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Simultaneous approximation of a real number by all conjugates of an algebraic number

by

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1. Introduction. An outstanding problem in Diophantine approximation, motivated initially by Mahler's and Koksma's classifications of numbers, is to provide sharp estimates for the approximation of a real number by algebraic numbers of bounded degree. Starting with the pioneer work [Wi] of E. Wirsing in 1961, this problem has been studied by many authors and extended in several directions. A good account of this can be found in Chapter 3 of [Bu]. For our purpose, let us simply mention that, in 1969, H. Davenport and W. M. Schmidt gave estimates for the approximation by algebraic integers [DS] and that, more recently, D. Roy and M. Waldschmidt looked at simultaneous approximations by several conjugate algebraic integers [RW]. While the latter work was limited to at most one quarter of the conjugates, we consider here the problem of simultaneous approximation of a real number by all (resp. all but one) conjugates of an algebraic number (resp. algebraic integer). Upon defining the *height* $H(P)$ of a polynomial $P \in \mathbb{R}[T]$ to be the largest absolute value of its coefficients, and the *height* $H(\alpha)$ of an algebraic number $\alpha \in \mathbb{C}$ to be the height of its irreducible polynomial in $\mathbb{Z}[T]$, our main result reads as follows.

Theorem A. *Let $\xi \in \mathbb{R}\backslash\mathbb{Q}$ and let $n \in \mathbb{N}^*$. There exist positive constants c_1, c_2, depending only on ξ and n, with the following properties*:

(i) *There are infinitely many algebraic numbers α of degree n such that*

$$\max_{\overline{\alpha}} |\xi - \overline{\alpha}| \le c_1 H(\alpha)^{-2/n} \tag{1}$$

where the maximum is taken over all conjugates $\overline{\alpha}$ of α.

(ii) *There are infinitely many algebraic integers α of degree $n+1$ such that*

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$$\max_{\overline{\alpha}\neq\alpha} |\xi - \overline{\alpha}| \le c_2 H(\alpha)^{-2/n} \tag{2}$$

where the maximum is taken over all conjugates $\overline{\alpha}$ different from α.

In the case $n = 2$, this improves the estimates of the Corollary in Section 1 of [AR]. In fact, as we will see in the next section, the statement of part (i) is optimal up to the value of c_1 for each $\xi \in \mathbb{R} \setminus \mathbb{Q}$, while the statement of part (ii) is optimal up to the value of c_2 at least for quadratic irrational values of ξ. This seems to be the first instance where an optimal exponent of approximation is known for all values of the degree n in this type of problem. The fact that we can control the degree of the approximations originates from an observation of Y. Bugeaud and O. Teulié in [BT].

An irrational real number ξ is said to be *badly approximable* if there exists a constant $c > 0$ such that $|\xi - p/q| \ge cq^{-2}$ for any rational number p/q. This is equivalent to asking that ξ has bounded partial quotients in its continued fraction expansion (see Theorem 5F in Chapter 1 of [Sc]). For these numbers, we can refine Theorem A as follows.

THEOREM B. *Let $\xi \in \mathbb{R} \setminus \mathbb{Q}$ be badly approximable and let $n \in \mathbb{N}^*$. Then there exist positive constants $c_1, \dots, c_4$, depending only on ξ and n, with the following properties*:

(i) *For each real number $X \ge 1$, there is an algebraic number α of degree n satisfying* (1) *and $c_3 X \le H(\alpha) \le c_4 X$.*

(ii) *For each real number $X \ge 1$, there is an algebraic integer α of degree $n+1$ satisfying* (2) *and $c_3 X \le H(\alpha) \le c_4 X$.*

The proof of both results follows the method introduced by Davenport and Schmidt in [DS]. Let $\mathbb{R}[T]_{\le n}$ denote the real vector space of polynomials of degree $\le n$ in $\mathbb{R}[T]$, and let $\mathbb{Z}[T]_{\le n}$ denote the subgroup of polynomials with integral coefficients in $\mathbb{R}[T]_{\le n}$. We first provide estimates for the last minimum of certain convex bodies of $\mathbb{R}[T]_{\le n}$ with respect to $\mathbb{Z}[T]_{\le n}$ and then deduce the existence of polynomials of $\mathbb{Z}[T]_{\le n}$ with specific inhomogeneous Diophantine properties. This is done in Section 3. In Section 4, we show that these polynomials have roots which meet the requirements of Theorem A or B.

Throughout this paper, all implied constants in the Vinogradov symbols $\gg$, $\ll$ and their conjunction $\asymp$ depend only on ξ and n.

2. Optimality of the exponents of approximation. Let $\xi \in \mathbb{R} \setminus \mathbb{Q}$ and let $n \in \mathbb{N}^*$. If $n \ge 2$, the result in part (i) of Theorem A is optimal up to the value of the implied constant since, for any algebraic number α of degree n with conjugates $\alpha_1, \dots, \alpha_n$, the discriminant $D(\alpha)$ of α satisfies

$$|D(\alpha)| \le H(\alpha)^{2(n-1)} \prod_{1\le i<j\le n} |\alpha_i - \alpha_j|^2 \le H(\alpha)^{2(n-1)} (2 \max_{1\le i\le n} |\xi - \alpha_i|)^{n(n-1)}.$$

Since $D(\alpha)$ is a non-zero integer, its absolute value is ≥ 1, and thus we deduce that

$$\max_{1\leq i\leq n} |\xi-\alpha_i| \geq \tfrac{1}{2}H(\alpha)^{-2/n}$$

(compare with §5 of [Wi]). If $n=1$, the result is optimal for any badly approximable ξ. Note that a similar argument also shows that, for any algebraic integer α of degree $n+1$ with conjugates $\alpha_1,\dots,\alpha_{n+1}$, we have

$$\max_{1\leq i\leq n} |\xi-\alpha_i| \geq \tfrac{1}{2}H(\alpha)^{-2/(n-1)}.$$

Similarly, the result in part (ii) of Theorem A is optimal up to the value of the implied constant when ξ is a quadratic irrational number. To prove this, suppose that an algebraic integer α of degree $n+1$ has conjugates $\alpha_1,\dots,\alpha_{n+1}$ distinct from ξ with the first n satisfying

$$\max_{1\leq i\leq n} |\xi-\alpha_i| \leq 1.$$

Let $Q(T)\in\mathbb{Z}[T]$ be the irreducible polynomial of ξ over $\mathbb{Z}$. Since α is an algebraic integer, the product $Q(\alpha_1)\cdots Q(\alpha_{n+1})$ is a rational integer, and since it is non-zero (because ξ is not a conjugate of α), we deduce that

$$1 \leq \prod_{i=1}^{n+1} |Q(\alpha_i)|.$$

For each $i=1,\dots,n$, we have $|Q(\alpha_i)| \ll |\xi-\alpha_i|$ since ξ is a root of Q and $|\xi-\alpha_i|\leq 1$. We also have $|Q(\alpha_{n+1})| \ll \max\{1,|\alpha_{n+1}|\}^2$ since Q has degree 2. This gives

$$1 \ll H(\alpha)^2 \prod_{i=1}^{n} |\xi-\alpha_i|$$

and consequently $\max_{1\leq i\leq n}|\xi-\alpha_i| \gg H(\alpha)^{-2/n}$.

Remark 1. It would be interesting to know if there exist as well transcendental numbers ξ for which the exponent $2/n$ for $H(\alpha)$ in Theorem A part (ii) is best possible.

Remark 2. The case where $\xi\in\mathbb{Q}$ is not interesting as it leads to much weaker estimates. In this case, one finds that, for each algebraic number α of degree n with $\alpha\neq\xi$, one has $\max_{\overline{\alpha}}|\xi-\overline{\alpha}| \gg H(\alpha)^{-1/n}$, and that, for each algebraic integer α of degree $n+1$ with $\alpha\neq\xi$, one has $\max_{\overline{\alpha}\neq\alpha}|\xi-\overline{\alpha}| \gg H(\alpha)^{-1/n}$.

3. Construction of polynomials. Throughout this section, we fix an irrational real number $\xi\in\mathbb{R}\setminus\mathbb{Q}$ and a positive integer $n\geq 1$. For each integer $q\geq 1$, we denote by $\mathcal{C}(q)$ the convex body of $\mathbb{R}[T]_{\leq n}$ which consists of all polynomials $P \in \mathbb{R}[T]_{\le n}$ satisfying

$$|P^{[k]}(\xi)| \le q^{2k-n} \quad (0 \le k \le n),$$

where $P^{[k]}(\xi) = P^{(k)}(\xi)/k!$ denotes the kth divided derivative of P at ξ (the coefficient of $(T-\xi)^k$ in the Taylor expansion of P at ξ). We first prove:

PROPOSITION 3.1. *Let q be the denominator of a convergent of ξ. Then the last minimum of $\mathcal{C}(q)$ with respect to the lattice $\mathbb{Z}[T]_{\le n}$ is $\le 2^n$, and its first minimum is $\ge (2^{n^2}(n+1)!)^{-1}$. Moreover, the convex body $2^n\mathcal{C}(q)$ contains a basis of $\mathbb{Z}[T]_{\le n}$ over $\mathbb{Z}$.*

Proof. Put $L_1 = qT - p$ where p/q denotes a convergent of ξ with denominator q. If $q > 1$, we also define $L_0 = q_0T - p_0$ where p_0/q_0 is the previous convergent of ξ (in reduced form). If $q = 1$, we simply take $L_0 = 1$. The theory of continued fractions tells us that these linear forms satisfy

$$|L_i(\xi)| \le q^{-1}, \quad |L_i'(\xi)| \le q \tag{3}$$

for $i = 0, 1$, and moreover that their determinant (or Wronskian) is ± 1 (see §4 in Chapter I of [Sc]). The latter fact means that $\{L_0, L_1\}$ spans $\mathbb{Z}[T]_{\le 1}$ over $\mathbb{Z}$. Therefore the products $P_j = L_0^j L_1^{n-j}$ $(0 \le j \le n)$ span $\mathbb{Z}[T]_{\le n}$ over $\mathbb{Z}$ and, since the rank of $\mathbb{Z}[T]_{\le n}$ is $n+1$, they form in fact a basis of $\mathbb{Z}[T]_{\le n}$ over $\mathbb{Z}$. Using (3), we also find that

$$|P_j^{[k]}(\xi)| \le \binom{n}{k} q^{2k-n} \le 2^n q^{2k-n} \quad (0 \le j, k \le n).$$

Thus $\{P_0, \ldots, P_n\}$ is a basis of $\mathbb{Z}[T]_{\le n}$ contained in $2^n\mathcal{C}(q)$. This proves the last assertion of the proposition as well as the fact that the last minimum of $\mathcal{C}(q)$ is $\le 2^n$.

Identify $\mathbb{R}[T]_{\le n}$ with $\mathbb{R}^{n+1}$ under the map which sends a polynomial $a_0 + a_1T + \cdots + a_nT^n$ to the point $(a_0, a_1, \ldots, a_n)$. Then the linear map $\theta : \mathbb{R}[T]_{\le n} \to \mathbb{R}^{n+1}$ given by $\theta(P) = (P(\xi), P^{[1]}(\xi), \ldots, P^{[n]}(\xi))$ has determinant 1 and so $\mathcal{C}(q)$ has volume $\prod_{k=0}^{n}(2q^{2k-n}) = 2^{n+1}$. Since the lattice $\mathbb{Z}[T]_{\le n}$ has co-volume 1 (it is identified with $\mathbb{Z}^{n+1}$), Minkowski's second convex body theorem shows that the successive minima $\lambda_1, \ldots, \lambda_{n+1}$ of $\mathcal{C}(q)$ with respect to $\mathbb{Z}[T]_{\le n}$ satisfy $((n+1)!)^{-1} \le \lambda_1 \cdots \lambda_{n+1} \le 1$. Since $\lambda_2 \le \cdots \le \lambda_{n+1} \le 2^n$, this implies that $\lambda_1 \ge (2^{n^2}(n+1)!)^{-1}$. ■

The construction of polynomials given by the next proposition uses only the last assertion of Proposition 3.1.

PROPOSITION 3.2. *Let q be the denominator of a convergent of ξ. There exist an irreducible polynomial $P(T) \in \mathbb{Z}[T]$ of degree n and an irreducible monic polynomial $Q(T) \in \mathbb{Z}[T]$ of degree $n+1$ satisfying*

$$c_5 q^{2k-n} \le |P^{[k]}(\xi)|, |Q^{[k]}(\xi)| \le 3c_5 q^{2k-n} \quad (0 \le k \le n)$$

where $c_5 = (n+1)2^{n+1}$.

Note that such polynomials have height $\asymp q^n$.

Proof. The last assertion of Proposition 3.1 states the existence of a basis $\{P_0, \dots, P_n\}$ of $\mathbb{Z}[T]_{\le n}$ satisfying

$$(4) \qquad |P_j^{[k]}(\xi)| \le 2^n q^{2k-n} \quad (0 \le j, k \le n).$$

Since $\{P_0, \dots, P_n\}$ is a basis of $\mathbb{Z}[T]_{\le n}$ over $\mathbb{Z}$, we can write $T^n + 2 = \sum_{j=0}^n b_j P_j(T)$ for some $b_0, \dots, b_n \in \mathbb{Z}$. Consider the polynomial

$$R(T) = 2c_5 \sum_{k=0}^n q^{2k-n}(T-\xi)^k$$

where $c_5 = (n+1)2^{n+1}$. Since $\{P_0, \dots, P_n\}$ is also a basis of $\mathbb{R}[T]_{\le n}$ over $\mathbb{R}$, we can also write $R(T) = \sum_{j=0}^n \theta_j P_j(T)$ for some $\theta_0, \dots, \theta_n \in \mathbb{R}$. Choose integers $a_0, \dots, a_n$ such that $a_j \equiv b_j \bmod 4$ and $|a_j - \theta_j| \le 2$ for $j = 0, \dots, n$, and define $P(T) = \sum_{j=0}^n a_j P_j(T)$.

By construction $P(T)$ belongs to $\mathbb{Z}[T]_{\le n}$ and is congruent to $T^n + 2$ modulo 4. Thus it is a polynomial of degree n over $\mathbb{Q}$ and it is irreducible by virtue of Eisenstein's criterion (for the prime 2). Since $P(T) - R(T) = \sum_{j=0}^n (a_j - \theta_j) P_j(T)$, we deduce from (4) that

$$|P^{[k]}(\xi) - R^{[k]}(\xi)| \le \sum_{j=0}^n |a_j - \theta_j|\, |P_j^{[k]}(\xi)| \le c_5 q^{2k-n} \quad (0 \le k \le n).$$

Since $R^{[k]}(\xi) = 2c_5 q^{2k-n}$, it follows that $c_5 q^{2k-n} \le |P^{[k]}(\xi)| \le 3c_5 q^{2k-n}$ for $k = 0, \dots, n$, as required.

The construction of $Q(T)$ is similar. Write

$$T^{n+1} + 2 = T^{n+1} + \sum_{j=0}^n b'_j P_j(T), \qquad (T-\xi)^{n+1} + R(T) = T^{n+1} + \sum_{j=0}^n \theta'_j P_j(T),$$

with $b'_0, \dots, b'_n \in \mathbb{Z}$ and $\theta'_0, \dots, \theta'_n \in \mathbb{R}$, and choose integers $a'_0, \dots, a'_n$ such that $a'_j \equiv b'_j \bmod 4$ and $|a'_j - \theta'_j| \le 2$ for $j = 0, \dots, n$. Then the polynomial

$$Q(T) = T^{n+1} + \sum_{j=0}^n a'_j P_j(T) \in \mathbb{Z}[T]$$

is irreducible (by virtue of Eisenstein's criterion for 2), monic of degree $n+1$, and also satisfies $|Q^{[k]}(\xi) - R^{[k]}(\xi)| \le c_5 q^{2k-n}$ for $k = 0, \dots, n$. ∎

4. Proof of Theorems A and B. In this section, we prove the main Theorems A and B of the introduction by combining Proposition 3.2 with the following result.

Proposition 4.1. *Let $\xi \in \mathbb{R}$, let $n \in \mathbb{N}^*$, let $\delta > 0$ and let $\mathcal{P}$ be a subset of $\mathbb{Z}[T]$. Suppose that the elements of $\mathcal{P}$ are either polynomials of degree n or monic polynomials of degree $n+1$. Then the following conditions are equivalent*:

(i) *There exists a constant $c_6 > 0$ such that $|P^{[k]}(\xi)| \le c_6 H(P)^{1-(n-k)\delta}$ for each $P \in \mathcal{P}$ and each $k = 0, 1, \dots, n$.*
(ii) *There exists a constant $c_7 > 0$ such that $|\xi - \alpha| \le c_7 H(P)^{-\delta}$ for each $P \in \mathcal{P}$ and for n of the roots α of P, counting multiplicity.*

Proof. Fix $P \in \mathcal{P}$ and write it in the form

$$P(T) = a_0(T-\alpha_1)\cdots(T-\alpha_m),$$

where $m = \deg P$ and $\alpha_1, \dots, \alpha_m$ are the roots of P ordered so that we have $|\xi - \alpha_1| \le \dots \le |\xi - \alpha_m|$. We put $\varepsilon = H(P)^{-\delta}$ and consider the polynomial

$$R(T) = P(\varepsilon T + \xi) = a_0 \varepsilon^m \prod_{k=1}^{m} (T + \varepsilon^{-1}(\xi - \alpha_k)).$$

The height of R is

$$H(R) = \max_{0 \le k \le m} |R^{[k]}(0)| = \max_{0 \le k \le m} |P^{[k]}(\xi)|\varepsilon^k,$$

and its Mahler measure is

$$M(R) = |a_0|\varepsilon^m \prod_{k=1}^{m} \max\{1, \varepsilon^{-1}|\xi - \alpha_k|\} = |a_0| \prod_{k=1}^{m} \max\{\varepsilon, |\xi - \alpha_k|\}.$$

For convenience, we also define

$$L = \begin{cases} |a_0| & \text{if } m = n, \\ \max\{\varepsilon, |\xi - \alpha_m|\} & \text{if } m = n+1, \end{cases}$$

so that the formula for $M(R)$ becomes

$$M(R) = L \prod_{k=1}^{n} \max\{\varepsilon, |\xi - \alpha_k|\}$$

(recall that $a_0 = 1$ when $m = n+1$). Our argument below is based on the standard inequalities relating these notions of heights, namely

$$M(R) \le (m+1)H(R) \quad \text{and} \quad H(R) \le 2^m M(R).$$

If condition (ii) holds, we find that $M(R) \ll \varepsilon^n L$. We also have $L \ll H(P)$ since $|a_0| \le H(P)$ and since $|\xi - \alpha| \ll \max\{1, |\alpha|\} \ll H(P)$ for any root α of P. Then, for each $k = 0, \dots, n$, we obtain

$$|P^{[k]}(\xi)| \ll \varepsilon^{-k} H(R) \ll \varepsilon^{-k} M(R) \ll \varepsilon^{n-k} H(P),$$

which shows that condition (i) holds.

Conversely assume that condition (i) holds. In this case we find that $H(R) \ll \varepsilon^n H(P)$. We claim that $H(P) \ll L$. If we take this for granted, we deduce that

$$L\varepsilon^{n-1}|\xi - \alpha_n| \leq M(R) \ll H(R) \ll \varepsilon^n L$$

which implies that condition (ii) holds.

To prove the claim, we observe that

$$H(P) \asymp H(P(T+\xi)) = \max_{0\leq k\leq m} |P^{[k]}(\xi)|.$$

By hypothesis, we have $|P^{[k]}(\xi)| \leq c_6 H(P)^{1-\delta}$ for $k = 0, \ldots, n-1$ and we also have $|P^{[m]}(\xi)| = 1$ if $m = n+1$. Finally, we have $|P^{[n]}(\xi)| = |a_0|$ if $m = n$, and $|P^{[n]}(\xi)| = |\sum_{k=1}^{m}(\xi - \alpha_k)| \leq m|\xi - \alpha_m|$ if $m = n+1$, showing that $|P^{[n]}(\xi)| \ll L$. All this implies that

$$H(P) \ll \max\{1, L\}.$$

Since $L \geq \varepsilon = H(P)^{-\delta}$, this in turn implies that $H(P) \ll L$. ∎

Proof of the theorems. Let $\xi \in \mathbb{R} \setminus \mathbb{Q}$ and $n \in \mathbb{N}^*$. We simply prove part (ii) of Theorems A and B since the proof of part (i) is similar and slightly easier.

For each denominator q of a convergent of ξ, Proposition 3.2 shows the existence of an irreducible monic polynomial $Q \in \mathbb{Z}[T]$ of degree $n+1$ satisfying $H(Q) \asymp q^n$ and

$$|Q^{[k]}(\xi)| \leq c_6 H(Q)^{(2k-n)/n} = c_6 H(Q)^{1-(n-k)(2/n)} \qquad (0 \leq k \leq n)$$

for some constant $c_6 = c_6(\xi, n)$. The family $\mathcal{P}$ of these polynomials satisfies condition (i) of Proposition 4.1 for the choice $\delta = 2/n$, and so it also satisfies condition (ii) of the same proposition for the same value of δ and for some constant c_7. For each $Q \in \mathcal{P}$, choose a root α of Q for which $|\xi - \alpha|$ is maximal. Since Q is irreducible, this root α is an algebraic integer of degree $n+1$ and height $H(\alpha) = H(Q)$ whose conjugates $\overline{\alpha}$ over $\mathbb{Q}$ are the $n+1$ distinct roots of Q. Therefore, we get $\max_{\overline{\alpha}\neq\alpha} |\xi - \overline{\alpha}| \leq c_7 H(\alpha)^{-2/n}$. This proves part (ii) of Theorem A since we find infinitely many such numbers α by varying Q.

If ξ is badly approximable, the ratios of the denominators of consecutive convergents of ξ are bounded. Thus, for each $X \geq 1$, there exists such a denominator q with $q \asymp X^{1/n}$, and so there exists a polynomial $Q \in \mathcal{P}$ with $H(Q) \asymp X$. Consequently, the root α of Q that we chose above satisfies $H(\alpha) \asymp X$ and this proves part (ii) of Theorem B. ∎

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