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Sums of two relatively prime cubes

by

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1. Introduction. Let $V(x)$ be the number of solutions (u,v) in $\mathbb{Z}^2$ of

$$|u|^3 + |v|^3 \le x, \quad (u,v) = 1,$$

and let

$$E(x) = V(x) - \frac{4\Gamma^2(1/3)}{\pi^2 \Gamma(2/3)}\, x^{2/3}$$

be the error term in the asymptotic formula for $V(x)$.

Recent progress in estimating $E(x)$ has been conditional on the Riemann hypothesis (R.H.). It is known that, for any $\varepsilon > 0$,

$$E(x) = O(x^{331/1254+\varepsilon}) \tag{1.1}$$

if R.H. holds (Zhai and Cao [26]). Earlier bounds are due to Moroz [15], Nowak [19], Müller and Nowak [16], Nowak [17–19] and Zhai [25]. I shall prove

Theorem 1. *We have*, *subject to R.H.*,

$$E(x) = O(x^{\theta+\varepsilon}), \tag{1.2}$$

where $\theta = 9581/36864$.

For comparison,

$$331/1254 = 0.26395\ldots, \quad 9581/36864 = 0.25990\ldots.$$

The correct exponent in this problem is likely to be $2/9$ (see for example, Zhai [25]), which would make (1.2) an improvement of over 9% on (1.1).

In the first instance, the improvement depends on a decomposition of sums

$$\sum_{D<n\le D'} \mu(n) f(n) \tag{1.3}$$

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into "Type I" and "Type II" sums. The decomposition is a slight variant of that of Heath-Brown [7] for sums

$$\sum_{D<n\le D'} \Lambda(n)f(n).$$

Here the complex function f is arbitrary, and $1 < D < D' \le 2D$. The decomposition is more flexible than that of Montgomery and Vaughan [14], which is used in [25, 26]. See §2 for details.

The second component of the method is a collection of exponential sum estimates in two integer variables, which we assemble in §§3–5. It is helpful to compare these, and the way they are applied, with [25, 26]. The proof of Theorem 1 reduces to the upper estimation of the quantities $E_1(x)$, $E_2(x)$ introduced in §6. Theorem 2 is used to dispatch $E_2(x)$. When I wrote the first version of this paper, this was a substantial improvement (based on [21]) of the treatment in Zhai [25]. While the first version was being refereed, I found that Zhai and Cao [26] had given a similar treatment of $E_2(x)$. Clearly, then, the present paper is stronger than [26] through the treatment of $E_1(x)$. Zhai and Cao use only one method to estimate Type II sums

$$S(M,N) = \sum_{\substack{m\sim M \\ D<mn\le D'}} \sum_{n\sim N} a_m b_n e\left(\frac{x^{1/3}}{mn}\right),$$

namely Theorem 2 of [1], a "three variable" method. Since one of the variables reduces to the value 1, a refinement of the theorem is possible (Theorem 6 below). I deploy three further estimates for Type II sums (Lemmas 5, 7 and Theorem 5).

When it comes to Type I sums ($b_n \equiv 1$ in $S(M,N)$), Zhai and Cao treat the variable m trivially. I supplement this with Theorems 4 and 8. Moreover, the decomposition of (1.3) in [26] requires the Type II method to work for

$$N \in [D^{1/3}, D^{1/2}]$$

and the Type I method only for the "easy" range $N > D^{2/3}$. In contrast, for a particular range of D in the relevant interval $[x^{0.13\ldots}, x^{0.22\ldots}]$, I examine what ranges of N are accessible for Types I and II, and then choose a "decomposition result" from §2 to take advantage of this information.

I now comment briefly on Theorems 3, 4, 5, 7 and 8. The approach in Theorems 3–5 resembles [3], but the outcome is different because in [3] a "degeneracy" occurs. Theorem 7 is essentially a generalization of [6, Theorem 6.12] while Theorem 8 is an application of Theorem 7 to Type I sums.

At one point in [6], there is an implicit use of a relation

$$u\,\frac{\partial^2 f_1}{\partial u^2} \asymp \frac{\partial f_1}{\partial u}$$

which I could not verify (see the appeal to Lemma 6.10 for T_0 on page 84). The proof of Theorem 6 bypasses this difficulty. The argument also allows for another lacuna in [6]: the proof will not work unless $R = \sqrt{ZY/X} \geq 1$. Hence, in optimizing the estimate

$$S^2 \ll N^2 Z^{-1} + FN^{1/12} Z^{1/2} + \cdots + F^{1/2} N^{1/4} Y Z^{3/4}$$

on page 85, extra terms $FN^{1/2}X^{1/2}Y^{-1/2}$, $F^{1/2}N^{1/4}Y^{1/4}X^{3/4}$ must appear. (It should be emphasized that [6] is much clearer than any discussion of similar two-dimensional sums elsewhere in the literature.)

In §6, I recapitulate from the literature a decomposition

$$E(x) = E_1(x) + E_2(x) + E_3(x)$$

and use R.H. to dispatch $E_3(x)$, essentially as in [25]. The treatment of $E_2(x)$ is also contained in §6. In §7, I complete the proof of Theorem 1 with the treatment of $E_1(x)$.

We conclude this section with a few remarks on notation. We assume, as we may, that ε is sufficiently small. In later sections, real constants α, β, γ appear.

The symbol c is reserved for a sufficiently small positive constant depending at most on α, β, γ. Constants implied by "O" and "$\ll$" notations depend at most on α, β, γ and also (in §§1, 2, 3, 6, 7) on ε. We write $A \asymp B$ if

$$A \ll B \ll A.$$

The cardinality of a finite set E is denoted by $|E|$. The symbol D always denotes a large positive number, and D' satisfies $D < D' \leq 2D$. We write "$n \sim N$" as an abbreviation for "$N < n \leq 2N$". We reserve the symbols I, J for bounded real intervals.

2. Decomposition of sums involving the Möbius function. Let

$$Y = (2D)^{1/k},$$

where k is a natural number, $k \leq \varepsilon^{-1}$. Let

$$M(s) = \sum_{n \leq Y} \mu(n) n^{-s}.$$

It is easy to verify the identity

$$\frac{1}{\zeta(s)} = \sum_{j=1}^{k} (-1)^{j-1} \binom{k}{j} \zeta(s)^{j-1} M(s)^j + \zeta(s)^{-1}(1 - \zeta(s)M(s))^k. \tag{2.1}$$

This is nearly the same as (6) of [7], which we can recover from (2.1) by multiplying by $\zeta'(s)$.

Since

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}=\frac{1}{\zeta(s)}\quad(\mathrm{Re}\, s>1),$$

we can express the coefficient of $f(n)$ in the sum

$$S(f)=\sum_{D<n\le D'}\mu(n)f(n) \tag{2.2}$$

by picking out the coefficient of n^{-s} on the right in (2.1). The last term makes no contribution, since

$$1-\zeta(s)M(s)=\sum_{n>Y}a(n)n^{-s}$$

for suitable $a(n)$. On splitting up the ranges of summation into ranges $(N,2N]$ $(N\ge 1/2)$, we find that $S(f)$ is a linear combination of $O((\log D)^{2k-1})$ sums of the form

$$\sum_{n_i\sim N_i,\, D<n_1\dots n_{2k-1}\le D'}\mu(n_k)\dots\mu(n_{2k-1})f(n_1\dots n_{2k-1}), \tag{2.3}$$

where $\prod_{i=1}^{2k-1}N_i\asymp D$ and

$$2N_i\le Y\quad \text{if } i\ge k. \tag{2.4}$$

We may allow one or more of the N_i to be $1/2$, so that $n_i=1$. This explains why k is the same in (2.1), (2.3).

We now define a *Type I sum* to be a sum of the form

$$S_1=S_1(M,N)=\sum_{\substack{m\sim M\\ D<mn\le D'}}a_m\sum_{n\sim N}f(mn) \tag{2.5}$$

in which $a_m\ll m^{\varepsilon}$ for every $\varepsilon>0$. A *Type II sum* is a sum of the form

$$S_2=S_2(M,N)=\sum_{\substack{m\sim M\\ D<mn\le D'}}\sum_{n\sim N}a_mb_nf(mn) \tag{2.6}$$

in which $a_m\ll m^{\varepsilon}$, $b_n\ll n^{\varepsilon}$ for every $\varepsilon>0$.

LEMMA 1. *Let $0\le\alpha_1\le\dots\le\alpha_r$, $\alpha_1+\dots+\alpha_r=1$. For $S\subseteq\{1,\dots,r\}$, we write $S'=\{1,\dots,r\}\setminus S$ and*

$$\sigma_S=\sum_{i\in S}\alpha_i.$$

(i) *Let h be an integer, $h\ge 3$. Suppose that $\alpha_r\le 2/(h+1)$. Then some $\sigma_S\in[1/h,2/(h+1)]$.*
(ii) *Let $\lambda\ge 2/3$ and suppose that $\alpha_r\le\lambda$. Then some $\sigma_S\in[1-\lambda,1/2]$.*
(iii) *Let $\varrho\in(1/3,2/5]$ and $\tau=\min(1-2\varrho,3/10)$. Suppose that $\alpha_r\le\varrho$. Then some $\sigma_S\in[\tau,1/3]\cup[2/5,1/2]$.*

(iv) *Let* $\chi \le 1/5$ *and*

$$\psi \ge \max(1/3, 1/5 + 4\chi/5).$$

Suppose that $\alpha_r \le 2\chi$. *Then some* $\sigma_S \in [\chi, \psi]$.

Proof. In each case, we suppose that the conclusion is false and obtain a contradiction.

(i) Let T be the set of i for which $\alpha_i \in [0, 2/(h+1) - 1/h]$. Then $\sigma_T < 1/h$, for otherwise the least σ_S with $S \subseteq T$, $\sigma_S \ge 1/h$ would have

$$\sigma_S \le \frac{1}{h} + \left(\frac{2}{h+1} - \frac{1}{h}\right) = \frac{2}{h+1}.$$

Our next step is to show that $|T'| = h$. If $|T'| < h$, then

$$1 = \sigma_T + \sigma_{T'} < |T'|\, h^{-1} + h^{-1} \le 1,$$

which is absurd. So $|T'| \ge h$.

Let i, i' be distinct elements of T'. Then

$$\alpha_i + \alpha_{i'} \ge 2\left(\frac{2}{h+1} - \frac{1}{h}\right) \ge \frac{1}{h}.$$

Consequently, $\alpha_i + \alpha_{i'} > 2/(h+1)$. It follows that

$$\sigma_{T'} > \frac{|T'|}{2}\,\frac{2}{h+1} = \frac{|T'|}{h+1}. \tag{2.7}$$

Clearly $|T'| = h$. Now (2.7) yields

$$\sigma_T < \frac{1}{h+1}.$$

We can improve this bound further. Let $\alpha_i = \min_{j \in T'} \alpha_j$. Then

$$h\alpha_i + \sigma_T \le \sigma_{T'} + \sigma_T = 1.$$

Adding on the inequality

$$(h-1)\sigma_T < \frac{h-1}{h+1}$$

we obtain

$$h\alpha_i + h\sigma_T < 1 + \frac{h-1}{h+1} = \frac{2h}{h+1}.$$

Of course it follows that $\alpha_i + \sigma_T < 1/h$. Now

$$\sigma_T < \frac{1}{h} - \alpha_i < \frac{1}{h} - \left(\frac{2}{h+1} - \frac{1}{h}\right) = \frac{2}{h} - \frac{2}{h+1}. \tag{2.8}$$

Now let $\alpha_u = \max_{j \in T'} \alpha_j$. From (2.8),

$$\alpha_u + \sigma_T < \frac{1}{h} + \left(\frac{2}{h} - \frac{2}{h+1}\right) \le \frac{2}{h+1},$$

and it follows that $\alpha_u + \sigma_T < 1/h$. But now $\sigma_{T'} + \sigma_T \le h\alpha_u + \sigma_T < 1$, which is absurd.

(ii) It is clear at once from complementation that no $\sigma_S \in [1-\lambda, \lambda]$. Hence $\alpha_r \le 1 - \lambda \le 1/3$. From part (i), some $\sigma_S \in [1/3, 1/2]$, which is absurd.

(iii) Let T be the set of all i for which $\alpha_i \in [0, \tau)$. Then $\sigma_T < 2/5$. To see this, we prove in succession that $\sigma_S < 2/5$ for $S \subseteq T$, $|S| = 2, 3, \ldots$. For $|S| = 2$, we have $\sigma_S \le 2\tau \le 3/5$. From the hypothesis, it is clear that $\sigma_S \notin [2/5, 3/5]$. So $\sigma_S < 2/5$.

Suppose $\sigma_S < 2/5$ whenever $S \subseteq T$, $|S| = j$ (where $j \ge 2$). For $S \subseteq T$, $|S| = j + 1$, then

$$\sigma_S < \frac{j+1}{j}\,\frac{2}{5} \le \frac{3}{5},$$

hence $\sigma_S < 2/5$. This proves our claim that $\sigma_T < 2/5$.

We now have $\sigma_{T'} > 3/5$ and also

$$1/3 < \alpha_i \le \varrho \quad (i \in T').$$

It follows that $|T'| = 2$. Hence $2/3 < \sigma_{T'} \le 2\varrho$, and so $1 - 2\varrho \le \sigma_T < 1/3$. This is absurd.

(iv) Let T be the set of i for which $\alpha_i \le \psi - \chi$. Arguing as in (i) yields $\sigma_T < \chi$. Let $U = \{i : \alpha_i \in (\psi - \chi, \chi)\}$, $V = \{i : \alpha_i \in (\psi, 2\chi]\}$. Then

$$\sigma_U + \sigma_V = 1 - \sigma_T > 1 - \chi.$$

We cannot have $|V| \ge 2$, for if $|V| \ge 2$, pick distinct i, j in V and let $W = \{i, j\}'$; then $\chi \le 1 - 4\chi \le \sigma_W < 1 - 2\psi \le \psi$, which is absurd.

Suppose that $|V| = 1$, $V = \{i\}$. Then

$$\sigma_U > 1 - \chi - \sigma_V \ge 1 - 3\chi \ge 2\chi.$$

Hence $|U| \ge 3$. Pick distinct j, k in U and let $W = \{i, j, k\}$. Then, since $\psi \ge 1/4 + \chi/2$ (as we easily verify), we have

$$1 - \psi \le 3\psi - 2\chi < \sigma_W \le 4\chi \le 1 - \chi, \qquad \chi \le \sigma_{W'} < \psi.$$

This is absurd, so V is empty. Now $\sigma_U > 1 - \chi \ge 4\chi$. So $|U| \ge 5$. Pick distinct i, j, k, l in U and let $W = \{i, j, k, l\}$. Then

$$1 - \psi \le 4\psi - 4\chi < \sigma_W < 4\chi \le 1 - \chi,$$

leading to a contradiction once more.

We use this combinatorial lemma in conjunction with the familiar notion of grouping variables in (2.3).

LEMMA 2. *Let h, λ, ϱ, τ, χ, ψ be as in Lemma* 1. *Let $B > 0$ and let f be a complex function on $\mathbb{Z} \cap (D, D']$.*

(i) *Suppose that every Type I sum with*

$$N \gg D^{2/(h+1)}$$

satisfies

$$S_1(M,N) \ll B \tag{2.9}$$

and every Type II sum with

$$D^{1/h} \ll N \ll D^{2/(h+1)}$$

satisfies

$$S_2(M,N) \ll B. \tag{2.10}$$

Then

$$S(f) \ll B(\log 3D)^A \tag{2.11}$$

with $A = 2h - 1$.

(ii) *Suppose that every Type I sum with*

$$M \gg D^{\lambda}$$

satisfies (2.9), *and every Type II sum with*

$$D^{1-\lambda} \ll M \ll D^{1/2}$$

satisfies (2.10). *Then* (2.11) *holds with* $A = 3$.

(iii) *Suppose that every Type I sum with*

$$M \gg D^{\varrho}$$

satisfies (2.9), *and every Type II sum with*

$$D^{\tau} \ll M \ll D^{1/3} \quad or \quad D^{2/5} \ll M \ll D^{1/2}$$

satisfies (2.10). *Then* (2.11) *holds with* $A = 5$.

(iv) *Suppose that every Type I sum with*

$$M \gg D^{2\chi}$$

satisfies (2.9), *and every Type II sum with*

$$D^{\chi} \ll M \ll D^{\psi}$$

satisfies (2.10). *Then* (2.11) *holds with* $A = 5$.

Proof. (i) Take $k = h$ in (2.3), so that $1/k < 2/(h+1)$. We must show that every sum (2.3) is $\ll B$. If some $N_i > \varepsilon D^{2/(h+1)}$, we must have $i < k$ from (2.4). Now we group the variables in (2.3) as

$$n = n_i, \qquad m = \prod_{\substack{j=1 \\ j \neq i}}^{2k-1} n_j$$

and appeal to (2.9).

Now suppose that

$$N_j \le \varepsilon D^{2/(h+1)} \qquad (1 \le i \le 2k-1). \tag{2.12}$$

Let $D_0 = 2^{2k-1} N_1 \ldots N_{2k-1}$ and write

$$2N_i = D_0^{\alpha_i}.$$

Then $\alpha_i \ge 0$, $\alpha_1 + \cdots + \alpha_{2k-1} = 1$, $D \ll D_0 \ll D$, and each $\alpha_i \le 2/(h+1)$. By Lemma 1(i), we have $\sigma_S \in [1/h, 2/(h+1)]$ for some $S \subseteq T$. Clearly

$$D^{1/h} \ll N := \prod_{i \in S} N_i \ll D^{2/(h+1)}. \tag{2.13}$$

Thus we may group the variables in such a way that the sum (2.3) becomes a linear combination of $O(1)$ Type II sums satisfying (2.10). The desired estimate follows at once.

(ii) Take $k = 2$ in (2.3), so that $1/k < \lambda$. The argument is very similar to the proof of (i), with $N_i \le \varepsilon D^{\lambda}$ $(1 \le i \le 3)$ in place of (2.12), and with $D^{1-\lambda} \ll M = \prod_{i \in S} N_i \ll D^{1/2}$ in place of (2.13).

(iii), (iv) Take $k = 3$ in (2.3). The argument follows the same lines as above, and we can omit the details.

We conclude this section by recording an elementary lemma that will be used for "optimizations" in §§3–5.

LEMMA 3 ([6, Lemma 2.4]). *Let $t \ll 1$, $u \ll 1$, and*

$$L(H) = \sum_{i=1}^{t} A_i H^{a_i} + \sum_{j=1}^{u} B_j H^{-b_j}$$

where A_i, B_j, a_i, b_j are positive. Let $0 < H_1 \le H_2$. Then there is some $H \in [H_1, H_2]$ with

$$L(H) \ll \sum_{i=1}^{t} \sum_{j=1}^{u} (A_i^{b_j} B_j^{a_i})^{1/(a_i+b_j)} + \sum_{i=1}^{t} A_i H_1^{a_i} + \sum_{j=1}^{u} B_j H_2^{-b_j}.$$

3. Estimates for exponential sums

LEMMA 4. *Let β be a real constant, $\beta \le 4$, $\beta(\beta-1) \ne 0$. Let $M > 1/2$, $\delta > 0$. Let $\mathcal{N}(M, \delta)$ denote the number of integer quadruples $(m_1, m_2, \widetilde{m}_1, \widetilde{m}_2)$, $1 \le m_i \le M$, $1 \le \widetilde{m}_i \le M$, such that*

$$|m_1^{\beta} + m_2^{\beta} - \widetilde{m}_1^{\beta} - \widetilde{m}_2^{\beta}| \le \delta M^{\beta}. \tag{3.1}$$

Then

$$\mathcal{N}(M, \delta) \ll M^{2+\varepsilon} + \delta M^{4+\varepsilon}.$$

Proof. Robert and Sargos ([21, Theorem 2]) give the corresponding result for quadruples satisfying (3.1) and

$$m_i \sim M, \qquad \widetilde{m}_i \sim M \qquad (i = 1, 2).$$

(The restriction $\beta \le 4$ does not occur in their result.) We indicate the details of their argument that have to be changed in order to get Lemma 4.

Clearly we may suppose that M is a power of 2. By Lemma 1 of [21],

$$\mathcal{N}(M,\delta) \ll \delta \int_0^{\delta^{-1}} \left| \sum_{m=1}^{M} e\left(x\left(\frac{m}{M}\right)^{\beta}\right)\right|^4 dx.$$

By a splitting-up argument combined with Minkowski's inequality, there is an interval I of the form $(H, 2H]$, $H \le M$, or $[1,2]$, such that

$$\mathcal{N}(M,\delta) \ll L^4\delta \int_0^{\delta^{-1}} \left| \sum_{m\in I} e\left(x\left(\frac{m}{M}\right)^{\beta}\right)\right|^4 dx. \tag{3.2}$$

Here $L = \log(M+2)$. If $I = [1,2]$, then trivially $\mathcal{N}(M,\delta) \ll L^4\delta\delta^{-1} = L^4$. So we may suppose that $I = (H, 2H]$.

Make a change of variable $y = x(HM^{-1})^{\beta}$ in the above integral. We obtain

$$\mathcal{N}(M,\delta) \ll L^4\delta(HM^{-1})^{-\beta} \int_0^{\delta^{-1}(HM^{-1})^{\beta}} \left| \sum_{m\in I} e\left(y\left(\frac{m}{H}\right)^{\beta}\right)\right|^4 dy. \tag{3.3}$$

If the upper limit of integration satisfies $\delta^{-1}(HM^{-1})^{\beta} \le H^2$, then it follows from [21, Lemma 7] that

$$\mathcal{N}(M,\delta) \ll L^4\delta(HM^{-1})^{-\beta}H^{4+\varepsilon/2} \ll L^4\delta M^{4+\varepsilon/2} \ll \delta M^{4+\varepsilon},$$

since $\beta \le 4$.

Suppose now that $\delta^{-1}(HM^{-1})^{\beta} > H^2$. Then

$$\begin{aligned} &\int_0^{\delta^{-1}(HM^{-1})^{\beta}} \left| \sum_{m\in I} e\left(y\left(\frac{m}{H}\right)^{\beta}\right)\right|^4 dy \\ &\ll L^4\left\{\frac{\delta^{-1}(HM^{-1})^{\beta}}{H^2}\right\} \int_0^{H^2} \left| \sum_{m\in I} e\left(y\left(\frac{m}{H}\right)^{\beta}\right)\right|^4 dy \qquad \text{([21, Lemma 3])} \\ &\ll \delta^{-1}(HM^{-1})^{\beta}M^{2+\varepsilon/2} \end{aligned} \tag{3.4}$$

by a further application of [21, Lemma 7]. Combining (3.3), (3.4), we obtain $\mathcal{N}(M,\delta) \ll M^{2+\varepsilon}$. This completes the proof of Lemma 4.

We introduce the notation

$$|(m_1,m_2)|_{\alpha} = (|m_1|^{\alpha} + |m_2|^{\alpha})^{1/\alpha}$$

for an integer pair (m_1, m_2). The following theorem is roughly comparable to Lemma 4.2 of [26].

THEOREM 2. *Let (κ,λ) be an exponent pair. Let γ,β be constants, $\gamma<1$, $\gamma\neq 0$, $1<\beta\le 4$. Let $M\ge 1/2$, $1/2\le M_1\le M_2$, $X\gg M_2^2$. Let $|a_m|\le 1$, $|b_{m_1,m_2}|\le 1$, and*

$$S=\sum_{m\sim M}\sum_{m_1\sim M_1}\sum_{m_2\sim M_2} a_m b_{m_1,m_2} e\left(\frac{Xm^\gamma|(m_1,m_2)|_\beta}{M^\gamma M_2}\right).$$

Then

$$S\ll_\varepsilon MM_2^{1+\varepsilon}+MM_2^{2+\varepsilon}\left(\frac{X}{M_2^2}\right)^{\kappa/(2+2\kappa)}M^{-(1+\kappa-\lambda)/(2+2\kappa)}. \tag{3.5}$$

We remark that if $M_1=M_2$, Theorem 2 is of the same strength as the estimate for trilinear sums

$$\sum_{m\sim M}\sum_{m_1\sim M_1}\sum_{m_2\sim M_2} a_m b_{m_1,m_2} e\left(\frac{Xm^\gamma m_1^{\alpha_1}m_2^{\alpha_2}}{M^\gamma M_1^{\alpha_1}M_2^{\alpha_2}}\right)$$

obtained by Heath-Brown's method [8]. See for example [1, Theorem 2]. The estimate (3.5) deteriorates for fixed M_1 and increasing M_2, but this will not cost us anything in the application in §6.

Proof of Theorem 2. For $m_1\sim M_1$, $m_2\sim M_2$, we have

$$|(m_1,m_2)|_\beta\in[c_1M_2,c_2M_2]$$

where c_1, c_2 are suitable positive constants. Let Q be an arbitrary natural number. We divide $[c_1M_2,c_2M_2]$ into intervals $I_1,\dots,I_Q$ of equal length, so that

$$|S|\le\sum_{m\sim M}\sum_{q=1}^{Q}\Bigg|\sum_{\substack{m_i\sim M_i\\ |(m_1,m_2)|_\beta\in I_q}} b_{m_1,m_2} e\left(\frac{Xm^\gamma|(m_1,m_2)|_\beta}{M^\gamma M_2}\right)\Bigg|.$$

Cauchy's inequality gives

$$|S|^2\le MQ\sum_{q=1}^{Q}\sum_{\substack{\mathbf{m}\\ (3.7)}}\Bigg|\sum_{m\sim M} e\left(\frac{Xm^\gamma D(\mathbf{m})}{M^\gamma M_2}\right)\Bigg|, \tag{3.6}$$

where

$$\mathbf{m}=(m_1,m_2,\widetilde{m}_1,\widetilde{m}_2)\quad (m_j,\widetilde{m}_j\sim M_j),\quad D(\mathbf{m})=|(m_1,m_2)|_\beta-|(\widetilde{m}_1,\widetilde{m}_2)|_\beta$$

and the sum over $\mathbf{m}$ in (3.6) is restricted by

$$|(m_1,m_2)|_\beta\in I_q,\qquad |(\widetilde{m}_1,\widetilde{m}_2)|_\beta\in I_q. \tag{3.7}$$

Clearly

$$|S|^2\le MQ\sum_{\substack{\mathbf{m}\\ (3.9)}}\Bigg|\sum_{m\sim M} e\left(\frac{Xm^\gamma D(\mathbf{m})}{M^\gamma M_2}\right)\Bigg|, \tag{3.8}$$

where $\mathbf{m}$ is restricted by

$$|D(\mathbf{m})| \le (c_2 - c_1)\,\frac{M_2}{Q}. \tag{3.9}$$

A splitting-up argument yields

$$|S|^2 \ll MQL \sum_{\substack{\mathbf{m}\\(3.11)}} \Big|\sum_{m\sim M} e\Big(\frac{Xm^\gamma D(\mathbf{m})}{M^\gamma M_2}\Big)\Big|. \tag{3.10}$$

Here $L = \log 3M_2$ and the sum over $\mathbf{m}$ is restricted by

$$\Big(\Delta - \frac{1}{M_2^2}\Big)M_2 \le |D(\mathbf{m})| < 2\Delta M_2. \tag{3.11}$$

The positive number Δ is of the form

$$\Delta = 2^h M_2^{-2}, \qquad \Delta \ll Q^{-1}, \qquad h \ge 0. \tag{3.12}$$

Now it is easy to see that (3.11) implies (3.1) with M_2 in place of M and a suitable $\delta \ll \Delta$. Accordingly, the number of quadruples satisfying (3.11) is

$$\ll M_2^{2+\varepsilon} + \Delta M_2^{4+\varepsilon}.$$

If $h = 0$ in (3.12), we use the trivial estimate for the inner sum in (3.10). It follows that

$$|S|^2 \ll M^2 QLM_2^{2+\varepsilon}. \tag{3.13}$$

If $h > 0$, the number of quadruples satisfying (3.11) is $\ll \Delta M_2^{4+\varepsilon}$. By the definition of an exponent pair, we have

$$\sum_{m\sim M} e\Big(\frac{Xm^\gamma D(\mathbf{m})}{M^\gamma M_2}\Big) \ll (X\Delta)^\kappa M^{\lambda-\kappa} + (X\Delta M^{-1})^{-1}$$

for any quadruple $\mathbf{m}$ counted in (3.10). Here the contribution to the right-hand side of (3.10) from $(X\Delta M^{-1})^{-1}$ is

$$\ll MQL\Delta M_2^{4+\varepsilon} X^{-1}\Delta^{-1}M \ll M_2^{4+2\varepsilon}M^2X^{-1}Q \ll M^2QM_2^{2+2\varepsilon} \tag{3.14}$$

since $X \gg M_2^2$. The remaining contribution is

$$\ll MQ\Delta M_2^{4+2\varepsilon}(X\Delta)^\kappa M^{\lambda-\kappa} \ll M^{1+\lambda-\kappa}Q^{-\kappa}M_2^{4+2\varepsilon}X^\kappa \tag{3.15}$$

from (3.12). Collecting (3.13)–(3.15) gives

$$S \ll MM_2^{1+\varepsilon}Q^{1/2} + M^{(1+\lambda-\kappa)/2}M_2^{2+\varepsilon}X^{\kappa/2}Q^{-\kappa/2}. \tag{3.16}$$

Now the theorem follows on applying Lemma 3 with $H_1 = 1$ and arbitrarily large H_2.

LEMMA 5. *Let α, β be real constants with $\alpha\beta(\alpha-1)(\beta-1) \ne 0$, $X > 0$, $M \ge 1$, $N \ge 1$, $|a_m| \le 1$, $|b_n| \le 1$, and $1 < D < D'$. Let $L = \log(2+XMN)$.*

Let (κ,λ) *be an exponent pair and*

$$S(M,N)=\sum_{\substack{m\sim M\\ D<mn\le D'}}\sum_{n\sim N} a_m b_n e\left(\frac{Xm^\alpha n^\beta}{M^\alpha N^\beta}\right). \tag{3.17}$$

Then

$$\begin{aligned} S(M,N)\ll L^3\{&(X^{2+4\kappa}M^{8+10\kappa}N^{9+11\kappa+\lambda})^{1/(12+16\kappa)}\\ &+X^{1/6}M^{2/3}N^{3/4+\lambda/(12+12\kappa)}+(XM^3N^4)^{1/5}\\ &+(XM^7N^{10})^{1/11}+M^{2/3}N^{11/12+\lambda/(12+12\kappa)}\\ &+MN^{1/2}+(X^{-1}M^{14}N^{23})^{1/22}+X^{-1/2}MN\}. \end{aligned}$$

Proof. At the cost of a factor L, we can remove the condition $D<mn\le D'$ from the sum in (3.17). See the discussion in the proof of [2, Lemma 11]. Now the result follows at once from Theorem 2 of Wu [24]. (As pointed out by Wu, his theorem is essentially an abstraction of an idea of Jia [11].)

LEMMA 6. *Let* α_j *be nonzero constants and* $M_j\ge 1$ $(1\le j\le 4)$. *Let* $X>0$ *and* $|a_{m_1m_2}|\le 1$, $|b_{m_3m_4}|\le 1$. *Let* $L=\log 2M_1M_2M_3M_4$. *We have*

$$\begin{aligned}\sum_{\substack{m_j\sim M\\ D<m_1m_2m_3m_4\le D'}} & a_{m_1m_2}b_{m_3m_4}e\left(\frac{Xm_1^{\alpha_1}m_2^{\alpha_2}m_3^{\alpha_3}m_4^{\alpha_4}}{M_1^{\alpha_1}M_2^{\alpha_2}M_3^{\alpha_3}M_4^{\alpha_4}}\right)\\ &\ll L^2\{(XM_1M_2M_3M_4)^{1/2}+M_1M_2(M_3M_4)^{1/2}\\ &\qquad+(M_1M_2)^{1/2}M_3M_4+X^{-1/2}M_1M_2M_3M_4\}.\end{aligned}$$

Proof. We remove the condition $D<m_1m_2m_3m_4\le D'$ as explained in the preceding proof. Now the result follows from Theorem 2 of Fouvry and Iwaniec [5].

The key element of the proof of Theorem 2 of [5] is the double large sieve of Bombieri and Iwaniec [4]. The same applies to the following result of Robert and Sargos [21], but they need the difficult "counting lemma" stated as Lemma 4, above.

LEMMA 7. *Let* α,β,γ *be constants*, $\alpha(\alpha-1)\beta\gamma\ne 0$. *Let* H,M,N *be positive integers and* $X>1$. *Let* $|a_{h,n}|\le 1$, $|b_m|\le 1$. *Then*

$$\begin{aligned}\sum_{h\sim H}\sum_{\substack{n\sim N\\ D<mn\le D'}} a_{h,n}\sum_{m\sim M} b_m e\left(\frac{Xh^\beta n^\gamma m^\alpha}{H^\beta N^\gamma M^\alpha}\right)\\ \ll (HNM)^{1+\varepsilon}\left\{\left(\frac{X}{HNM^2}\right)^{1/4}+\frac{1}{(HN)^{1/4}}+\frac{1}{M^{1/2}}+\frac{1}{X^{1/2}}\right\}.\end{aligned}$$

Proof. After the preliminary removal of the condition $D < mn \le D'$ as above, this reduces to Theorem 1 of [21].

4. Applications of the theory of exponent pairs. We begin with a simple lemma from [12] (Lemma 2.8).

LEMMA 8. *Let $K > 0$ and let f be a continuously differentiable function on $[N, 2N]$ with*

$$F_0 = \max_{N \le t \le 2N} |f(t)|, \qquad F_1 = \max_{N \le t \le 2N} |f'(t)| > 0.$$

Then

$$\sum_{n \sim N} \min\left\{K, \frac{1}{\|f(n)\|}\right\} \ll (F_0 + 1)(K + F_1^{-1} \log 2N).$$

We next state a version of the "B-process".

LEMMA 9. *Suppose that f has four continuous derivatives on $[N, 2N]$, $f'' > 0$ on $[N, 2N]$, and for some $F > 0$,*

$$|f^{(j)}(t)| \asymp FN^{-j} \qquad (t \sim N,\ 2 \le j \le 4).$$

Let t_ν be defined by $f'(t_\nu) = \nu$, and let $\phi(\nu) = -f(t_\nu) + \nu t_\nu$. Let $L = \log(FN^{-1} + 2)$. Then for $[a, b] \subseteq [N, 2N]$,

$$\sum_{a \le n \le b} e(f(n)) = \sum_{f'(a) \le \nu \le f'(b)} \frac{e(\phi(\nu) + 1/8)}{f''(t_\nu)^{1/2}} + O(L + \min(F^{-1/2}N, \|f'(a)\|^{-1}) + \min(F^{-1/2}N, \|f'(b)\|^{-1})). \tag{4.1}$$

Proof. This can easily be obtained by an elaboration of the proof of [6, Lemma 3.6] with $-f$ in place of f, and $-\nu$ in place of ν. We simply separate the smallest term from the sum

$$\sum_{H_1 \le \nu \le H_2} \min(|-f'(a) + \nu|^{-1}, F^{-1/2}N)$$

on p. 29, and proceed similarly with b in place of a.

We now add to the hypothesis of Lemma 9 the assumption that

$$f^{(j)}(t) = \binom{\alpha}{j} A t^{\alpha - j}(1 + R_j(t)) \qquad (0 \le j \le 2)$$

where $\alpha < 0$, A is positive and independent of t, and

$$|R_j(t)| < c^2 \qquad (a \le t \le b,\ 0 \le j \le 2).$$

Then

$$|\alpha| A t_\nu^{\alpha - 1}(1 + R_1(t_\nu)) = -\nu > 0.$$

Hence

$$t_\nu = (|\alpha| A)^{-1/(\alpha-1)}(1+R(t_\nu))(-\nu)^{1/(\alpha-1)},$$

with $R(t) = (1+R_1(t))^{-1/(\alpha-1)} - 1$. So

$$\begin{aligned} f''(t_\nu) &= \frac{1}{2}\,\alpha(\alpha-1)At_\nu^{\alpha-2}(1+R_2(t_\nu)) \\ &= \tau A^{1/(\alpha-1)}(-\nu)^{(\alpha-2)/(\alpha-1)}(1+R^*(t_\nu)) \end{aligned}$$

with $\tau = -|\alpha|^{1/(\alpha-1)}(\alpha-1)/2,\ R^*(t) = (1+R_2(t))(1+R(t))^{\alpha-2} - 1$.

Similarly,

$$\begin{aligned} (4.2)\quad \phi(\nu) &= -At_\nu^\alpha(1+R_0(t_\nu)) - (|\alpha|A)^{-1/(\alpha-1)}(1+R(t_\nu))(-\nu)^{\alpha/(\alpha-1)} \\ &= \gamma A^{-1/(\alpha-1)}(-\nu)^{\alpha/(\alpha-1)}(1+\widehat{R}(t_\nu)) \end{aligned}$$

with $\gamma = -|\alpha|^{-\alpha/(\alpha-1)} - |\alpha|^{-1/(\alpha-1)}$ and

$$\gamma(1+\widehat{R}(t)) = -|\alpha|^{-\alpha/(\alpha-1)}(1+R_0(t))(1+R(t))^\alpha - |\alpha|^{-1/(\alpha-1)}(1+R(t)).$$

The point is that

$$\max(|R^*(t)|, |\widehat{R}(t)|) < c \quad (a \le t \le b).$$

We now permit A to depend on a variable u:

$$A = Cg(u) \qquad (u \sim M,\ M \ll N)$$

where $C > 0$,

$$(4.3)\qquad g(u) = u^\beta\Big(1 + \sum_{j\ge1} d_j(B/u)^j\Big) \qquad (u \sim M)$$

where β is a nonzero constant,

$$(4.4)\qquad \alpha+\beta<1, \qquad 0<B<cM,$$

and the power series $\sum_j d_j z^j$ converges in the unit disc. Writing $h(t) = t^\alpha(1+R_0(t))$, we have

$$f(t,u) = Ch(t)g(u)$$

in place of $f(t)$. Let $F = CM^\beta N^\alpha$. We rewrite (4.1), with $[a(m), b(m)]$ in place of $[a,b]$, as

$$\begin{aligned} (4.5)\qquad & \sum_{a(m)\le n\le b(m)} e(f(n,m)) \\ &= \sum_{A(m)\le\nu\le B(m)} \frac{W(\nu)e(G(\nu,m))}{g(m)^{1/(2\alpha-2)}} \\ &\quad + O(L + \min(F^{-1/2}N, \|A(m)\|^{-1}) + \min(F^{-1/2}N, \|B(m)\|^{-1})). \end{aligned}$$

Here

$$W(\nu) = (\tau C^{1/(\alpha-1)}(1+R^*(t_\nu))(-\nu)^{(\alpha-2)/(\alpha-1)})^{-1/2}e(1/8),$$
$$G(\nu,m) = \gamma(-\nu)^{\alpha/(\alpha-1)}C^{-1/(\alpha-1)}(1+\widehat{R}(t_\nu))g(m)^{-1/(\alpha-1)},$$
$$A(m) = Cg(m)h'(a(m)), \qquad B(m) = Cg(m)h'(b(m)).$$

We apply this formula to the sum

$$S(h,g,C) = \sum_{m\sim M}\ \sum_{a(m)\le n\le b(m)} e(Ch(n)g(m))$$

where $[a(m),b(m)] \subset (N,2N]$. Summing over m in (4.5) and interchanging summations yields

$$S(h,g,C) = \sum_{\nu\ll FN^{-1}} W(\nu) \sum_{m\in E_\nu} \frac{e(G(\nu,m))}{g(m)^{1/(2\alpha-2)}} + O\Big(ML + \sum_{m\sim M} \min\Big(F^{-1/2}N, \frac{1}{\|G(m)\|}\Big)\Big). \tag{4.6}$$

Here $G(m)$ is one of $A(m)$, $B(m)$, and

$$E_\nu = \{m\sim M : A(m)\le \nu\le B(m)\}.$$

Let us suppose that

(4.7) E_ν is a union of $O(1)$ disjoint intervals,

(4.8) G is continuously differentiable and $G'(m) \gg F(MN)^{-1}$,

(4.9) $F \gg N$.

In view of Lemma 8 and the monotonicity of g, (4.6) yields

$$S(h,g,C) \ll (FN^{-2})^{-1/2}FN^{-1} \max_{\substack{\nu\ll FN^{-1}\\ I\subseteq(M,2M]}} \Big|\sum_{m\in I} e(G_\nu(m))\Big| + ML + F^{1/2} \ll F^{1/2+\kappa}M^{\lambda-\kappa} + ML \tag{4.10}$$

for any exponent pair (κ,λ). It is clear that $\partial G(\nu,m)/\partial m^j$ satisfies the required conditions ([6, pp. 30–31]) for the last bound, for $j = 1,2,\ldots$. (The exponent in $m^{-\beta/(\alpha-1)}$ is less than 1, by hypothesis.)

We summarize our conclusions in the following theorem. In the language of [6], the theorem asserts that $(1/2,1/2;\kappa,\lambda)$ is an *exponent quadruple.*

THEOREM 3. *Let (κ,λ) be an exponent pair. Define $S(h,g,C)$ by* (4.6), *with the assumptions on h and g made above. Let $F = CM^\beta N^\alpha$, $M \ll N$, $L = \log(FN^{-1}+2)$. Suppose further that* (4.7) *and* (4.8) *hold. Then*

$$S(h,g,C) \ll F^{1/2+\kappa}M^{\lambda-\kappa} + ML + MNF^{-1}. \tag{4.11}$$

Note that the condition (4.9) has been dropped and the term MNF^{-1} incorporated in (4.11). This is justified since, if $F < cN$, the Kuz'min–Landau theorem ([6, Theorem 2.1]) gives

$$S(h,g,C) = \sum_{m\sim M} O(F^{-1}N) \ll MNF^{-1}. \tag{4.12}$$

We now apply Theorem 3 to Type I sums.

Lemma 10. *Let* $K \geq 1$, $M \geq 1$, $|u_{k,m}| \leq 1$ $(k \sim K, m \sim M)$. *Let* $I_m \subseteq (K, 2K]$. *There is a real number* t *such that*

$$\sum_{m\sim M} \Big|\sum_{k\in I_m} u_{k,m}\Big| \ll (\log 2K) \sum_{m\sim M} \Big|\sum_{k\sim K} u_{k,m} e(kt)\Big|. \tag{4.13}$$

Proof. From Lemma 2.2 of [4], there is a positive continuous function $F_K(t)$ on $\mathbb{R}$ such that $\int_{\mathbb{R}} F_K(t)\,dt \ll \log 2K$ and

$$\Big|\sum_{k\in I_m} u_{k,m}\Big| \leq \int_{-\infty}^{\infty} F_K(t)\Big|\sum_{k\sim K} u_{k,m} e(kt)\Big|\,dt.$$

Thus the left-hand side of (4.13) is

$$\leq \int_{-\infty}^{\infty} F_K(t) \sum_{m\sim M} \Big|\sum_{k\sim K} u_{k,m} e(kt)\Big|\,dt \ll \log 2K \max_t \sum_{m\sim M} \Big|\sum_{k\sim K} u_{k,m} e(kt)\Big|,$$

as required.

Theorem 4. *Let* (κ,λ) *be an exponent pair. Let* α,β *be constants*, $\alpha \neq 0$, $\alpha < 1$, $\beta < 0$. *Let* $X > 0$, $M \geq 1/2$, $N \geq 1/2$, $MN \asymp D$, $N_0 = \min(M,N)$, $L = \log(D+2)$. *Let* $|a_m| \leq 1$, $I_m \subseteq (N, 2N]$, *and*

$$S_1 = \sum_{m\sim M} a_m \sum_{n\in I_m} e\Big(\frac{Xm^\beta n^\alpha}{M^\beta N^\alpha}\Big). \tag{4.14}$$

Then

$$S_1 \ll L^2\{DN^{-1/2} + DX^{-1} + (D^{4+4\kappa} X^{1+2\kappa} N^{-(1+2\kappa)} N_0^{2(\lambda-\kappa)})^{1/(6+4\kappa)}\}. \tag{4.15}$$

Proof. We may suppose that N is large. If $X < cN$, we proceed as in (4.12). Now suppose that $X \geq cN$. Let Q be a natural number, $Q < c^2 N$.

By Lemma 10, there is a real number t such that

$$S_1 \ll L \sum_{m\sim M}\Big|\sum_{n\sim N} e\Big(\frac{Xm^\beta n^\alpha}{M^\beta N^\alpha} + tn\Big)\Big|.$$

By the Cauchy–Schwarz inequality and the Weyl–van der Corput inequality ([6, (2.3.4)]), and writing $I(q) = (N, 2N - q]$, we have

$$S_1^2 \ll L^2 M \sum_{m\sim M}\bigg|\sum_{n\sim N} e\bigg(\frac{Xm^\beta n^\alpha}{M^\beta N^\alpha} + tn\bigg)\bigg|^2$$
$$\ll L^2D^2Q^{-1} + L^2DQ^{-1}\sum_{q=1}^{Q}\sum_{m\sim M}\sum_{n\in I(q)} e\bigg(\frac{Xm^\beta((n+q)^\alpha - n^\alpha)}{M^\beta N^\alpha} + tq\bigg).$$

After applying a splitting-up argument to the sum over q, we find that there is a $q \in [1, Q]$ for which

$$S_1^2 \ll L^2D^2Q^{-1} + L^3DqQ^{-1}\bigg|\sum_{m\sim M}\sum_{n\in I(q)} e\bigg(\frac{Xm^\beta((n+q)^\alpha - n^\alpha)}{M^\beta N^\alpha}\bigg)\bigg|.$$

After a straightforward verification that the conditions are satisfied, we may apply Theorem 3 to the above double sum, with either $(n^\beta, (m+q)^\alpha - m^\alpha)$ or $((n+q)^\alpha - n^\alpha, m^\beta)$ in the role of $h(n)$, $g(m)$ (depending on whether N_0 is N or M). Thus

$$F \asymp XN^{-1}q,$$

$$L^{-4}S_1^2 \ll D^2Q^{-1} + D(X^{1/2+\kappa}N^{-1/2-\kappa}N_0^{\lambda-\kappa}Q^{1/2+\kappa} + N_0 + DX^{-1}NQ^{-1}).$$

We can drop the last two terms since

$$DN_0 \ll D^2N^{-1} \ll D^2Q^{-1}, \quad D^2X^{-1}Q^{-1}N \ll D^2Q^{-1}.$$

The resulting bound for $L^{-4}S_1^2$ holds, in fact, for $0 < Q < c^2N$. An application of Lemma 3 completes the proof.

We now adapt this proof to estimate a Type II sum.

Theorem 5. *Make the hypothesis of Theorem 4 and suppose that* $|b_n| \le 1$. *Let*

$$S_2 = \sum_{m\sim M} a_m \sum_{\substack{n\sim N\\ D<mn\le D'}} b_n e\bigg(\frac{Xm^\beta n^\alpha}{M^\beta N^\alpha}\bigg).$$

Suppose further that

$$N \ll M, \quad X \gg D. \tag{4.16}$$

Then

$$S_2 \ll L^{7/4}(DN^{-1/2} + DM^{-1/4} + (D^{11+10\kappa}X^{1+2\kappa}N^{2(\lambda-\kappa)})^{1/(14+12\kappa)}). \tag{4.17}$$

Proof. We remove the condition $D < mn \le D'$ at the cost of a factor L. Let Q be a positive integer, $Q < c^2N$. As in the preceding proof, there is a

$q \in [1, Q]$ such that (writing $I_{m,q} = I_m \cap (I_m - q)$)

$$(4.18)\quad L^{-2}S_2^2 \ll D^2Q^{-1} + \frac{DqL}{Q}\Big|\sum_{m\sim M}\sum_{n\sim N}\bar{b}_n b_{n+q} e\Big(\frac{Xm^\beta((n+q)^\alpha - n^\alpha)}{M^\beta N^\alpha}\Big)\Big|$$

$$\ll D^2Q^{-1} + \frac{DqL}{Q}\sum_{n\sim N}\Big|\sum_{m\sim M} e\Big(\frac{Xm^\beta((n+q)^\alpha - n^\alpha)}{M^\beta N^\alpha}\Big)\Big|.$$

Suppose further that $Q < cM^{1/2}$. Using the Weyl–van der Corput inequality again, we obtain

$$\Big|\sum_{m\sim M} e\Big(\frac{Xm^\beta((n+q)^\alpha - n^\alpha)}{M^\beta N^\alpha}\Big)\Big|^2$$
$$\le \frac{M^2}{Q^2} + \frac{LMq'}{Q^2}\sum_{m\in J(q')} e\Big(\frac{X((m+q')^\beta - m^\beta)((n+q)^\alpha - n^\alpha)}{M^\beta N^\alpha}\Big)$$

for some $q' \le Q^2$, with $J(q') = (M, 2M - q']$. Combining this with (4.18) and Cauchy's inequality, we have

$$L^{-4}S_2^4 \ll \frac{D^4L^2}{Q^2} + \frac{D^3q^2q'L^3}{Q^4}\, S_{M,N}$$

where

$$S_{M,N} = \sum_{n\sim N}\sum_{m\in J(q')} e\Big(\frac{X((m+q')^\beta - m^\beta)((n+q)^\alpha - n^\alpha)}{M^\beta N^\alpha}\Big).$$

An application of Theorem 3 to $S_{M,N}$ now yields

$$L^{-7}S_2^4 \ll \frac{D^4}{Q^2} + D^3\Big(\Big(\frac{XQ^3}{D}\Big)^{1/2+\kappa} N^{\lambda-\kappa} + N + \frac{D^2}{XQ^3}\Big)$$

(this is of course also true for $0 < Q < 1$).

We may discard the term D^5/XQ^3 since $D^5/XQ^3 \le D^5/XQ^2 \ll D^4/Q^2$. An application of Lemma 3 with $H_1 \to 0+$, $H_2 = \min(c^2N, cM^{1/2})$ yields

$$L^{-7}S_2^4 \ll D^4N^{-2} + D^4M^{-1} + (D^{11+10\kappa}X^{1+2\kappa}N^{2(\lambda-\kappa)})^{2/(7+6\kappa)} + D^3N.$$

Since $D^3N \ll D^4M^{-1}$, the theorem follows at once.

We now pursue a variant of the above arguments.

Theorem 6. *Let (κ, λ) be an exponent pair. Let α, β be constants, $\alpha \ne 1$, $\beta < 0$, $\alpha + \beta < 2$. Let $N \ge 1/2$, $X \gg N$, $M \gg N$, $MN \asymp D$, $L = \log(XD + 2)$. Let $|a_m| \le 1$, $|b_n| \le 1$. Let*

$$S_2 = \sum_{\substack{m\sim M\\ D<mn\le D'}} a_m \sum_{n\sim N} b_n e\Big(\frac{Xm^\beta n^\alpha}{M^\beta N^\alpha}\Big).$$

Then

$$S_2 \ll L^2(DN^{-1/2} + DM^{-1/4} + X^{1/6}(D^{4+5\kappa}N^{\lambda-\kappa})^{1/(6+6\kappa)}).$$

Proof. Let Q be a positive integer, $Q < c^2 \min(N, M^{1/2})$. As in (4.18), there is a q, $1 \le q \le Q$, for which

$$L^{-2}S_2^2 \ll D^2Q^{-1} + \frac{DqL}{Q} \sum_{n\sim N}\Big|\sum_{m\sim M} e\Big(\frac{Xm^\beta((n+q)^\alpha - n^\alpha)}{M^\beta N^\alpha}\Big)\Big|. \tag{4.19}$$

We apply (4.5) to the inner sum, with the roles of n, m reversed, so that (α,β) is replaced by $(\beta,\alpha-1)$:

$$\sum_{m\sim M} e\Big(\frac{Xm^\beta((n+q)^\alpha - n^\alpha)}{M^\beta N^\alpha}\Big) = \sum_{\nu\in I_n} \frac{W(\nu)}{g(n)^{1/(2\beta-2)}}\, e(G(\nu,n)) \tag{4.20}$$

$$+O\bigg(L+\min\bigg(\Big(\frac{Xq}{N}\Big)^{-1/2}, \|A(n)\|^{-1}\bigg)+\min\bigg(\Big(\frac{Xq}{N}\Big)^{-1/2}, \|B(n)\|^{-1}\bigg)+\frac{D}{Xq}\bigg).$$

Here

$$B(n) = \frac{X}{M^\beta N^\alpha}\,((n+q)^\alpha - n^\alpha)\beta M^{\beta-1}, \qquad A(n) = 2^{\beta-1}B(n),$$

and $I_n = [A(n), B(n)]$. The last term on the right-hand side of (4.20) allows for a possible application of the Kuz'min–Landau inequality.

Combining (4.19), (4.20) shows that there are numbers w_ν, $|w_\nu| \le 1$, with

$$L^{-2}S_2^2 \ll D^2Q^{-1} + \frac{LqD}{Q}\Big(\frac{Xq}{DM}\Big)^{-1/2} \sum_n\Big|\sum_{\nu\in I_n} w_\nu e(G(\nu,n))\Big| + L^2DN + LDX^{1/2}Q^{1/2}N^{-1/2}.$$

Since $DN \ll D^2Q^{-1}$ from $Q < N \ll D^{1/2}$, we have

$$L^{-4}S_2^2 \ll D^2Q^{-1} + DX^{1/2}Q^{1/2}N^{-1/2} + Q^{-1/2}X^{-1/2}D^{3/2}M^{1/2}\sum_n\Big|\sum_{\nu\in I_n} w_\nu e(G(\nu,n))\Big|.$$

We apply the Cauchy and Weyl–van der Corput inequalities to obtain

$$L^{-8}S_2^4 \ll D^4Q^{-2} + D^2XQN^{-1} + (QX)^{-1}D^4\bigg(\frac{(Xq/D)^2}{H} + \frac{Xq}{D}\Big|\sum_n \sum_{\nu\in I_n\cap(I_n-h)} e(G_1(\nu,n))\Big|\bigg) \tag{4.21}$$

with H (specified below) satisfying $0 < H < cXq/D$ and some h, $1 \le h \le H$. Here

$$G_1(\nu,n) = G(\nu+h,n) - G(\nu,n).$$

We make the obvious choice $H = Qq^2NX/D^2$; the assumption $Q < c^2M^{1/2}$ yields $H < c(Xq/D)$.

We interchange the summations over n and ν in (4.21). Once ν is fixed, n runs over a single interval. We apply the method of exponent pairs to $\sum_n e(G_1(\nu, n))$; the order of size of G_1 is

$$\frac{h}{Xq/D}\,\frac{Xq}{N} \asymp hM \gg N.$$

Thus

$$\sum_n e(G_1(\nu, n)) \ll (hM)^\kappa N^{\lambda-\kappa} \ll (Q^3X/D)^\kappa N^{\lambda-\kappa}.$$

Combining this with (4.21) gives

$$\begin{aligned} L^{-8}S_2^4 &\ll D^4Q^{-2} + D^2XQN^{-1} + D^2XQ(Q^3X/D)^\kappa N^{\lambda-\kappa} \\ &\ll D^4Q^{-2} + D^2XQN^{-1} + D^{2-\kappa}X^{1+\kappa}Q^{1+3\kappa}N^{\lambda-\kappa}. \end{aligned}$$

An application of Lemma 3 yields

$$\begin{aligned} L^{-8}S_2^4 &\ll D^{4/3}(D^2XN^{-1})^{2/3} + (D^4)^{(1+3\kappa)/(3+3\kappa)}(D^{2-\kappa}X^{1+\kappa}N^{\lambda-\kappa})^{2/(3+3\kappa)} \\ &\quad + D^4M^{-1} + D^4N^{-2} \\ &\ll D^{4/3}X^{2/3}N^{-2/3} + X^{2/3}(D^{8+10\kappa}N^{\lambda-\kappa})^{2/(3+3\kappa)} + D^4M^{-1} + D^4N^{-2}. \end{aligned}$$

In the last expression, the first term is clearly dominated by the second, and Theorem 6 follows.

5. The AB theorem. Let $X \ge 1$, $Y \ge 1$, $N = XY$. Let $\mathbf{D}$ be a subset of $\mathbf{R} = [X, 2X] \times [Y, 2Y]$ satisfying some mild restrictions discussed below. Let α, β be real with

(5.1) $$(\alpha)_3\,(\beta)_3(\alpha+\beta+1)_2 \ne 0,$$

where $(\alpha)_0 = 1$, $(\alpha)_s = (\alpha + s - 1)(\alpha)_{s-1}$ for $s = 1, 2, \ldots$.

Theorem 6.12 of [6] states that, for $F > 0$, $L = \log(FN + 2)$,

$$\begin{aligned} S_f := \sum_{(m,n)\in\mathbf{D}} & e(FX^\alpha Y^\beta m^{-\alpha} n^{-\beta}) \\ &\ll F^{1/3}N^{1/2} + N^{5/6}L^{2/3} + F^{-1/8}N^{15/16}L^{3/8} + F^{-1/4}NL^{1/2} \end{aligned}$$

(the "AB theorem"). In the present section, I extend this by replacing $u^{-\alpha}$, $v^{-\beta}$ by more general functions $h_1(u)$, $h_2(v)$, with

(5.2) $$(h_1(u) - u^{-\alpha})^{(j)} \ll \eta|(u^{-\alpha})^{(j)}| \quad (u \sim X),$$

(5.3) $$(h_2(v) - v^{-\beta})^{(j)} \ll \eta|(v^{-\beta})^{(j)}| \quad (v \sim Y),$$

where η is a sufficiently small positive quantity (in terms of α, β) and $j = 0, 1, \ldots, j \ll 1$. We write $f_0(u, v) = Ah_1(u)h_2(v)$ for $(u, v) \in \mathbf{R}$.

We must deal with some monotonicity conditions for

$$f_1(u,v;q,r) = f_0(u+q,v+r) - f_0(u,v) = \int_0^1 \frac{\partial}{\partial t} f_0(u+qt,v+rt)\,dt.$$

These are a technical nuisance rather than a serious obstacle. We shall see that $\mathbf{R}$ can be partitioned into $O(1)$ rectangles $\mathbf{R}' = I \times J$ such that one of $f_1^{(2,0)}$ or $f_1^{(0,2)}$ has no zero in each $\overline{\mathbf{R}'}$. This allows us to impose helpful conditions in Lemmas 11–14 below. In these lemmas, let $f(u,v)$ be a real function on $\overline{I} \times \overline{J}$ and $\mathbf{D} \subseteq \overline{I} \times \overline{J}$. We write $f^{(a,b)}$ for $\partial^{a+b} f/\partial u^a \partial v^b$. Suppose that $f^{(2,0)}$ is nonzero on $\mathbf{R}$. Let $\psi(w,v)$ denote the solution of

$$f^{(1,0)}(\psi(w,v),v) = w. \tag{5.4}$$

For any function ϕ having second order partial derivatives on $\mathbf{D}$, let $H\phi = \phi^{(2,0)}\phi^{(0,2)} - \{\phi^{(1,1)}\}^2$. We need to state some "omega conditions" on f, which we assume to be true for the duration of these lemmas.

(Ω_1) f has partial derivatives of all orders. For a suitable $F > 0$ there is a constant C_1 such that

$$|f^{(a,b)}(u,v)| \le C_1 F X^{-a} Y^{-b} \quad ((u,v) \in \overline{I} \times \overline{J},\ 0 \le a,b \le 4).$$

(Ω_2) There is a constant C_2 such that the set

$$U(v) = \{u : (u,v) \in \mathbf{D}\}$$

is the union of at most C_2 intervals for each v.

(Ω_3) There is a constant C_3 such that the set

$$V(l) = \{v : (\psi(l,v),v) \in \mathbf{D}\}$$

is the union of at most C_3 intervals for each l.

A function $f : I \to \mathbb{R}$ is said to be C-*monotonic* if I can be partitioned into C intervals on each of which f is monotonic.

(Ω_4) There is a constant C_4 such that, for each fixed l, $f^{(2,0)}(\psi(l,v),v)$ is C_4-monotonic on $\overline{J}$.

In Lemmas 11–14, implied constants depend at most on $C_1,\ldots,C_4$. In Theorem 7, implied constants depend at most on $C_1,\ldots,C_4$, α and β.

LEMMA 11 ([6, Lemma 6.6]). *Suppose that* $|f^{(2,0)}| \asymp \Lambda$ *on* $\mathbf{D}$. *Then*

$$S_f \ll |\mathbf{D}|\Lambda^{1/2} + \Lambda^{-1/2} Y.$$

We shall write

$$g(w,v) = f(\psi(w,v),v) - w\psi(w,v).$$

LEMMA 12 ([6, proof of Lemma 6.7]). *We have*

(i) $$\frac{\partial\psi}{\partial v}(w,v) = -f^{(1,1)}(\psi(w,v),v)/f^{(2,0)}(\psi(w,v),v),$$

(ii) $$\frac{\partial g}{\partial v} = f^{(0,1)}(\psi(w,v),v), \qquad \frac{\partial^2 g}{\partial v^2} = \frac{(Hf)(\psi(w,v),v)}{f^{(2,0)}(\psi(w,v),v)}.$$

LEMMA 13. *Suppose that*

$$|f^{(j,0)}| \asymp FX^{-j}, \qquad |f^{(0,j)}| \asymp FY^{-j} \quad (j=1,2), \qquad |Hf| \asymp F^2N^{-2}$$

and that (Ω_2)–(Ω_4) *hold, and remain valid with the roles of the variables interchanged. Then*

$$S_f \ll F + F^{-1/2}NL.$$

Proof. We may follow the proof of [6, Lemma 6.11] almost verbatim.

We now give a variant of [6], Lemma 6.8.

LEMMA 14. *Suppose that* $|f^{(2,0)}| \asymp \Lambda$ *and* $|Hf| \asymp M$ *on* $\mathbf{D}$. *Then*

$$S_f \ll |\mathbf{D}|M^{1/2} + FM^{-1/2}X^{-1} + M^{-1/2} + Y\Lambda^{-1/2} + YL.$$

Proof. In view of Lemma 11, we may suppose that $\Lambda \geq M$. Replacing f by $-f$ if necessary, we may suppose that $f^{(2,0)} < 0$ on $\mathbf{D}$. Lemma 3.6 of [6] gives

$$\begin{aligned} S_f &= \sum_{n\sim Y}\sum_{m\in I(n)} e(f(m,n)) \\ &= \sum_{n\sim Y}\sum_{k\in K(n)} e\Big(-\frac{1}{8}+g(k,n)\Big)|f^{(2,0)}(\psi(k,n),n)|^{-1/2} + O(Y\Lambda^{-1/2}+YL). \end{aligned}$$

Here $K(n) = \{k : (\psi(k,n),n) \in \mathbf{D}\}$. Changing the order of summation yields

$$S_f = \sum_{k\ll FX^{-1}}\sum_{n\in V(k)} e\Big(-\frac{1}{8}+g(k,n)\Big)|f^{(2,0)}(\psi(k,n),n)|^{-1/2} + O(Y\Lambda^{-1/2}+YL).$$

Thanks to (Ω_3), (Ω_4), we may now apply partial summation to conclude that

$$S_f \ll \sum_{k\ll FX^{-1}} \Lambda^{-1/2}\Big|\sum_{n\in J(k)} e(g(k,n))\Big| + O(Y\Lambda^{-1/2}+YL) \tag{5.5}$$

for an interval $J(k) \subset V(k)$. Now an application of [6, Theorem 2.2], in conjunction with (5.5) and Lemma 12, gives

$$S_f \ll \sum_{k\ll FX^{-1}} \Lambda^{-1/2}\bigg\{\frac{M^{1/2}}{\Lambda^{1/2}}|V(k)| + \Big(\frac{M}{\Lambda}\Big)^{-1/2}\bigg\} + Y\Lambda^{-1/2} + YL. \tag{5.6}$$

Moreover,

$$\sum_k |V(k)| = \sum_{n} \sum_{\substack{k \\ \psi(k,n)\in U(n)}} 1.$$

Now the inner sum is the number of integer values assumed by $f^{(1,0)}(m,n)$ as m runs over $U(n)$. Recalling (Ω_2), the inner sum is

$$\ll \int\limits_{U(n)} f^{(2,0)}(t,n)\,dt + 1 \ll |U(n)|\Lambda + 1.$$

Hence

$$\sum_k |V(k)| \ll \sum_{n\sim Y} (|U(n)|\Lambda + 1) \ll |\mathbf{D}|\Lambda + Y. \tag{5.7}$$

Combining (5.6) and (5.7) gives

$$S \ll |\mathbf{D}|M^{1/2} + \Lambda^{-1}M^{1/2}Y + M^{-1/2}(FX^{-1}+1) + Y\Lambda^{-1/2} + YL.$$

Since $Y\Lambda^{-1/2} \ge \Lambda^{-1}M^{1/2}Y$, the lemma follows.

For convenience, we record three more lemmas from [6].

Lemma 15. *Let P be a polynomial over $\mathbb{C}$ having distinct zeros, with $P(0)\ne 0$. Let $\delta>0$. Let q, r be integers, $r\ne 0$. Let*

$$E = \left\{(m,n) : m\sim X,\ n\sim Y,\ \left|P\left(\frac{qn}{rm}\right)\right| < \delta\right\}.$$

Then

$$|E| < C(P)\{\delta N + 1\}.$$

Proof. For $q\ne 0$, this follows from [6, Lemma 6.4]. For $q=0$, E is empty if δ is sufficiently small, and otherwise the result is trivial.

Lemma 16. *Let P, Q be polynomials over $\mathbb{C}$ having no common zero. Let q, r, m, n be integers, $rm\ne 0$. Then*

$$\max\left(\left|P\left(\frac{qn}{rm}\right)\right|, \left|Q\left(\frac{qn}{rm}\right)\right|\right) > C(P,Q) > 0.$$

Proof. This follows from [6, Lemma 6.5].

Lemma 17 ([6, p. 76]). *For $1\le Q\le X$, $1\le R\le Y$, we have*

$$S_f^2 \ll \frac{N}{QR} \sum_{|q|\le Q} \sum_{|r|\le R} \sum_{(m,n)\in\mathbf{D}(q,r)} e(f_1(m,n;q,r)).$$

Here $\mathbf{D}(q,r) = \mathbf{D}\cap(\mathbf{D}-(q,r))$ and

$$f_1(m,n;q,r) = f(m+q,n+r) - f(m,n). \tag{5.8}$$

In Theorem 7, we write $f_0\in\mathcal{E}$ as an abbreviation for the following hypothesis. If f_0 is restricted to a rectangle $I\times J$ with the property that

$f_1^{(2,0)} \neq 0$ in $E = \bar{I} \times \bar{J}$, then (Ω_2), (Ω_3), (Ω_4) hold for f_1 when the domain of f_1 is $\mathbf{D}(q,r,\theta)$ or $\mathbf{D}'(q,r,\theta)$ (for all $(q,r) \in \mathbb{Z}^2$, $\theta > 0$). Here

$$\begin{aligned}\mathbf{D}(q,r) &= \{(u,v) \in (E \cap \mathbf{D}) \cap (E \cap \mathbf{D} - (q,r))\},\\ \mathbf{D}(q,r,\theta) &= \{(u,v) \in \mathbf{D}(q,r) : \theta \le |Hf_1| < 2\theta\},\\ \mathbf{D}'(q,r,\theta) &= \{(u,v) \in \mathbf{D}(q,r) : |Hf_1| < \theta\}.\end{aligned}$$

Moreover, f_0 has the same property when the variables interchange roles.

Theorem 7. *Let $0 < \eta < c(\alpha,\beta)$ where c is sufficiently small. Let f_0 satisfy* (5.2), (5.3) *and suppose that $f_0 \in \mathcal{E}$. Let $F = AX^{-\alpha}Y^{-\beta} \gg N^{1/6}$ and $Y \le X$. Then*

$$\begin{aligned}(5.9)\qquad S_{f_0} \ll L\{&F^{1/3}N^{1/2} + N^{5/6} + F^{-1/8}N^{15/16} + F^{1/2}N^{1/2}Y^{-1/2}\\ &+ F^{1/12}N^{1/2}Y^{5/12} + \eta^{2/5}N^{1/2}F^{1/5}Y^{2/5}\\ &+ \eta^{1/4}N^{3/4}Y^{1/4} + \eta^{1/2}F^{1/4}N^{1/2}Y^{1/4}\}.\end{aligned}$$

Proof. We may assume that $F > N^{5/6}$ and $Y > N^{1/4}$. For suppose that $N^{1/6} \ll F \le N^{5/6}$. Since

$$(-\alpha)(-\alpha-1)(-\beta)(-\beta-1) - \alpha^2\beta^2 = \alpha\beta(\alpha+\beta+1),$$

it is easy to deduce from (5.2), (5.3) that $Hf_0 \asymp F^2N^{-2}$. The omega conditions are rather straightforward to check for f_0. Hence Lemma 13 gives

$$S_{f_0} \ll F + F^{-1/2}NL \ll N^{5/6} + F^{-1/8}N^{15/16}L,$$

as required. Now suppose that $F > N^{5/6}$ and $Y \le N^{1/4}$. We note that (5.9) is trivial for $F > N^{3/2}$, so we suppose that $F \le N^{3/2}$. Then Lemma 11 gives

$$\begin{aligned}S_{f_0} &\ll N(FX^{-2})^{1/2} + (FX^{-2})^{-1/2}Y \ll F^{1/2}Y + F^{-1/8}N^{15/16}\\ &\ll F^{1/3}N^{1/2} + F^{-1/8}N^{15/16},\end{aligned}$$

since $F^{1/6}YN^{-1/2} \ll YN^{-1/4} \ll 1$.

We write $f_1(u,v)$ rather than $f_1(u,v;q,r)$ for the function in (5.8) with $f = f_0$. Let $S = S_{f_0}$. From Lemma 17,

$$(5.10)\qquad S^2 \ll \frac{N^2}{Z} + \frac{N}{Z}\sum_{\substack{|q|\le Q\\ (q,r)\neq(0,0)}}\sum_{|r|\le R} S(q,r)$$

where $S(q,r) = \sum_{(m,n)\in\mathbf{D}(q,r)} e(f_1(m,n))$. Here Z is at our disposal subject to $X/Y \le Z \le c^2N$, and we choose

$$Q = \sqrt{ZX/Y}, \qquad R = \sqrt{ZY/X}.$$

Note that $Q/X = R/Y = \sqrt{Z/N} \le c$.

For a fixed pair q, r,

$$\varrho := \max(|q|/X, |r|/Y) \le c.$$

We consider the contribution to the right-hand side of (5.10) from terms with $\varrho = |r|/Y$ (in particular, $r \neq 0$). The remaining terms can be estimated similarly.

The hypotheses of the theorem imply that, for bounded a, b,

$$f_1^{(a,b)}(m,n) = (-1)^{a+b+1} A\, m^{-\alpha-a} n^{-\beta-b}\, \frac{r}{n} \left\{ T_{a,b}\left(\frac{qn}{rm}\right) + O(\varrho+\eta) \right\} \tag{5.11}$$

where

$$T_{a,b}(z) = (\alpha)_{a+1}(\beta)_b z + (\alpha)_a(\beta)_{b+1}.$$

Moreover,

$$Hf_1(m,n) = A^2 m^{-2\alpha-2} n^{-2\beta-2}\, \frac{r^2}{n^2} \left\{ U\left(\frac{qn}{rm}\right) + O(\varrho+\eta) \right\}$$

where

$$U(z) = \alpha\beta(\alpha+\beta+2)\{(\alpha)_2 z^2 + 2(\alpha+1)(\beta+1)z + (\beta)_2\}.$$

As pointed out on p. 84 of [6], $U(z)$ has degree 2 and has distinct zeros. We also need the observation that no two of $T_{0,2}$, $T_{1,1}$ and $T_{2,0}$ have a common zero; nor does $T_{0,2}$ or $T_{2,0}$ share a zero with U.

Because of these observations, it suffices to prove (5.9) with $\mathbf{D}$ replaced by a domain $\mathbf{D} \cap (I \times J)$ with the property that $f_1^{(2,0)} \neq 0$ in $\bar{I} \times \bar{J}$ or $f_1^{(0,2)} \neq 0$ in $\bar{I} \times \bar{J}$. Let us suppose, say, that $f_1^{(2,0)} \neq 0$ in $\bar{I} \times \bar{J}$.

Let δ be a small positive number, to be chosen later. Consider the domains (possibly empty)

$$\mathbf{D}_{0,j} = \Big\{ (m,n) \in \mathbf{D}(q,r) : 2^j \delta \varrho^2 F^2 N^{-2} \le |Hf_1| < 2^{j+1} \delta \varrho^2 F^2 N^{-2} \text{ and } \left| T_{2,0}\left(\frac{qn}{rm}\right) \right| \ge c \Big\} \quad (j \ge 0),$$

$$\mathbf{D}_1 = \left\{ (m,n) \in \mathbf{D}(q,r) : \left| T_{2,0}\left(\frac{qn}{rm}\right) \right| < c \right\},$$

$$\mathbf{D}_2 = \{ (m,n) \in \mathbf{D}(q,r) : |Hf_1| < \delta \varrho^2 F^2 N^{-2} \}$$

and set

$$S_{0,j} = \sum_{(m,n)\in \mathbf{D}_{0,j}} e(f(m,n)),$$

$$S_1 = \sum_{(m,n)\in \mathbf{D}_1} e(f(m,n)), \qquad S_2 = \sum_{(m,n)\in \mathbf{D}_2} e(f(m,n)).$$

By Lemma 16, the sets $\mathbf{D}_{0,j}$ ($0 \le j \ll L$), $\mathbf{D}_1$, $\mathbf{D}_2$ form a partition of $\mathbf{D}(q,r)$. Clearly

$$|Hf_1| \asymp 2^j \delta \varrho^2 F^2 N^{-2} \quad \text{and} \quad |f_1^{(2,0)}| \asymp \varrho F^2 X^{-2} \quad \text{on } \mathbf{D}_{0,j}. \tag{5.12}$$

Moreover, from Lemma 16,

$$|Hf_1| \asymp \varrho^2 F^2 N^{-2} \quad \text{and} \quad |f_1^{(0,2)}| \asymp \varrho F X^{-2} \quad \text{on } \mathbf{D}_1, \tag{5.13}$$

$$|f_1^{(2,0)}| \asymp \varrho F X^{-2} \quad \text{on } \mathbf{D}_2. \tag{5.14}$$

We may estimate the terms on the right in the decomposition

$$S(q,r) = \sum_{0 \le j \ll L} S_{0,j} + S_1 + S_2$$

by applying Lemmas 11 and 14. For $\mathbf{D}_{0,j}$, $\mathbf{D}_1$, $\mathbf{D}_2$ are domains $\mathbf{D}(q,r,\theta)$ or $\mathbf{D}'(q,r,\theta)$. From Lemma 14 and (5.12),

$$\begin{aligned} S_{0,j} &\ll N(\varrho^2 F^2 N^{-2})^{1/2} + \varrho F(\delta \varrho^2 F^2 N^{-2})^{-1/2} X^{-1} \\ &\quad + (\delta \varrho^2 F^2 N^{-2})^{-1/2} + Y(\varrho F X^{-2})^{-1/2} + YL \\ &\ll \varrho F + \varrho^{-1/2} F^{-1/2} N + \delta^{-1/2} Y L + \delta^{-1/2} \varrho^{-1} F^{-1} N. \end{aligned} \tag{5.15}$$

Similarly, Lemma 14 and (5.13) give

$$\begin{aligned} S_1 &\ll N(\varrho^2 F^2 N^{-2})^{1/2} + \varrho F(\varrho^2 F^2 N^{-2})^{-1/2} Y^{-1} \\ &\quad + (\varrho^2 F^2 N^{-2})^{-1/2} + X(\varrho F Y^{-2})^{-1/2} + XL \\ &\ll \varrho F + \varrho^{-1} F^{-1} N + XL + \varrho^{-1/2} F^{-1/2} N. \end{aligned} \tag{5.16}$$

By Lemma 15, and since $\varrho N \ge N/X$, the number of points in $\mathbf{D}_2$ is $\ll (\delta + \varrho + \eta)N$. From Lemma 11 and (5.14),

$$S_2 \ll (\delta + \varrho + \eta) \varrho^{1/2} F^{1/2} Y + \varrho^{-1/2} F^{-1/2} N. \tag{5.17}$$

Collecting (5.15)–(5.17) yields

$$\begin{aligned} S(q,r) &\ll L\varrho F + L\varrho^{-1/2} F^{-1/2} N + L^2 \delta^{-1/2} Y \\ &\quad + L\delta^{-1/2} \varrho^{-1} F^{-1} N + LNY^{-1} + (\delta + \varrho + \eta) \varrho^{1/2} F^{1/2} Y. \end{aligned}$$

Note that, since $F \ge N^{5/6}$, we have $\varrho F \gg F X^{-1} \gg 1$. We may take $\delta = c(\varrho F)^{-1/3}$ to obtain

$$\begin{aligned} L^{-2} S(q,r) &\ll \varrho F + \varrho^{-1/2} F^{-1/2} N + \varrho^{1/6} F^{1/6} Y \\ &\quad + \varrho^{-5/6} F^{-5/6} N + NY^{-1} + \varrho^{3/2} F^{1/2} Y + \eta \varrho^{1/2} F^{1/2} Y. \end{aligned} \tag{5.18}$$

Now if a is a constant, $a > -1$, we have

$$\frac{1}{Z} \sum_{(q,r) \in \mathbf{Q}} \varrho^a \ll \frac{1}{Z} \sum_{q=1}^{Q} \sum_{r=1}^{R} \left(\frac{q^a}{X^a} + \frac{r^a}{Y^a} \right) \ll \left(\frac{Z}{N} \right)^{a/2},$$

where $\mathbf{Q} = \{(q,r) : |q| \le Q,\ |r| \le R,\ (q,r) \ne (0,0)\}$. We combine this with

(5.10), (5.18) to obtain

$$(5.19)\quad L^{-2}S^2 \ll N^2Z^{-1} + FN^{1/2}Z^{1/2} + F^{-1/2}N^{9/4}Z^{-1/4} + F^{1/6}N^{11/12}YZ^{1/12} + F^{-5/6}N^{29/12}Z^{-5/12} + N^2Y^{-1} + F^{1/2}N^{1/4}YZ^{3/4} + \eta F^{1/2}N^{3/4}YZ^{1/4}.$$

We may discard $F^{-5/6}N^{29/12}Z^{-5/12}$ since $F \gg N^{5/6}$:

$$F^{-5/6}N^{29/12}Z^{-5/12} = (N^2Z^{-1})^{11/18}(FN^{1/2}Z^{1/2})^{7/18}NF^{-11/9} \ll (N^2Z^{-1})^{11/18}(FN^{1/2}Z^{1/2})^{7/18}.$$

Applying Lemma 3, we find that

$$L^{-2}S^2 \ll F^{2/3}N + F^{2/13}NY^{12/13} + F^{2/7}NY^{4/7} + \eta^{4/5}NF^{2/5}Y^{4/5} + N^{5/3} + N^{5/4}Y^{3/4} + F^{-1/4}N^{7/4}Y^{1/4} + \eta^{1/2}N^{3/2}Y^{1/2} + FN^{1/2}X^{1/2}Y^{-1/2} + F^{1/6}N^{11/12}X^{1/12}Y^{11/12} + F^{1/2}N^{1/4}X^{3/4}Y^{1/4} + F^{-1/2}N^2 + N^2Y^{-1} + \eta F^{1/2}N^{3/4}X^{1/4}Y^{3/4}.$$

Clearly we may suppose that $F \ll N^{3/2}$. We use $N^{5/6} \ll F \ll N^{3/2}$, $N^{1/4} \ll Y \ll N^{1/2}$ to obtain

$$F^{-1/4}N^{7/4}Y^{1/4} \ll F^{-1/4}N^{15/8}, \quad F^{-1/2}N^2 \ll F^{-1/4}N^{15/8},$$
$$F^{1/2}N^{1/4}X^{3/4}Y^{1/4} \ll NX^{3/4}Y^{1/4} \ll N^{5/3}, \quad N^{5/4}Y^{3/4} \ll N^{5/3}.$$

Moreover,

$$F^{2/7}NY^{4/7} \ll F^{2/7}N^{9/7} \le (F^{2/3}N)^{33/49}(F^{-1/2}N^2)^{16/49},$$
$$F^{2/13}NY^{12/13} \ll F^{2/13}N^{19/13} \le (F^{2/3}N)^{63/143}(F^{-1/4}N^{15/8})^{80/143}.$$

Hence

$$(5.20)\quad L^{-2}S^2 \ll F^{2/3}N + F^{-1/4}N^{15/8} + \eta^{4/5}NF^{2/5}Y^{4/5} + N^{5/3} + \eta^{1/2}N^{3/2}Y^{1/2} + N^2Y^{-1} + \eta F^{1/2}NY^{1/2} + FNY^{-1} + F^{1/6}NY^{5/6}.$$

We noted above that

$$S \ll F^{1/2}Y + NF^{-1/2}, \quad S^2 \ll FY^2 + N^2F^{-1}.$$

If N^2Y^{-1} is the maximum term in (5.20)

$$S^2 \ll (N^2Y^{-1})^{2/3}(FY^2)^{1/3} + N^2F^{-1} \ll N^{4/3}F^{1/3} + F^{-1/4}N^{15/8} \ll (F^{2/3}N)^{1/2}(N^{5/3})^{1/2} + F^{-1/4}N^{15/8},$$

which yields (5.9). We conclude that (5.9) always holds.

Let us now specialize h_1 and h_2 for application to Type I sums. We suppose that

(a) Either

$$h_1(u) = u^{-\alpha} \quad \text{or} \quad h_1(u) = \frac{u^{1-\alpha} - (u+p)^{1-\alpha}}{(1-\alpha)p},$$

where $p > 0$ and p/X is sufficiently small.

(b) Either

$$h_2(v) = v^{-\beta} \quad \text{or} \quad h_2(v) = \frac{v^{1-\beta} - (v+s)^{1-\beta}}{(1-\beta)s},$$

where $s > 0$ and s/Y is sufficiently small.

Thus $h_1(u)$ is a holomorphic function in $G = \{u \in \mathbb{C} : \operatorname{Re} u \in (X/2, 3X)\}$ satisfying the approximation (5.2) in G; and similarly for $h_2(v)$ and $G' = \{v \in \mathbb{C} : \operatorname{Re} v \in (Y/2, 3Y)\}$.

We further suppose that $\mathbf{D}$ is a rectangle.

We can now make some observations useful for verification of the omega conditions, with f_1, ϱF in place of f, F. The condition (Ω_1) gives no difficulty. Interchanging α, β if necessary, we suppose that

$$|f_1^{(2,0)}(u,v)| \gg 1 \quad \text{on } \mathbf{D}.$$

Let us define ψ as in (5.4), with f_1 in place of f. Then:

(i) For fixed real k and l, the equation

$$f_1^{(1,0)}(k,v) = l \tag{5.21}$$

has $O(1)$ solutions $v \in J$.

Take a suitable rectangle R in G' containing J in its interior. We readily obtain a holomorphic function g on G such that $|g(v)| < |f_1^{(1,0)}(k,v)|$ on R, namely

$$f_1^{(1,0)}(k,v) + g(v) = Ak^{-\alpha-1}v^{-\beta}\,\frac{r}{v}\,T_{1,0}\left(\frac{qv}{rk}\right)$$

for $v \in G$. From Rouché's theorem, the equation (5.21) has $O(1)$ solutions inside R.

(ii) For fixed real k and l, the equation

$$\psi(l,v) = k$$

has $O(1)$ solutions $v \in J$.

For if $\psi(l,v) = k$, then $f_1^{(1,0)}(k,v) = l$. This equation has $O(1)$ solutions $v \in J$ from (i).

(iii) For fixed real k' and l, the equation

$$\frac{\partial \psi}{\partial v}\,(l,v) = k' \tag{5.22}$$

has $O(1)$ solutions $v \in J$.

For (5.22) implies

$$f_1^{(1,1)}(\psi(l,v),v)+k'f_1^{(2,0)}(\psi(l,v),v)=0.$$

In view of (ii), we need only show that

$$f_1^{(1,1)}(k,v)+k'f_1^{(2,0)}(k,v)=0$$

has $O(1)$ solutions $v\in J$. This is accompanied by an application of Rouché's theorem much as above.

(iv) Let q be a given function on $\mathbb{R}^2$. Suppose that

$$h(v):=-q^{(1,0)}(k,v)f_1^{(1,1)}(k,v)+q^{(0,1)}(k,v)f_1^{(2,0)}(k,v)$$

is holomorphic in G for any fixed k in I. Suppose further that h has only $O(1)$ zeros on the interval J. Then the equation

$$(q(\psi(l,v),v))'=0 \tag{5.23}$$

has $O(1)$ solutions v in J for fixed l. To see this, we apply Lemma 12 again. Abbreviating $\psi(l,v)$ to ψ, we have

$$\begin{aligned}(q(\psi(l,v),v))' &= q^{(1,0)}(\psi,v)\,\frac{\partial\psi}{\partial v}+q^{(0,1)}(\psi,v)\\ &= \frac{-q^{(1,0)}(\psi,v)f_1^{(1,1)}(\psi,v)+q^{(0,1)}(\psi,v)f_1^{(2,0)}(\psi,v)}{f_1^{(2,0)}(\psi,v)}.\end{aligned}$$

Our claim now follows from observation (ii) and the hypothesis concerning the zeros of h.

The domains $\mathbf{D}(q,r,\theta)$, $\mathbf{D}'(q,r,\theta)$ take the form

$$\{(u,v)\in\mathbf{D}: Hf_1\in I'\}$$

where $\mathbf{D}$ has the property assumed above and I' is an interval. Thus for (Ω_2), we need to show that $(Hf_1)(u,k)$ is C-monotonic in u for fixed k, $C=O(1)$. For (Ω_3), (Ω_4) we need statements of the form "$q(\psi(l,v),v)$ is C-monotonic in v for fixed l" with $C=O(1)$. For (Ω_3), we must take $q=Hf_1$, and for (Ω_4), $q=f_1^{(2,0)}$. Thus the verification of these two conditions can be completed by showing (for both choices of q) that the equation $h(v)=0$ has $O(1)$ solutions v in J. As above, all we need is a suitably chosen rectangle R, containing J in its interior, and a holomorphic function g, $|g(v)|<|h(v)|$ on R, such that $g+h$ is of a simple form and can be seen to have finitely many zeros in G. The case $q=Hf_1$ is distinctly more difficult.

Looking ahead to Theorem 8, we now take $(\alpha,\beta)=(1,2)$. Routine calculations, using the approximations to Hf_1 and $f_1^{(a,b)}$ already found above, give the desired approximation $g(v)+h(v)$ to $h(v)$. No matter what the value of k, the rational function $g+h$ cannot vanish identically. This is a matter of examining the roots of certain quadratic and linear polynomials, which we leave to the reader. In this way we can verify (Ω_3). The corresponding tasks for (Ω_4) and (Ω_2) are similar but simpler.

We now apply Theorem 7 to Type I sums.

THEOREM 8. *Let* $D \geq 1$, $MN \asymp D$, $X \gg ND^{1/6}$, $L = \log(XD+2)$, $|a_n| \leq 1$. *Let* $I_m \subseteq (N, 2N]$ *and*

$$S_1 = \sum_{m\sim M} a_m \sum_{n\in I_m} e\left(\frac{XMN}{mn}\right).$$

Then

$$S_1 \ll L^{3/2}\{D^{11/12} + DN^{-1/2} + X^{-1/14}D^{27/28}N^{1/14} + X^{1/8}D^{13/16}N^{-1/8} + X^{1/16}D^{27/32}N^{-1/16} + X^{1/14}N^{-1/7}D^{6/7} + X^{1/6}D^{5/6}N^{-1/6}N_0^{-1/6} + X^{1/26}D^{10/13}N^{2/13}\}.$$

Proof. Let Q be a positive integer, $Q < cN$. Just as in the proof of Theorem 4, there is a $q \in [1, Q]$ and a rectangle $\mathbf{D} \subseteq [M, 2M] \times [N, 2N]$ for which

$$S_1^2 \ll LD^2Q^{-1} + \frac{L^2Dq}{Q} \sum_{(m,n)\in\mathbf{D}} e(XMNm^{-1}((n+q)^{-1} - n^{-1})). \tag{5.24}$$

Let $N_0 = \min(M, N)$. We apply Theorem 7 to the sum on the right-hand side of (5.24), replacing F by Xq/N, (X, Y, N) by $(D/N_0, N_0, D)$ and η by Q/N. (We note that $Xq/N \gg D^{1/6}$.) Thus

$$L^{-3}S_1^2 \ll D^2Q^{-1} + D\{(XQ/N)^{1/3}D^{1/2} + D^{5/6} + (XQ/N)^{-1/8}D^{15/16} + (Q/N)^{2/5}D^{1/2}(XQ/N)^{1/5}N_0^{2/5} + (Q/N)^{1/4}D^{3/4}N_0^{1/4} + (Q/N)^{1/2}(XQ/N)^{1/4}D^{1/2}N_0^{1/4} + D^{1/2}(XQ)^{1/2}N^{-1/2}N_0^{-1/2} + D^{1/2}(XQ)^{1/12}N^{-1/12}N_0^{5/12}\}.$$

We simplify this bound using $N_0 \leq N$. We further restrict Q by

$$Q < X^{1/7}D^{1/14}N^{-1/7},$$

so that

$$(XQ)^{-1/8}D^{31/16}N^{1/8} < D^2/Q.$$

It follows that, for $0 < Q < \min(cN, X^{1/7}D^{1/14}N^{-1/7})$, we have

$$L^{-3}S_1^2 \ll D^2Q^{-1} + X^{1/3}D^{3/2}N^{-1/3}Q^{1/3} + D^{11/6} + D^{11/8}X^{1/2}Q^{1/2}N^{-1/2}N_0^{-1/4} + X^{1/5}D^{3/2}N^{-1/5}Q^{3/5} + D^{7/4}Q^{1/4} + X^{1/4}D^{3/2}N^{-1/2}Q^{3/4} + D^{3/2}X^{1/2}Q^{1/2}N^{-1/2}N_0^{-1/2} + D^{3/2}X^{1/12}Q^{1/12}N^{1/3}.$$

Applying Lemma 3, we find that

$$\begin{aligned}L^{-3}S_1^2 \ll D^{11/6} + X^{1/4}D^{13/8}N^{-1/4} + D^{19/12}X^{1/3}N^{-1/3}N_0^{-1/6} \\ + X^{1/8}D^{27/16}N^{-1/8} + D^2N^{-1} + D^{27/14}X^{-1/7}N^{1/7} + X^{1/7}N^{-2/7}D^{12/7} \\ + D^{5/3}X^{1/3}N^{-1/3}N_0^{-1/3} + D^{20/13}X^{1/13}N^{4/13}.\end{aligned}$$

Theorem 8 follows at once.

6. Proof of Theorem 1: initial steps. In this section, let $s = \sigma + it$ denote a complex variable.

LEMMA 18. *Assume the Riemann hypothesis, and let* $\sigma \in (1/2, 2]$. *For* $y \geq 1$, *we have*

$$\sum_{n \leq y} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)} + O(y^{1/2-\sigma+\varepsilon}(|t|^{\varepsilon} + 1)).$$

Proof. This is proved in all essentials in [22, §14.25].

We shall write

$$r(n) = \sum_{\substack{(u,v) \in \mathbb{Z}^2 \\ |u|^3+|v|^3=n}} 1, \qquad b = \frac{2\Gamma^2(1/3)}{3\Gamma(2/3)},$$

$$\Delta(x) = \sum_{n \leq x} r(n) - bx^{2/3}, \qquad Z(s) = \sum_{n \geq 1} \frac{r(n)}{n^s} \qquad (\sigma > 1).$$

Nowak [17] showed (in a more general context) that $Z(s)$ has an analytic continuation to $\sigma > 2/9$ with the exception of a simple pole at $s = 2/3$, with residue $2b/3$. His discussion yields the estimate

$$Z(s) \ll |t|^{9/7(1-\sigma+\varepsilon)} \qquad (\sigma \geq 2/9 + \varepsilon,\ |t| \geq 1). \tag{6.1}$$

LEMMA 19. *Let* λ *be a constant*, $2/9 < \lambda < 1/2$, *and suppose that*, *for every* $\varepsilon > 0$,

$$\int_T^{2T} |Z(\lambda + it)|\, dt \ll T^{1+\varepsilon} \tag{6.2}$$

for $T \geq 1$. *Then for* $\varepsilon > 0$, $1 \leq y < x^{1/3}$, *we have*

$$E(x) = \sum_{d \leq y} \mu(d) \Delta\left(\frac{x}{d^3}\right) + O(x^{\lambda+\varepsilon}y^{1/2-3\lambda}).$$

Proof. This is essentially stated in [25]. We give details for the convenience of the reader, following [17]. First of all,

$$(6.3)\qquad V(x) = \sum_{|m|^3+|n|^3\le x} \sum_{\substack{d\ge 1\\ d|m,\, d|n}} \mu(d) = \sum_{\substack{d\ge 1\\ d^3(|m|^3+|n|^3)\le x}} \mu(d)$$
$$= \sum_{\substack{d\ge 1,\, t\ge 1\\ d^3 t\le x}} \mu(d) r(t) = bx^{2/3} \sum_{d\le y} \frac{\mu(d)}{d^2} + \sum_{d\le y} \mu(d) \Delta\left(\frac{x}{d^3}\right) + Q(x),$$

where

$$Q(x) = \sum_{\substack{d>y,\, t\ge 1\\ d^3 t\le x}} \mu(d) r(t).$$

Now let

$$f(s) = \frac{1}{\zeta(s)} - \sum_{n\le y} \mu(n) n^{-s} \qquad (\sigma > 1/2),$$

so that

$$f(3s)Z(s) = \sum_{n\ge 1} \frac{a(n)}{n^s}, \qquad a(n) = \sum_{\substack{d>y,\, t\ge 1\\ d^3 t = n}} \mu(d) r(t).$$

An application of Perron's formula gives

$$Q(x) = \frac{1}{2\pi i} \int_{2-ix^C}^{2+ix^C} f(3s)Z(s)\, \frac{x^s}{s}\, ds + O(1)$$

for any constant $C > 2$. We move the vertical segment to the left. For a sufficiently large C, we obtain

$$(6.4)\qquad Q(x) = \frac{1}{2\pi i} \int_{\lambda-ix^C}^{\lambda+ix^C} f(3s)Z(s)\, \frac{x^s}{s}\, ds + \mathrm{Res}\left(\frac{f(3s)Z(s)x^s}{s}, \frac{2}{3}\right) + O(1)$$
$$= \frac{1}{2\pi i} \int_{\lambda-ix^C}^{\lambda+ix^C} f(3s)Z(s)\, \frac{x^s}{s}\, ds + bx^{2/3} \sum_{d>y} \frac{\mu(d)}{d^2} + O(1)$$

on applying (6.1) on the horizontal segments.

By a splitting-up argument, there is a T, $1 \le T \le x^C$, such that

$$(6.5)\qquad \left| \int_{\lambda-ix^C}^{\lambda+ix^C} f(3x)Z(s)\, \frac{x^s}{s}\, ds \right| \ll \frac{x^\lambda \log x}{T} \int_{T-1}^{2T} |f(3\lambda+3it)Z(\lambda+it)|\, dt$$
$$\ll x^{\lambda+\varepsilon} y^{1/2-3\lambda}.$$

The last estimate follows from Lemma 18 and (6.2). The lemma follows at once on combining (6.3)–(6.5), since

$$b\sum_{d\ge 1}\frac{\mu(d)}{d^2}=\frac{6b}{\pi^2}=\frac{4\Gamma^2(1/3)}{\pi^2\Gamma(2/3)}.$$

The best available value of λ at present is 4/9:

LEMMA 20. *For $T\ge 1$,*

$$\int_T^{2T}|Z(4/9+it)|\,dt \ll T\log T.$$

Proof. This follows from Lemma 3.1 of Zhai [25] on applying the Cauchy–Schwarz inequality.

Let $\psi(t)=\{t\}-1/2$, where $\{\ \}$ denotes the fractional part.

LEMMA 21. *We may write*

$$\Delta(x)=\Delta_1(x)+\Delta_2(x)$$

where, *for a positive constant* c_1,

$$\Delta_1(x)=c_1x^{2/9}\sum_{l=1}^{\infty}\frac{1}{l^{4/3}}\cos 2\pi(lx^{1/3}-1/3)+O(1),$$

$$\Delta_2(x)=-8\sum_{(x/2)^{1/3}\le n\le x^{1/3}}\psi((x-n^3)^{1/3})+O(1).$$

Proof. See Krätzel [12, Chapter 3].

On combining the last three lemmas, we obtain the decomposition $E(x)=E_1(x)+E_2(x)+E_3(x)$ where (for a parameter y in $[1,x^{1/3}]$ which is at our disposal)

$$E_1(x)=c_1x^{2/9}\sum_{d\le y}\frac{\mu(d)}{d^{2/3}}\sum_{k=1}^{\infty}\frac{1}{k^{4/3}}\cos 2\pi\left(\frac{kx^{1/3}}{d}-\frac{1}{3}\right),$$

$$E_2(x)=-8\sum_{d\le y}\mu(d)\sum_{x^{1/3}/(2^{1/3}d)\le n\le x^{1/3}/d}\psi\left(\left(\frac{x}{d^3}-n^3\right)^{1/3}\right),$$

$$E_3(x)=O(x^{4/9+\varepsilon}y^{-5/6}+y).$$

To obtain (1.2), we choose $y=x^{8/15-6\theta/5}$. It now suffices to show, for any D, D' with $1\le D\le y$, $D<D'\le 2D$, and any $K\ge 1$, that

$$\sum_{k\sim K}\sum_{D\le d\le D'}\mu(d)e\left(\frac{kx^{1/3}}{d}\right)\ll K^{4/3}D^{2/3}x^{\theta-2/9+\varepsilon} \tag{6.6}$$

and

$$\sum_{d\sim D}\mu(d)\sum_{n\sim x^{1/3}/(2^{1/3}d)}\psi\Big(\Big(\frac{x}{d^3}-n^3\Big)^{1/3}\Big)\ll x^{\theta+\varepsilon}. \tag{6.7}$$

We complete this section with a proof of a stronger result than (6.7), namely

$$\begin{aligned} S(D) := \sum_{d\sim D}\mu(d)\sum_{n\sim x^{1/3}/(2^{1/3}d)}&\psi\Big(\Big(\frac{x}{d^3}-n^3\Big)^{1/3}\Big)\\ &\ll x^{7/27+\varepsilon}\quad (D\ll x^{2/9}). \end{aligned}\tag{6.8}$$

(This is the bound corresponding to (6.7) if θ is replaced by $7/27$.)

Lemma 22. *For $H\ge 1$, we have a representation*

$$\psi(u)=\sum_{1\le|h|\le H}a(h)e(hu)+O\Big(\sum_{1\le h\le H}b(h)e(hu)\Big)+O(H^{-1})$$

with coefficients $a(h)\ll 1/|h|$, $b(h)\ll 1/H$.

Proof. See Vaaler [23], or the appendix to [6].

We now split up $S(D)$ as follows. For $d\sim D$ (suppressing dependence on d), let

$$N_j=\frac{x^{1/3}}{d(1+2^{-3j/2})^{1/3}},\qquad j=0,1,\dots,J.$$

Here J is the least integer such that $x^{1/3}/d-N_J\le x^{\varepsilon}$. Thus $J\ll\log x$, $N_{j+1}-N_j\asymp x^{1/3}2^{-3j/2}D^{-1}\gg x^{\varepsilon}$, and in particular

$$x^{1/3}2^{-3j/2}D^{-1}\gg x^{\varepsilon} \tag{6.9}$$

for $j=0,1,\dots,J$.

It suffices to show that for each $j=0,\dots,J$,

$$\sum_{d\sim D}\mu(d)\sum_{n\in I_d}\psi\Big(\Big(\frac{x}{d^3}-n^3\Big)^{1/3}\Big)\ll x^{7/27+\varepsilon}, \tag{6.10}$$

where $I_d=I_d(j)=[N_j,N_{j+1}]$.

It is convenient to write P for 2^j. We apply Lemma 22 with

$$H=\max(x^{2/27}P^{-3/2},1). \tag{6.11}$$

Thus the sum in (6.10) can be rewritten as

$$\sum_{d\sim D}\mu(d)\sum_{1\le|h|\le H}a(h)\sum_{n\in I_d}e\biggl(h\biggl(\frac{x}{d^3}-n^3\biggr)^{1/3}\biggr)$$
$$+O\biggl(\sum_{d\sim D}\sum_{1\le h\le H}b(h)\sum_{n\in I_d}e\biggl(h\biggl(\frac{x}{d^3}-n^3\biggr)^{1/3}\biggr)\biggr)+O(x^{7/27}).$$

We need only show that, for $1\le K\le H$ and $|a_h|\le 1$,

$$S(D,K,P):=K^{-1}\sum_{d\sim D}\mu(d)\sum_{h\sim K}a_h\sum_{n\in I_d}e\biggl(h\biggl(\frac{x}{d^3}-n^3\biggr)^{1/3}\biggr) \ll x^{7/27+\varepsilon}. \tag{6.12}$$

The corresponding result with 1 in place of $\mu(d)$ is, of course, easier.

We apply the B-process to the sum over n in (6.10). We may quote the result from Kühleitner [13, (3.5)]:

$$S(D,K,P)\ll\frac{x^{1/6}}{P^{5/4}D^{1/2}K^{3/2}}\,|S'(D,K,P)|+D\log x, \tag{6.13}$$

with

$$S'(D,K,P)=\sum_{d\sim D}\mu(d)\biggl(\frac{D}{d}\biggr)^{1/2}\sum_{(h,m)\in\mathbf{T}}b(h,m)e\biggl(\frac{-x^{1/3}|(h,m)|_{3/2}}{d}\biggr).$$

Here $b(h,m)\ll 1$ and $\mathbf{T}=\{(h,m):h\sim K,\ Ph\le m\le 2Ph\}$. Thus we must show that

$$\sum_{D<d\le D'}\mu(d)\sum_{(h,m)\in\mathbf{T}}b(h,m)e\biggl(\frac{-x^{1/3}|(h,m)|_{3/2}}{d}\biggr) \ll x^{5/54+\varepsilon}P^{5/4}D^{1/2}K^{3/2}. \tag{6.14}$$

If $K<x^{5/27}P^{1/2}D^{-1}$, then (6.14) is trivial. We now assume that

$$H\ge K\ge x^{5/27}P^{1/2}D^{-1}. \tag{6.15}$$

We next dispose of the case where

$$K<\min(x^{-2/27}P^{-1/4}D^{1/2},x^{5/27}P^{1/2}D^{-2/3}) \tag{6.16}$$

by treating the variables h,m trivially in (6.14). In view of Lemma 2(ii), we need only show that, for $|a_m|\le 1$, $|b_n|\le 1$,

$$\sum_{\substack{m\sim M,\,n\sim N\\ D<mn\le D'}}a_mb_ne\biggl(\frac{Y}{mn}\biggr)\ll x^{5/54+\varepsilon}P^{1/4}D^{1/2}K^{-1/2} \tag{6.17}$$

whenever

$$D^{1/3}\ll N\ll D^{1/2},\qquad Y\asymp x^{1/3}PK, \tag{6.18}$$

and that for $|a_m| \le 1$,

$$\sum_{\substack{m\sim M,\, n\sim N\\ D<mn\le D'}} a_m e\left(\frac{Y}{mn}\right) \ll x^{5/54+\varepsilon} P^{1/4} D^{1/2} K^{-1/2} \tag{6.19}$$

whenever

$$N \gg D^{2/3}, \qquad Y \asymp x^{1/3} PK. \tag{6.20}$$

For (6.17), we use Lemma 5 with $M_1 = M$, $M_2 = 1$, $M_3 = N$, $M_4 = 1$. The left-hand side of (6.17) is

$$\begin{aligned}
&\ll (\log x)^2\{Y^{1/2} + M^{1/2}N + MN^{1/2} + MN(Y/D)^{-1/2}\}\\
&\ll (\log x)^2\{x^{1/6}P^{1/2}K^{1/2} + D^{5/6} + D^{3/2}x^{-1/6}P^{-1/2}K^{-1/2}\}\\
&\ll (\log x)^2\{x^{1/6}P^{1/2}K^{1/2} + D^{5/6}\} \ll x^{5/54+\varepsilon}P^{1/4}D^{1/2}K^{-1/2},
\end{aligned}$$

where we appeal to (6.16) in the last step.

For (6.19), we treat m trivially and estimate the sum over n using the exponent pair $(1/2, 1/2)$. The left-hand side of (6.19) is

$$\ll M\left(\frac{Y}{D}\right)^{1/2} + M\left(\frac{Y}{DN}\right)^{-1} \ll MD^{-1/2}x^{1/6}P^{1/2}K^{1/2} + D^2x^{-1/3}P^{-1}K^{-1}.$$

Certainly $D^2x^{-1/3}P^{-1}K^{-1} \ll x^{5/54}P^{1/4}D^{1/2}K^{-1/2}$, and we obtain

$$MD^{-1/2}x^{1/6}P^{1/2}K^{1/2} \ll x^{5/54}P^{1/4}D^{1/2}K^{-1/2}$$

by appealing to (6.16) and (6.18). This completes the treatment of the case (6.16).

We note in particular that (6.14) holds whenever $H = 1$, for in this case $K = 1$, while (6.15) gives $D \ge x^{5/27}P^{1/2}$. Now (6.16) is easily verified. We may now suppose that $H = x^{2/27}P^{-3/2}$. Since $K \le H$, we have

$$KP^{3/2} \le x^{2/27}, \qquad P \le x^{4/81}. \tag{6.21}$$

We are now in a position to apply Theorem 2 with $(\kappa, \lambda) = (1/2, 1/2)$ and essentially $(D, K, PK, x^{1/3}PKD^{-1})$ in place of (M, M_1, M_2, X). We may suppose, in addition to (6.21), that

$$K \ge DP^{-1/2}x^{-5/27}, \tag{6.22}$$

for in the contrary case we note that (6.16) holds, since

$$DP^{-1/2}x^{-5/27}(x^{-2/27}P^{-1/4}D^{1/2})^{-1} \ll D^{1/2}x^{-3/27} \ll 1,$$

$$DP^{-1/2}x^{-5/27}(x^{5/27}P^{1/2}D^{-2/3})^{-1} \ll D^{5/3}x^{-10/27} \ll 1.$$

The condition $X \gg M_2^2$ in Theorem 2 reduces to $x^{1/3}PKD^{-1} \gg P^2K^2$, that is, $DPK \ll x^{1/3}$. This is a consequence of (6.21). Thus the left-hand side of (6.14) is

$$\begin{aligned}&\ll x^{\varepsilon}(DPK + D^{2/3}(PK)^2(x^{1/3}/(DPK))^{1/6})\\&\ll x^{\varepsilon}DPK + D^{1/2}x^{1/18+\varepsilon}(PK)^{11/6}.\end{aligned}$$

It remains to show that

$$DPK \le x^{5/54}P^{5/4}D^{1/2}K^{3/2}$$

(which is simply (6.22)), and that

$$D^{1/2}x^{1/18}(PK)^{11/6} \ll x^{5/54}P^{5/4}D^{1/2}K^{3/2},$$

that is, $P^{7/12}K^{1/3} \ll x^{1/27}$. This is an easy consequence of (6.21):

$$P^{7/12}K^{1/3} = P^{1/12}(P^{3/2}K)^{1/3} \ll x^{1/243+2/81}.$$

This completes the proof of (6.14).

7. Completion of the proof of Theorem 1. It remains to prove (6.6). We write $D = x^{\phi}$. Since the trivial bound gives (6.6) for $\phi \le 3\theta - 2/3$, we assume that

$$0.113\ldots = 3\theta - \frac{2}{3} < \phi \le \frac{8}{15} - \frac{6\theta}{5} = 0.221\ldots. \tag{7.1}$$

We fix $K \ge 1$ and D', $D \le D' < 2D$. Let

$$S_1 = \sum_{l\sim K}\ \sum_{\substack{m\sim M\\ D\le mn<D'}}\ \sum_{n\sim N} a_m e\Big(\frac{lx^{1/3}}{mn}\Big), \qquad S_2 = \sum_{l\sim K}\ \sum_{\substack{m\sim M\\ D\le mn<D'}}\ \sum_{n\sim N} a_m b_n e\Big(\frac{lx^{1/3}}{mn}\Big)$$

with coefficients satisfying $|a_m| \le 1$, $|b_n| \le 1$.

Lemma 23. *Suppose that*

$$N \gg D^{2/3}x^{4/9-2\theta}. \tag{7.2}$$

Then

$$S_1 \ll K^{4/3}D^{2/3}x^{\theta-2/9+\varepsilon} \tag{7.3}$$

provided that either

$$N \gg D^{-25/21}x^{-50\theta/7+19/9} \tag{7.4}$$

or

$$N \gg D^{-944/267}x^{-1888\theta/89+4898/801}. \tag{7.5}$$

Proof. It suffices to show that

$$S_1' := \sum_{\substack{m\sim M\\ D\le mn<D'}}\ \sum_{n\sim N} a_m e\Big(\frac{lx^{1/3}}{mn}\Big) \ll l^{1/3}D^{2/3}x^{\theta-2/9+\varepsilon}. \tag{7.6}$$

We appeal to Theorem 4 with $X \asymp lx^{1/3}D^{-1}$. In (4.15), the terms $DN^{-1/2}$, $DX^{-1/2}$ are acceptable because of (7.2). As $N_0 \le N$ and $(1+2\kappa)/(6+4\kappa) \le 1/4$, we need only show that

$$(D^{4+4\kappa}(x^{1/3}D^{-1})^{1+2\kappa}N^{-(1+4\kappa-2\lambda)})^{1/(6+4\kappa)} \ll D^{2/3}x^{\theta-2/9}.$$

The condition (7.4) arises on choosing $(\kappa,\lambda) = (2/7, 4/7) = BA^2B(0,1)$, while (7.5) arises from $(\kappa,\lambda) = (89/570, 1/2 + 89/570)$. The latter exponent pair requires lengthy arguments (Huxley [9, Chapter 17]).

We remark that the slightly stronger exponent pair in [10], $(32/205, 1/2 + 32/205)$, would not significantly "reduce θ".

LEMMA 24. *Suppose that* (7.2) *holds*, *and that* $\phi \le 4(\theta - 2/9) = 0.151\ldots$,

$$N \le D^{-25/21}x^{-50\theta/7+19/9}, \tag{7.7}$$

and

$$N \ge x^{19/9-8\theta}D^{1/6}. \tag{7.8}$$

Then (7.3) *holds.*

Proof. It suffices to prove (7.6). Since $N \ll D \ll x^{1/3}D^{-7/6}$, we may appeal to Theorem 8 with $X \asymp lx^{1/3}D^{-1}$. Thus

$$\begin{aligned} S_1 \ll (\log x)^{3/2} l^{1/6}\{&D^{11/12} + DN^{-1/2} + x^{-1/42}D^{29/28}N^{1/14} \\ &+ x^{1/24}D^{11/16}N^{-1/8} + x^{1/48}D^{25/32}N^{-1/16} + x^{1/42}D^{11/14}N^{-1/7} \\ &+ D^{2/3}x^{1/18}N^{-1/3} + x^{1/18}D^{1/2} + x^{1/78}D^{19/26}N^{2/13}\}. \end{aligned} \tag{7.9}$$

The first term on the right-hand side of (7.9) is acceptable as $\phi \le 4(\theta - 2/9)$. The second term is acceptable because of (7.2). The third term is acceptable because of (7.7), which is stronger than the required condition

$$N \le D^{-31/6}x^{14\theta-25/9}$$

since $\phi < 0.152$.

The fourth, fifth, sixth and seventh terms are acceptable because of (7.8). The eighth term is acceptable since $D > x^{0.11}$. The last term is also acceptable because of (7.7). This completes the proof of the lemma.

LEMMA 25. *Let* $\phi < 1/6$. *We have*

$$S_2 \ll K^{4/3}D^{2/3}x^{\theta-2/9+\varepsilon} \tag{7.10}$$

provided that (7.2) *holds*, *and either*

$$N \ll \min(x^{276\theta/5-14}D^{-2}, D^{1/2}) \tag{7.11}$$

or

$$N \ll \min(D^{-2/3}x^{1508\theta/95-226/57}, D^{1/2}). \tag{7.12}$$

Proof. It suffices to show that

$$S_2' := \sum_{\substack{m\sim M\\ D\le mn<D'}} \sum_{n\sim N} a_m b_n e\left(\frac{lx^{1/3}}{mn}\right) \ll l^{1/3} D^{2/3} x^{\theta-2/9+\varepsilon}. \tag{7.13}$$

We appeal to Theorem 5 with $X \asymp lx^{1/3}/D \gg D$. Again, $DN^{-1/2}$ is acceptable. Since $M \gg D^{1/2}$ and $\phi < 1/6$, we have

$$DM^{-1/4}(D^{2/3}x^{\theta-2/9})^{-1} \ll D^{5/24}x^{-\theta+2/9} \ll 1,$$

so the term $DM^{-1/4}$ is acceptable. Since $(1+2\kappa)/(14+12\kappa) < 1/3$, it remains to show that

$$(D^{11+10\kappa}(x^{1/3}/D)^{1+2\kappa}N^{2(\lambda-\kappa)})^{1/(14+12\kappa)} \ll D^{2/3}x^{\theta-2/9}.$$

The condition (7.11) arises on choosing $(\kappa,\lambda)=(11/30,16/30)=BA^3B(0,1)$, while (7.12) arises from $(\kappa,\lambda)=(89/570, 1/2+89/570)$.

LEMMA 26. *We have* (7.10) *provided that* (7.2) *holds and either*

$$\phi < 26/15 - 6\theta = 0.17\ldots, \qquad N \ll D^{13/6}x^{10\theta-26/9}, \tag{7.14}$$

or

$$\phi \ge 26/15 - 6\theta, \qquad N \ll D^{1/2}. \tag{7.15}$$

Proof. Again, we need only prove (7.13). Lemma 5 with $(\kappa,\lambda)=(1/2,1/2)$ yields

$$\begin{aligned} S_2' \ll (\log x)^3 l^{1/5}\{&x^{1/15}D^{9/20}N^{1/10} + x^{1/18}D^{1/2}N^{1/9} + x^{1/15}D^{2/5}N^{1/5} \\ &+ x^{1/33}D^{6/11}N^{3/11} + D^{2/3}N^{5/18} \\ &+ DN^{-1/2} + x^{-1/66}D^{15/22}N^{9/22} + D^{3/2}x^{-1/6}\}. \end{aligned}$$

The last four terms are easily seen to be acceptable in view of (7.2). Since $N \ll D^{1/2}$, we have

$$x^{1/15}D^{2/5}N^{1/5} \ll x^{1/15}D^{9/20}N^{1/10},$$
$$x^{1/33}D^{6/11}N^{3/11} \ll x^{1/33}D^{15/22} \ll D^{2/3}x^{\theta-2/9}.$$

Moreover,

$$x^{1/15}D^{9/20}N^{1/10} \ll D^{2/3}x^{\theta-2/9}$$

for $N \ll D^{13/6}x^{10\theta-26/9}$, and certainly if (7.15) holds.

The remaining term $x^{1/18}D^{1/2}N^{1/9}$ is $\ll D^{2/3}x^{\theta-2/9}$ for $N \ll D^{3/2}x^{9\theta-5/2}$, which holds if either (7.14) or (7.15) is assumed.

LEMMA 27. *We have* (7.10) *provided that*

$$N \gg D^{4/3}x^{8/9-4\theta}, \tag{7.16}$$

and either

$$\phi < 22/21 - 24\theta/7 = 0.155\ldots, \qquad N \ll D^{5/3}x^{4\theta-11/9}, \tag{7.17}$$

or

$$\phi \geq 22/21 - 24\theta/7, \qquad N \ll D^{1/2}. \tag{7.18}$$

Proof. From Lemma 7, essentially with $(K, N, M, Kx^{1/3}/D)$ in place of (H, N, M, X), we have

$$S_2 \ll x^\varepsilon K\{x^{1/12}D^{1/4}N^{1/4} + DN^{-1/4} + D^{1/2}N^{1/2} + D^{3/2}x^{-1/6}\}.$$

The last term has already been discussed above. The second term is acceptable since (7.16) holds. The first term is acceptable since (7.17) (or, if $\phi \geq 22/21 - 24\theta/7$, the stronger condition (7.18)) holds. Finally, the third term is acceptable since $N \ll D^{1/2}$.

LEMMA 28. *Let*

$$\phi < \frac{24}{5}\left(\theta - \frac{2}{9}\right) = 0.18\ldots. \tag{7.19}$$

We have (7.10) *provided that* (7.2) *holds and*

$$N \ll \min((D^{659}x^{5076\theta-1410})^{1/187}, D^{1/2}). \tag{7.20}$$

Proof. We apply Theorem 6 with $\alpha = \beta = -1$, $X \asymp x^{1/3}D^{-1}$, taking $(\kappa, \lambda) = BA(89/570, 1/2 + 89/570) = (187/659, 374/659)$. The term $L^2DN^{-1/2}$ is acceptable because of (7.2). The term $L^2DN^{-1/4}$ is acceptable since

$$DM^{-1/4} \ll D^{7/8} \ll D^{2/3}x^{\theta-2/9}$$

from (7.19). Finally, the term

$$X^{1/6}(D^{4+5\kappa}N^{\lambda-\kappa})^{1/(6+6\kappa)} \asymp x^{1/18}(D^{2725}N^{187})^{1/5076}$$

is acceptable because of (7.20).

Completion of the proof of Theorem 1. We assume (7.1) and show that (6.6) holds.

Suppose first that $\phi > 26/15 - 6\theta$. By Lemma 25, we have (7.13) for

$$D^{2/3}x^{4/9-2\theta} \ll N \ll D^{1/2}$$

and moreover, $D^{2/3}x^{4/9-2\theta} \ll D^{1/3}$.

By Lemma 23, we have (7.3) for

$$N \gg D^{-25/21}x^{-50\theta/7+19/9}.$$

We note that

$$D^{-25/21}x^{-50\theta/7+19/9} \ll D^{3/5} \quad \text{for } \phi > 0.142. \tag{7.21}$$

Now (6.6) follows from Lemma 2(ii).

Suppose next that

$$0.161\ldots = \frac{30}{7}\left(\theta - \frac{2}{9}\right) < \phi \leq \frac{26}{15} - 6\theta. \tag{7.22}$$

We claim that (7.10) holds for $D^{1/4} \ll N \ll D^{2/5}$ and (7.3) holds for $N \gg D^{2/5}$. This is sufficient for (6.6) in view of Lemma 2(i) with $h = 4$.

For S_2, we use Lemma 28. We have

$$D^{2/3}x^{4/9-2\theta} < D^{1/4},$$

since $\phi \le 26/15 - 6\theta < (24/5)(\theta - 2/9)$. We also have

$$(D^{659}x^{5076\theta-1410})^{1/187} > D^{2/5};$$

this requires only $\phi > 0.156$. Moreover,

$$\min(D^{1/2}, (D^{659}x^{5076\theta-1410})^{1/187}) > D^{-25/21}x^{-50\theta/7+19/9};$$

this requires only $\phi > 0.157$. In view of Lemma 25, we conclude that (7.13) holds for $D^{1/4} \ll N \ll D^{2/5}$ and (7.3) holds for $N \gg D^{2/5}$, as required for the range (7.22).

Suppose next that

$$0.156\ldots = \frac{22}{21} - \frac{24\theta}{7} < \phi \le \frac{30}{7}\left(\theta - \frac{2}{9}\right). \tag{7.23}$$

We claim that (7.10) holds for $D^{1/5} \ll N \ll D^{1/3}$ and $D^{2/5} \ll N \ll D^{1/2}$, while (7.3) holds for $N \gg D^{2/5}$. This is sufficient for (6.6), in view of Lemma 2(iii).

We have

$$D^{2/3}x^{4/9-2\theta} \le D^{1/5},$$

since $\phi \le (30/7)(\theta - 2/9)$, while

$$(D^{659}x^{5076\theta-1410})^{1/187} > D^{1/3};$$

this requires only $\phi > 0.153$. Thus Lemma 28 gives (7.10) for $D^{1/5} \ll N \ll D^{1/3}$. Moreover, Lemma 27 gives (7.10) for $D^{2/5} \ll N \ll D^{1/2}$, and indeed for $D^{2/5} \ll N \ll D^{3/5}$. Now, recalling (7.21), we have (7.3) for $N \gg D^{2/5}$, and we have established (6.6) in the range (7.23).

Suppose next that

$$0.150\ldots = 4\left(\theta - \frac{2}{9}\right) < \phi \le \frac{22}{21} - \frac{24\theta}{7}. \tag{7.24}$$

We now use Lemma 25. This yields (7.10) for

$$D^{2/3}x^{4/9-2\theta} \ll N \ll D^{-2/3}x^{1508\theta/95-226/57},$$

while Lemma 27 yields (7.10) for

$$D^{4/3}x^{8/9-4\theta} \ll N \ll D^{5/3}x^{4\theta-11/9}.$$

Note that

$$D^{5/3}x^{4\theta-11/9} \ge D^{-944/267}x^{-1888\theta/89+4898/801},$$

since

$$\phi\left(\frac{5}{3}+\frac{944}{267}\right) \ge 4\left(\theta-\frac{2}{9}\right)\left(\frac{5}{3}+\frac{944}{267}\right) = \frac{4898}{801}+\frac{11}{9}-\theta\left(\frac{1888}{89}+4\right).$$

In view of Lemma 23, we have (7.10) for $N \gg D^{4/3}x^{8/9-4\theta}$. We now apply Lemma 2(iv) with

$$D^{\chi} = D^{2/3}x^{4/9-2\theta}, \qquad D^{\psi} = D^{-2/3}x^{1508\theta/95-226/57}.$$

Since $4(\theta-2/9) < \phi \le (30/7)(\theta-2/9)$, we have $1/6 < \chi \le 1/5$, and

$$\max\left(\frac{1}{3}, \frac{1}{5}+\frac{4\chi}{5}\right) = \frac{1}{5}+\frac{4\chi}{5}.$$

Moreover,

$$D^{-2/3}x^{1508\theta/95-226/57} > D^{1/5}(D^{2/3}x^{4/9-2\theta})^{4/5};$$

this requires only $\phi < 0.157$. Thus Lemma 2(iv) is applicable, and (6.6) holds in the range (7.24).

Suppose now that

$$0.138\ldots = \frac{2}{5}\left(\frac{276\theta}{5}-14\right) < \phi \le 4\left(\theta-\frac{2}{9}\right). \tag{7.25}$$

Then (7.3) holds for

$$N \gg x^{19/9-8\theta}D^{1/6}. \tag{7.26}$$

To see this, we appeal to Lemma 24. We may suppose that (7.7) holds, in view of Lemma 23.

In order to apply Lemma 2 with $h=5$, we need only verify that

$$\max(D^{1/3}, x^{19/9-8\theta}D^{1/6}) < D^{-2/3}x^{1508\theta/95-226/57}.$$

This requires only $\phi < 0.154$, and we have established (6.6) in the range (7.25).

Suppose finally that

$$\phi \le \frac{2}{5}\left(\frac{276\theta}{5}-14\right). \tag{7.27}$$

From Lemma 25, (7.10) holds for $D^{2/3}x^{4/9-2\theta} \ll N \ll D^{1/2}$. Moreover, $D^{1/3}x^{2\theta-4/9} > D^{2/3}$. In view of Lemma 2(ii), it suffices to establish (7.6) for

$$N > D^{1/3}x^{2\theta-4/9}.$$

We estimate the sum over m in (7.6) trivially and apply the exponent pair $(1/6, 2/3) = AB(0,1)$ to the sum over n. Since $lx^{1/3}/D > D \gg N$, this gives

$$S_1' \ll M\left(\frac{lx^{1/3}}{D}\right)^{1/6}N^{1/2} \ll l^{1/6}x^{1/18}D^{5/6}N^{-1/2} \ll l^{1/6}D^{2/3}x^{\theta-2/9}$$

for $N \gg D^{1/3}x^{5/9-2\theta}$. This is stronger than we need, and we have (6.6) for the range (7.27). This completes the proof of Theorem 1.

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