Proof of Theorem 6. Without loss of generality we may assume $|U_i|<h$, $i=1,\ldots,n$, since the only summand possible with $|U_i|=h$ is $\zeta(s,N_{\mathrm{Cl}(S),0})=\zeta_K(s)$, which has no singularities other than the pole at $s=1$, hence does not affect the assertions. Let

$$Y_i=\prod_{X\notin U_i}X^{A_i(X)},\qquad i=1,\ldots,n.$$

The assumption $N_{U_i,A_i}\ne\emptyset$ implies that $Y_i\in\langle U_i\rangle$. We have

$$Z(s)=\frac{1}{h}\sum_{i=1}^{n}\sum_{\chi}\alpha_i\chi(Y_i)\Big(\prod_{\psi}\zeta(s,\psi)^{\langle\chi\overline{\psi}|U_i\rangle}\Big)\Big(\prod_{X\in U_i}F_{X,\chi(X)}(s)\Big)$$
$$\times\prod_{X\notin U_i}P_{X,A_i(X)}(\log\zeta(s,\chi_0),\ldots,\log\zeta(s,\chi_{h-1}),s),\qquad \sigma>1,$$

by Lemma 4. To simplify notation we write formally $P_i(\boldsymbol{l}_0,\ldots,\boldsymbol{l}_{h-1},s)$ instead of

$$\prod_{X\notin U_i} P_{X,A_i(X)}(\log\zeta(s,\chi_0),\ldots,\log\zeta(s,\chi_{h-1}),s).$$

Put $d=\max_{|U_i|=M}\sum_{X\notin U_i} A_i(X)$ and suppose that $d>0$ and, contrary to the first assertion, $Z(s)$ is regular in a region Ω containing the set

$$\{s\in\mathbb{C}:\sigma\ge 1/2,\ |t|\ge T_1\}\cup\{s\in\mathbb{C}:\sigma>1,\ |t|<T_1\},$$

for a $T_1>0$. Taking

$$I=\Big\{(i,\chi)\in\{1,\ldots,n\}\times\widehat{\mathrm{Cl}(S)}:\sum_{\psi} m(\varrho,\psi)\langle\chi\overline{\psi}|U_i\rangle\in\mathbb{Z},\ \varrho\in\Omega\Big\}$$

we have

$$Z(s)=\frac{1}{h}\sum_{(i,\chi)\in I}\alpha_i\chi(Y_i)\Big(\prod_{\psi}\zeta(s,\psi)^{\langle\chi\overline{\psi}|U_i\rangle}\Big)\Big(\prod_{X\in U_i}F_{X,\chi(X)}(s)\Big)H_{U_i,A_i}(s),\qquad \sigma>1,$$

where $H_{U_i,A_i}(s)=\prod_{X\notin U_i}P_{X,A_i(X)}(0,\ldots,0,s)$, by Lemma 2. Therefore, in the neighbourhood of $s=1$, we get

$$\sum_{(i,\chi)\in I}\alpha_i\chi(Y_i)(s-1)^{-\langle\chi|U_i\rangle}G_{i,\chi}(s)H_{U_i,A_i}(s) =\sum_{i=1}^{n}\sum_{\chi}\alpha_i\chi(Y_i)(s-1)^{-\langle\chi|U_i\rangle}G_{i,\chi}(s)P_i(\boldsymbol{l}_0,\ldots,\boldsymbol{l}_{h-1},s),$$

where $G_{i,\chi}(s)=(s-1)^{\langle\chi|U_i\rangle}\prod_{\psi}\zeta(s,\psi)^{\langle\chi\overline{\psi}|U_i\rangle}\prod_{X\in U_i}F_{X,\chi(X)}(s)$ is regular and non-vanishing in the neighbourhood of 1. We have $\langle\chi|U_i\rangle\le M/h$, $i=1,\ldots,n$, $\chi\in\widehat{\mathrm{Cl}(S)}$, and $\langle\chi|U_i\rangle=M/h$ if and only if $|U_i|=M$ and $\langle U_i\rangle\subseteq\ker\chi$, hence, by Lemma 1, we get

$$\sum_{\substack{(i,\chi)\in I\\ |U_i|=M\\ \langle U_i\rangle\subseteq\ker\chi}}\alpha_i(s-1)^{-M/h}G_{i,\chi}(s)H_{U_i,A_i}(s) +\sum_{\substack{(i,\chi)\in I\\ \langle\chi|U_i\rangle=M/h-1}}\alpha_i\chi(Y_i)(s-1)^{1-M/h}G_{i,\chi}(s)H_{U_i,A_i}(s)$$

$$=\sum_{\substack{|U_i|=M\\ \langle U_i\rangle\subseteq\ker\chi}}\alpha_i(s-1)^{-M/h}G_{i,\chi}(s)P_i(\boldsymbol{l}_0,\ldots,\boldsymbol{l}_{h-1},s) +\sum_{\langle\chi|U_i\rangle=M/h-1}\alpha_i\chi(Y_i)(s-1)^{1-M/h}G_{i,\chi}(s)P_i(\boldsymbol{l}_0,\ldots,\boldsymbol{l}_{h-1},s)$$

and consequently

$$\sum_{\substack{|U_i|=M\\ \langle U_i\rangle\subseteq\ker\chi}} \alpha_i G_{i,\chi}(s)(P_i(\boldsymbol{l}_0,\ldots,\boldsymbol{l}_{h-1},s)-\mathrm{char}_I(i,\chi)H_{U_i,A_i}(s)) \tag{6}$$
$$+(s-1)\sum_{\langle\chi|U_i\rangle=M/h-1}\alpha_i\chi(Y_i)G_{i,\chi}(s)$$
$$\times(P_i(\boldsymbol{l}_0,\ldots,\boldsymbol{l}_{h-1},s)-\mathrm{char}_I(i,\chi)H_{U_i,A_i}(s))=0.$$

The left side of (6) is a polynomial in $\log(s-1)$ with coefficients regular in the neighbourhood of 1. The value at $s=1$ of its coefficient at $\log^d(s-1)$ is

$$c=(-1)^d\sum_{\substack{|U_i|=M\\ \langle U_i\rangle\subseteq\ker\chi\\ \sum A_i(X)=d}}\frac{\alpha_i G_{i,\chi}(1)}{h^d}\prod_{X\notin U_i}(A_i(X)!)^{-1}.$$

For all i,χ such that $|U_i|=M$ and $\langle U_i\rangle\subseteq\ker\chi$ we have $\alpha_i>0$ and, for $\sigma>1$,

$$(\sigma-1)^{-\langle\chi|U_i\rangle}G_{i,\chi}(\sigma)=\Big(\prod_{\psi}\zeta(\sigma,\psi)^{\langle\overline{\psi}|U_i\rangle}\Big)\Big(\prod_{X\in U_i}F_{X,1}(\sigma)\Big)$$
$$=\sum_{\substack{\mathfrak{a}\in\mathcal{I}(\mathcal{O}_K)\\ \mathfrak{p}|\mathfrak{a}\Rightarrow[\mathfrak{p}]\in U_i}}\mathrm{N}(\mathfrak{a})^{-\sigma}>0,$$

where the last equality follows from Lemma 3. Since $G_{i,\chi}(1)\neq0$, the above implies $G_{i,\chi}(1)>0$. Therefore $c\neq0$, contradicting (6) in view of Lemma 1. The first assertion must therefore be true.

Assume now that $m(S)$ is not a multiple of $h/(h,M)$ and let $\varrho_1,\ldots,\varrho_q\in\mathbb{C}\setminus\mathbb{R}$ and $k_1,\ldots,k_q\in\mathbb{Z}$ be such that

$$\sum_{j=1}^{q}k_j m(\varrho_j,\chi)=\begin{cases}m(S), & \chi=\chi_0,\\ 0, & \chi\in\widehat{\mathrm{Cl}(S)},\ \chi\neq\chi_0.\end{cases}$$

We are free to assume $\mathrm{Re}\,\varrho_j\geq1/2$, $j=1,\ldots,q$, since $m(\varrho,\chi)=m(1-\varrho,\overline{\chi})$, $\chi\in\widehat{\mathrm{Cl}(S)}$, by the functional equation (cf. e.g. [15]). We also assume that there are no zeros ϱ of $\prod_\chi\zeta(s,\chi)$ other than $\varrho_1,\ldots,\varrho_q$ such that $\mathrm{Im}\,\varrho=\mathrm{Im}\,\varrho_j$ and $\mathrm{Re}\,\varrho>\mathrm{Re}\,\varrho_j$ for any j (if there are, we append them to $\varrho_1,\ldots,\varrho_q$). We are going to show that $Z(s)$ must have a singularity at one of the ϱ_j's at least.

To this end assume the converse and put

$$I'=\Big\{(i,\chi)\in\{1,\ldots,n\}\times\widehat{\mathrm{Cl}(S)}:\sum_{\psi}m(\varrho_j,\psi)\langle\chi\overline{\psi}|U_i\rangle\in\mathbb{Z},\ j=1,\ldots,q\Big\}.$$

We have

$$Z(s)=\frac{1}{h}\sum_{(i,\chi)\in I'}\alpha_i\chi(Y_i)\Big(\prod_{\psi}\zeta(s,\psi)^{\langle\chi\overline{\psi}|U_i\rangle}\Big)\times\Big(\prod_{X\in U_i}F_{X,\chi(X)}(s)\Big)P_i(\boldsymbol{l}_0,\ldots,\boldsymbol{l}_{h-1},s),\qquad \sigma>1,$$

using Lemma 2 again. Therefore, in a neighbourhood of $s=1$, we have

$$\sum_{(i,\chi)\notin I'}\alpha_i\chi(Y_i)(s-1)^{-\langle\chi|U_i\rangle}G_{i,\chi}(s)P_i(\boldsymbol{l}_0,\ldots,\boldsymbol{l}_{h-1},s)=0 \tag{7}$$

with $G_{i,\chi}(s)$ as before. Lemma 1 and (7) imply

$$\sum_{\substack{(i,\chi)\notin I'\\ |U_i|=M\\ \langle U_i\rangle\subseteq\ker\chi}}\alpha_iG_{i,\chi}(s)P_i(\boldsymbol{l}_0,\ldots,\boldsymbol{l}_{h-1},s)+(s-1)\sum_{\substack{(i,\chi)\notin I'\\ \langle\chi|U_i\rangle=M/h-1}}\alpha_i\chi(Y_i)G_{i,\chi}(s)P_i(\boldsymbol{l}_0,\ldots,\boldsymbol{l}_{h-1},s)=0 \tag{8}$$

for s close to 1. The left side of (8) is again a polynomial in $\log(s-1)$ and the value at 1 of its coefficient at $\log^d(s-1)$ is

$$c'=(-1)^d\sum_{\substack{(i,\chi)\notin I'\\ |U_i|=M,\ \sum A_i(X)=d\\ \langle U_i\rangle\subseteq\ker\chi}}\frac{\alpha_iG_{i,\chi}(1)}{h^d}\prod_{X\notin U_i}(A_i(X)!)^{-1}. \tag{9}$$

Each summand in (9) is positive. On the other hand, $c'=0$ by (8) and Lemma 1. Therefore, for each i_0 such that $|U_{i_0}|=M$ and $\sum A_{i_0}(X)=d$ we must have $(i_0,\chi_0)\in I'$, i.e.

$$\sum_{\psi}m(\varrho_j,\psi)\langle\overline{\psi}|U_{i_0}\rangle\in\mathbb{Z},\qquad j=1,\ldots,q.$$

However, there is at least one such i_0 and we have

$$\sum_{j=1}^{q}k_j\sum_{\psi}m(\varrho_j,\psi)\langle\overline{\psi}|U_{i_0}\rangle=\sum_{\psi}\Big(\sum_{j=1}^{q}k_jm(\varrho_j,\psi)\Big)\langle\overline{\psi}|U_{i_0}\rangle=m(S)\langle\chi_0|U_{i_0}\rangle=\frac{m(S)M}{h}\notin\mathbb{Z},$$

a contradiction. ∎

3. The constant $m(S)$. First we show that, indeed, $m(S) < \infty$.

LEMMA 5 (Kaczorowski, Perelli [12]). *Let* $\log F_1, \ldots, \log F_N \in \log \mathcal{S}$ *be linearly independent over* $\mathbb{Q}$ *and let* $\nu(\varrho) = (m(\varrho, F_1), \ldots, m(\varrho, F_N))$ *for every* $\varrho \in \mathbb{C}$. *Then there exist infinitely many disjoint* N*-tuples* $(\varrho_1, \ldots, \varrho_N)$ *of non-trivial zeros of* $\prod_{j=1}^{N} F_j(s)$, *with* $\operatorname{Re} \varrho_j \geq 1/2$ *for* $j = 1, \ldots, N$, *such that the vectors* $\nu(\varrho_1), \ldots, \nu(\varrho_N)$ *form a basis of* $\mathbb{R}^N$.

COROLLARY 2. *Let* $F_1, \ldots, F_N \in \mathcal{S}$ *and let* $\log F_1$ *be linearly independent of* $\log F_2, \ldots, \log F_N$ *over* $\mathbb{Q}$. *Then there exist some complex non-real zeros* $\varrho_1, \ldots, \varrho_q$ *of* $\prod_{i=1}^{N} F_i(s)$ *and* $k_1, \ldots, k_q \in \mathbb{Z}$ *such that*

$$\sum_{j=1}^{q} k_j m(\varrho_j, F_i) = \begin{cases} m, & i = 1, \\ 0, & i = 2, \ldots, N, \end{cases}$$

for certain $m \in \mathbb{N}$.

Proof. We may assume that $\log F_1, \ldots, \log F_r$ are linearly independent and $\log F_{r+1}, \ldots, \log F_N$ depend on $\log F_2, \ldots, \log F_n$. Then there must be some $\varrho_1, \ldots, \varrho_q \in \mathbb{C} \setminus \mathbb{R}$, $k_1, \ldots, k_q \in \mathbb{Z}$, and $m \in \mathbb{N}$ such that

$$\sum_{j=1}^{q} k_j m(\varrho_j, F_i) = \begin{cases} m, & i = 1, \\ 0, & i = 2, \ldots, r, \end{cases}$$

by Lemma 5. The remaining equalities follow from linear dependence. ■

COROLLARY 3. *We have* $m(S) < \infty$.

Proof. Because of the pole at 1, $\log \zeta(s, \chi_0)$ is linearly independent of $\log \zeta(s, \chi_1), \ldots, \log \zeta(s, \chi_{h-1})$, and we can apply the previous corollary. ■

In order to prove Theorem 4 we need some effective upper bounds for the derivatives of the Hecke zeta functions involved. For fields with a large discriminant one could obtain better asymptotic estimates using the method of K. Wiertelak [31].

LEMMA 6. *Let* n *be the degree of* K, d_K *the absolute value of the discriminant of* K, *and* χ *a character of the class group* $H(K)$. *Then we have*

$$\left| \frac{d^2}{ds^2} (s-1) \zeta_K(s) \right| \leq 4 \max(d_K \pi^{-n}, 2^n)(|t| + 3)^{n+1}$$

and

$$|\zeta'(s, \chi)| \leq \frac{4}{3} \max(d_K \pi^{-n}, 2^n)(|t| + 3)^n, \qquad \chi \neq \chi_0,$$

in the strip $1/4 \leq \sigma \leq 3/4$.

Proof. Let $r_1, 2r_2$ be the number of real, respectively complex, embeddings of K in $\mathbb{C}$. For all $\chi \in \widehat{H(K)}$ we have

$$|\zeta(3/2 + it, \chi)| \leq \zeta_K(3/2) \leq 2^n, \qquad t \in \mathbb{R},$$

and by the functional equation (cf. e.g. [15])

$$\begin{aligned}
&|\zeta(-1/2+it,\chi)| \\
&\quad = 2^{-2r_2} d_K \pi^{-n} \left|\frac{\Gamma\big(3/4-\frac{1}{2}it\big)}{\Gamma\big(-1/4+\frac{1}{2}it\big)}\right|^{r_1} \left|\frac{\Gamma(3/2-it)}{\Gamma(-1/2+it)}\right|^{r_2} |\zeta(3/2-it,\overline{\chi})| \\
&\quad = 2^{-2r_2} d_K \pi^{-n} \big|-1/4+\tfrac{1}{2}it\big|^{r_1} |(1/2+it)(-1/2+it)|^{r_2} \zeta(3/2-it,\overline{\chi}) \\
&\quad \le d_K \pi^{-n} (t^2+1/4)^{n/2}, \qquad t\in\mathbb{R}.
\end{aligned}$$

The function

$$F(s) = (s-5/2)^{-n-1}(s-1)\zeta_K(s)$$

is of finite order, regular in the strip $-1/2 \le \sigma \le 3/2$, and we have

$$|F(3/2+it,\chi)| \le 2^n, \qquad |F(-1/2+it,\chi)| \le d_K\pi^{-n}$$

for all $t\in\mathbb{R}$. Using the Phragmèn–Lindelöf theorem we get

$$|F(s)| \le \max(2^n, d_K\pi^{-n}), \qquad -1/2 \le \sigma \le 3/2.$$

Hence

$$|(s-1)\zeta_K(s)| \le \max(2^n, d_K\pi^{-n})(|t|+3)^{n+1}, \qquad -1/2 \le \sigma \le 3/2.$$

In a similar way we obtain

$$|\zeta(s,\chi)| \le \max(2^n, d_K\pi^{-n})(|t|+3)^{n}, \qquad -1/2 \le \sigma \le 3/2,\ \chi\ne\chi_0.$$

Using the formula

$$f^{(k)}(s_0) = \frac{k!}{2\pi i}\int_{\mathcal{C}} \frac{f(s)}{(s-s_0)^{k+1}}\,ds$$

for $f(s) = \zeta(s,\chi)$, $\chi \in \widehat{H(K)}$, s_0 in the strip $1/4 \le \operatorname{Re} s_0 \le 3/4$, and $\mathcal{C}$ a circle of radius $3/4$ and centre s_0 we obtain the assertions. ∎

Lemma 7. *Let $f(s)$ be a function regular at $s_0\in\mathbb{C}$, $f'(s_0)\ne 0$, and suppose $f(s)$ is regular in an open set containing the disc*

$$|s-s_0| \le 2\,\frac{|f(s_0)|}{|f'(s_0)|}$$

and $|f''(s)| \le M$, $M>0$, for all s in the disc. If

$$|f(s_0)| < \frac{|f'(s_0)|^2}{2M}$$

then $f(s)$ has a simple zero in the disc.

Proof. For $|s-s_0| = 2|f(s_0)|/|f'(s_0)|$ we have

$$|f(s)-(s-s_0)f'(s_0)| \le \tfrac{1}{2}M|s-s_0|^2 + |f(s_0)| < |(s-s_0)f'(s_0)|,$$

so the assertion follows from Rouché's Theorem. ∎

Using the PARI/GP system by C. Batut, K. Belabas, D. Bernardi, H. Cohen, and M. Olivier [30] and the ComputeL package by T. Dokchitser [2] we can find a zero of the appropriate Dedekind zeta function. The location of such zeros is given in Table 1. We list the generating polynomial of

Table 1. Zeros of $\zeta_K(s)$ ($K = \mathbb{Q}(\alpha)$, $f(\alpha) = 0$)

$f(x)$	$H(K)$	$\operatorname{Im} \varrho$
$x^2 - 26$	C_2	$1.370583964578\ldots$
$x^2 + 65$	$C_2 \oplus C_4$	$1.05325893699922446326153\ldots$
$x^2 + 9982$	$C_8 \oplus C_2^2$	$0.27659701748718108818108\ldots$
$x^3 - x^2 + 7x + 8$	C_6	$1.35047419556160885557154\ldots$
$x^3 - x^2 - 97x - 384$	$C_4 \oplus C_2$	$0.43063928124489314683107\ldots$

the field, the imaginary part of the first zero of $\zeta_K(s)$ (the real part was always 0.5 ± 10^{-18}), and the class group structure. The five cases studied include three quadratic fields and two non-normal, cubic fields. With Lemmas 6 and 7 we can verify (all the required inequalities being satisfied with ample margin of error) that each Dedekind zeta function considered has a simple zero close to the point we have found and that none of the other functions $\zeta(s, \chi)$ have zeros close to that point. Thus Theorem 4 is demonstrated. The PARI scripts used in the computations can be found at http://www.amu.edu.pl/~maciejr.

4. Applications. In this section we prove Theorems 1, 2, 3 and 5. We use an earlier result:

Theorem 8 ([23]). *Let $f(x)$ be a real, piecewise continuous function, defined for $x > 0$. Suppose the integral $\int_0^\infty f(x)x^{-s-1}\,dx$ is absolutely convergent in a half-plane $\sigma \geq \sigma_1$ with $\sigma_1 \in \mathbb{R}$. Let $F(s) = \int_0^\infty f(x)x^{-s-1}\,dx$ in that half-plane and let $\theta \in \mathbb{R}$ be the smallest number such that $F(s)$ can be continued analytically to a function regular in the half-plane $\sigma > \theta$. Assume that $F(s)$ can be analytically continued to a function regular in a larger half-plane $\sigma > \theta - c_0$ ($c_0 > 0$) with the exclusion of some horizontal cuts starting at its edge. The right ends of the cuts, denoted ϱ, contained in the strip $\theta - c_0 \leq \sigma \leq \theta$, having non-zero imaginary parts and no point of accumulation, are assumed to be singular points of $F(s)$, i.e., $F(s)$ cannot be extended further to a function regular at any of the ϱ. In the neighbourhood of radius $\eta_\varrho > 0$ of a singularity ϱ assume that, off the cut,*

$$F(s) = \sum_{j=1}^{m_\varrho} (s-\varrho)^{w_{\varrho,j}} P_{\varrho,j}(\log(s-\varrho)), \tag{10}$$

where $m_\varrho \geq 1$, $w_{\varrho,j} \in \mathbb{C}$, and $P_{\varrho,j}$ are polynomials with coefficients regular in the entire η_ϱ-neighbourhood of ϱ, $j = 1, \ldots, m_\varrho$. Let $\gamma = \min_{\mathrm{Re}\,\varrho=\theta} |\mathrm{Im}\,\varrho|$ and $\gamma = \infty$ if there are no singularities on the line $\sigma = \theta$. Then $f(x)$ is subject to oscillations of lower logarithmic frequency greater than or equal to γ/π and size $x^{\theta-\varepsilon}$.

Let E denote the neutral element of $\mathrm{Cl}(S)$. Consider the set $\mathcal{A}=\mathcal{A}(\mathrm{Cl}(S))$ of irreducible elements (atoms) of the block monoid $\mathcal{B}(\mathrm{Cl}(S))$. Let $\mathcal{A}'_k = \mathcal{A} \cdot \ldots \cdot \mathcal{A}$ (k times), $k \in \mathbb{N}$, $\mathcal{A}_k = \mathcal{A}'_1 \cup \cdots \cup \mathcal{A}'_k$, and let $\mathcal{A}_{a,b}$, $a, b \in \mathbb{N}$, $a \leq b$, be the set of the elements of $\mathcal{A}_b$ not contained in any $\mathcal{A}'_k$ for $k \notin [a, b]$. The set $\mathcal{A}$ is finite and so are $\mathcal{A}_k$, $\mathcal{A}'_k$, and $\mathcal{A}_{a,b}$. It is obvious that $\mathcal{A}$, $\mathcal{A}_k$ and $\mathcal{A}'_k$ are non-empty. $\mathcal{A}_{a,b}$ is also non-empty, as it contains E^a. Moreover, treating the blocks formally as functions from $\mathrm{Cl}(S)$ to $\mathbb{N} \cup \{0\}$, we get

$$M = \sum_{A\in\mathcal{A}} N_{\emptyset,A}, \quad M_k = \sum_{A\in\mathcal{A}_k} N_{\emptyset,A}, \quad M'_k = \sum_{A\in\mathcal{A}'_k} N_{\emptyset,A}, \quad k \in \mathbb{N},$$

$$M_{a,b} = \sum_{A\in\mathcal{A}_{a,b}} N_{\emptyset,A}, \quad a, b \in \mathbb{N},\ a \leq b.$$

From [28], [29], and [11] it follows (the arguments work in our, slightly more general, case without change) that, for $a, b \in \mathbb{N}$, $a \leq b$, there exist systems (U_i, A_i) and integers α_i, $i = 1, \ldots, m$, such that

$$\mathrm{char}_{G_{a,b}} = \sum_{i=1}^{m} \alpha_i \mathrm{char}_{N_{U_i,A_i}}, \tag{11}$$

$\alpha_{i_0} > 0$ for all i_0 such that $|U_{i_0}| = \max_i |U_i|$, and each U_i is half-factorial (cf. [27] or [28]).

If B is one of the sets M, M_k, M'_k, $M_{a,b}$, or $G_{a,b}$, then the above statements imply that $\zeta(s, B)$ is a finite combination of zeta functions of type $\zeta(s, N_{U,A})$ associated to systems (U, A). The proofs of Lemmas 3 and 4 show that for any system (U, A) the function $\zeta(s, N_{U,A})$ admits an analytic continuation to the half-plane $\sigma > 1/3$ with cuts from possible singularities (located at the zeros of $\prod_{\chi\in\widehat{\mathrm{Cl}(S)}} \zeta(s, \chi)\zeta(2s, \chi)$ or at 1 or 1/2) to the edge of the half-plane. The type of singularities is as described in Lemma 1. Now it suffices to see that, for the main term $\mathcal{B}(x)$ defined as before and the error term $E(x) = B(x) - \mathcal{B}(x)$, we have (cf. [14])

$$\int_0^\infty E(x)x^{-s-1}\,dx = \frac{1}{s}\,\zeta(s, B) - \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{1}{s-z}\,\frac{\zeta(z, B)}{z}\,dz, \quad \sigma > 1.$$

The function $\int_0^\infty E(x)x^{-s-1}\,dx$ is regular inside $\mathcal{C}$ and the difference

$$\frac{1}{s}\,\zeta(s, B) - \int_0^\infty E(x)x^{-s-1}\,dx$$

is regular outside $[1/2-\delta,1]$, therefore it suffices to prove the existence of a singularity of $\zeta(s,B)$ in $\{s\in\mathbb{C}:\sigma\geq 1/2,\ t\neq 0\}$ to prove the assertions of Theorems 1, 2, 3 and 5.

For Theorem 1 this is immediate from Theorem 6.

In the case $a\geq 2$ of Theorem 2 we use (11) and notice that for i_0 such that $|U_{i_0}|=\max_i|U_i|$ we must have $N_{U_{i_0},A_{i_0}}\cap G_{a,b}\neq\emptyset$. If we had $\sum_{X\notin U_{i_0}}A_{i_0}(X)=0$, then $N_{U_{i_0},A_{i_0}}\subseteq G_{1,1}$ by half-factoriality of U_{i_0}, hence $G_{1,1}\cap G_{a,b}\neq\emptyset$, a contradiction. Let us take a $U\subseteq\mathrm{Cl}(S)$ half-factorial, $|U|=\mu(\mathrm{Cl}(S))$, and any non-zero A: $\mathrm{Cl}(S)\setminus U\to\mathbb{N}\cup\{0\}$. We have $N_{U,A}\subseteq G_{1,b_0}$ for a $b_0\geq 1$ (cf. [28]). For all $b\geq b_0$ we have $N_{U,A}\subseteq G_{1,b}$ and $N_{U,A}$ has the maximum possible dimension, so it must be one of the summands of (11), and we have $\sum_{X\notin U}A(X)>0$ again. Theorem 2 is thus proven.

Theorem 3 is immediate from Theorem 6.

To prove Theorem 5 we note that if $\psi(\mathrm{Cl}(S),b)>0$ and if $U\subseteq\mathrm{Cl}(S)$ and $F=\prod_{g\in G\setminus U}g^{\alpha_g}$ are as in the definition of $\psi(\mathrm{Cl}(S),b)$, $\sum_{g\in G\setminus U}\alpha_g>0$, then $N_{U,F}\subseteq G_{1,b}$ and the assertion follows as in the proof of Theorem 2. ∎

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Faculty of Mathematics and Computer Science
Adam Mickiewicz University
Umultowska 87
61-614 Poznań, Poland
E-mail: maciejr@amu.edu.pl

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