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Some results concerning certain periodic continued fractions

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1. Introduction. One of the most important invariants of an order of an algebraic number field is the regulator of that order. It has been known for over two hundred years that the fundamental unit of such an order can be determined from the periodic continued fraction expansion of the generator of the order. In this case the regulator is simply the logarithm of the fundamental unit.

For most real quadratic orders the regulator tends to be rather large, but in certain, unusual and infrequent cases it is small, ensuring that the ideal class number will be large. Such orders are of very great interest to number theorists because the corresponding ideal class groups often have exotic structures. The search for such orders is a very old problem in number theory (see [3]). This is done in the context of determining families of values of non-square D for which the period length ℓ of the regular continued fraction of $\sqrt{D}$ is small. Since the regulator of the order $\mathcal{O}$ $(= \mathbb{Z} + \mathbb{Z}\sqrt{D})$ can be easily bounded above by $\ell(\log 2\sqrt{D})$, this means that the regulator of $\mathcal{O}$ will be small.

In this paper, we study classes of integer-valued quadratic polynomials $D(X)$ for which the regular continued fraction expansion of $\sqrt{D(X)}$ exhibits a predictable pattern as X varies. For instance, see [2], [5], [10] and [14].

The most important results concerning the regular continued fraction expansion of $\sqrt{D(X)}$ were established by Schinzel [11], [12]. He showed that if $D(X) = a^2X^2 + bX + c$, where $a, b, c \in \mathbb{Z}$, $X \in \mathbb{N}$ and $a \neq 0$, then $\sqrt{D(X)}$ has bounded period length if and only if the discriminant $b^2 - 4ac$ divides $4\gcd(2a^2, b)^2$. Also, Louboutin [7] and Farhane [4] later gave lower bounds for the case where the divisibility condition fails.

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Schinzel elegantly established this result without computing the actual continued fraction expansion of $\sqrt{D(X)}$. This prompted van der Poorten and Williams [9] to study the continued fraction expansion of $\sqrt{D(X)}$ that has bounded period length. They first established that by examining the cases of even X and odd X separately, there is no loss of generality in considering only quadratics of the form

$$D(X) = A^2X^2 + 2BX + C,$$

where $A > 0$, B and C are integers. Under this consideration, Schinzel's divisibility condition becomes

$$B^2 - A^2C \mid 4\gcd(A^2, B)^2,$$

the *Schinzel condition*. Assuming $\gcd(A^2, 2B, C)$ to be squarefree, van der Poorten and Williams obtained the exact expansion of $\sqrt{D(X)}$ for sufficiently large X. Moreover, the period length of the expansion of $\sqrt{D(X)}$ in this case is constant and is independent of X. Besides van der Poorten and Williams, Mollin [8] also studied the actual continued fraction expansion of $\sqrt{D(X)}$ from a different perspective.

In this article, we drop the condition that $\gcd(A^2, 2B, C)$ be squarefree but maintain the divisibility condition $B^2 - A^2C \mid 4\gcd(A^2, B)^2$, and seek the continued fraction expansion of $\sqrt{D(X)}$ for sufficiently large X. We take a constructive approach in explicitly computing the expansion of $\sqrt{D(X)}$. As a consequence of this approach, we find the period length of the expansion to be constant with respect to the congruence classes of X modulo a certain integer whose value depends only on A, B, C. Also, we are able to establish an upper bound for the period length of the expansion of $\sqrt{D(X)}$ and produce a closed form for the fundamental unit (and consequently the regulator) of the real quadratic order $[1, \sqrt{D(X)}] = \mathbb{Z} + \mathbb{Z}\sqrt{D(X)}$ when X is sufficiently large.

We begin with some preliminary remarks about continued fractions. If θ is any real number, we can find the regular continued fraction (RCF) expansion of θ, denoted as

$$\theta = (a_0, a_1, \ldots, a_{n-1}, \theta_n),$$

by putting $\theta_0 = \theta$, $a_0 = \lfloor \theta_0 \rfloor$ and defining the *complete quotient* as $\theta_{i+1} = 1/(\theta_i - a_i)$ and the *partial quotient* as $a_{i+1} = \lfloor \theta_{i+1} \rfloor$ for $i = 0, 1, \ldots$.

If θ is an irrational, we know that the RCF expansion of θ contains an infinite number of partial quotients and that the ordered set of partial quotients is unique to θ. If θ is a rational, we know that the RCF expansion is finite. That is,

$$\theta = (a_0, a_1, \ldots, a_m),$$

where $a_0 \in \mathbb{Z}$, $a_i \in \mathbb{N}$ $(i = 1, \ldots, m)$. If $\theta \neq 1$, there is some ambiguity about m, however, as $a_m = (a_m - 1, 1)$ $(a_m > 1)$ and $(a_{m-1}, 1) = a_{m-1} + 1$. This ambiguity can be easily resolved if we specify the parity of m or insist that the last partial quotient be greater than 1.

If θ is a real quadratic irrational, we may assume with no loss of generality that $\theta = (P + \sqrt{D})/Q$, where $D, P, Q \in \mathbb{Z}$, $D > 0$, $\sqrt{D} \notin \mathbb{Q}$ and $Q \mid D - P^2$. In this case, we can put $P_0 = P$, $Q_0 = Q$ and $\theta_i = (P_i + \sqrt{D})/Q_i$. The values of P_i and Q_i can be computed recursively by

$$P_{i+1} = a_i Q_i - P_i, \qquad Q_{i+1} = (D - P_{i+1}^2)/Q_i$$

for $i = 0, 1, \ldots$. It is well known that the RCF expansion of a quadratic irrational ultimately becomes periodic. We signal the periodic part of the continued fraction expansion by putting a bar over it. For example,

$$\sqrt{7} = (2, 1, 1, 1, 4, 1, 1, 1, 4, 1, 1, 1, 4, \ldots) = (2, \overline{1, 1, 1, 4}).$$

We define the *convergents* C_i $(i = 0, 1, \ldots)$ of $(a_0, a_1, \ldots, a_{n-1}, \theta_n)$ to be the values of the continued fraction $(a_0, a_1, \ldots, a_i)$ $(i = 0, 1, \ldots)$. These can be easily computed by defining $A_{-2} = 0$, $A_{-1} = 1$, $B_{-2} = 1$, $B_{-1} = 0$, and computing

$$A_{j+1} = a_{j+1} A_j + A_{j-1}, \qquad B_{j+1} = a_{j+1} B_j + B_{j-1}$$

recursively for $j = -1, 0, 1, \ldots$. Then $C_i = A_i/B_i$. Also, it is well known that $A_i B_{i-1} - B_i A_{i-1} = (-1)^{i-1}$.

Notice that in our definition of a RCF expansion of θ, we have all partial quotients $a_i > 0$, where $i \geq 1$. However, this condition can be relaxed in a formal continued fraction. In this case we simply write

$$\theta = \langle a_0, a_1, \ldots, a_{n-1}, \theta_n \rangle,$$

where $a_i \in \mathbb{Z}$ $(i = 0, 1, \ldots, n-1)$. Here we have $\theta_{i+1} = 1/(\theta_i - a_i)$ as before. The convergents are defined similarly to the case of the RCF and

$$\theta_n = -\frac{B_{n-2}\theta - A_{n-2}}{B_{n-1}\theta - A_{n-1}},$$

just as in the RCF case. Here and in what follows, the bracket notation $\langle\ \rangle$ stands for the formal continued fraction while the parenthesis notation ( ) continues to represent the regular continued fraction.

2. Preliminary results. In this section we present two general results needed for our work. The first result deals with $\lfloor \sqrt{D(X)} \rfloor$ and the second result deals with the embedding of a sequence $s_0, s_1, \ldots, s_{m-1}$ in a continued fraction.

For any quadratic $D(X) = A^2X^2 + 2BX + C$, not necessarily satisfying the Schinzel condition, we may write

$$D(X) = \left(\frac{A^2X + B}{A}\right)^2 - \frac{\Delta}{A^2},$$

where $\Delta = B^2 - A^2C$. A simple verification will secure the following result.

THEOREM 2.1. *Write $B = Aq + r$ with $0 \le r < A$. Then*

$$\lfloor\sqrt{D(X)}\rfloor = \begin{cases} AX + q - 1 & \text{if } \Delta > 0,\ r = 0 \text{ and } X > \dfrac{\Delta}{2A^3} - \dfrac{2B - A}{2A^2}, \\ AX + q & \text{if } \Delta > 0,\ r > 0 \text{ and } X > \dfrac{\Delta}{2A^2 r} - \dfrac{2B - r}{2A^2}, \\ AX + q & \text{if } \Delta < 0,\ r = 0 \text{ and } X > \dfrac{-\Delta}{2A^3} - \dfrac{2B + A}{2A^2}, \\ AX + q & \text{if } \Delta < 0,\ r > 0 \text{ and } X > \dfrac{-\Delta}{2A^2(A - r)} - \dfrac{2B + (A - r)}{2A^2}. \end{cases}$$ ■

Let $s_0, s_1, \ldots, s_{m-1}$ be any sequence of integers and $\theta = (P + \sqrt{D})/Q$ be a quadratic irrational. Further, let $\theta^* = (\mathbf{P} + \sqrt{D})/\mathbf{Q}$ be defined by

$$\theta = \langle s_0, s_1, \ldots, s_{m-1}, \theta^* \rangle.$$

As indicated earlier, we have

$$\theta^* = -\frac{B_{m-2}\theta - A_{m-2}}{B_{m-1}\theta - A_{m-1}}. \tag{1}$$

After some simple algebraic manipulations, we obtain

$$\mathbf{P} = (-1)^m(B_{m-1}B_{m-2}Q' - A_{m-1}A_{m-2}Q + (A_{m-1}B_{m-2} + A_{m-2}B_{m-1})P),$$
$$\mathbf{Q} = (-1)^m(A_{m-1}^2 Q - B_{m-1}^2 Q' - 2A_{m-1}B_{m-1}P),$$

where $Q' = (D - P^2)/Q \in \mathbb{Z}$.

Let $d \in \mathbb{N}$ and put $F = dA_{m-1}$ and $E = dB_{m-1}$. Since $\gcd(A_{m-1}, B_{m-1}) = 1$, we get $d = \gcd(E, F)$ and

$$\mathbf{P} = \frac{(-1)^m}{d}(B_{m-2}EQ' - A_{m-2}FQ + (B_{m-2}F + A_{m-2}E)P), \tag{2}$$

$$\mathbf{Q} = \frac{(-1)^m}{d^2}(F^2Q - E^2Q' - 2EFP). \tag{3}$$

THEOREM 2.2. *Let $\theta = (P_0 + \sqrt{D})/Q_0$ be a quadratic irrational and put*

$$a_0 = \left\lfloor \frac{P_0 + \sqrt{D}}{Q_0} \right\rfloor, \qquad P_1 = a_0Q_0 - P_0, \qquad Q_1 = \frac{D - P_1^2}{Q_0}.$$

Let $L^2D = M^2 - N$, where $L, M \in \mathbb{N}$ and $N \in \mathbb{Z}$, and put $F = LQ_0$, $E = M - LP_1 \ne 0$ and $d = \gcd(E, F)$. Let the regular continued fraction expansion of F/E be given by

$$(s_0, s_1, \ldots, s_{m-1}),$$

where m is chosen to be odd if $N > 0$ and even if $N < 0$. If $(\mathbf{P} + \sqrt{D})/\mathbf{Q}$ $(\mathbf{P}, \mathbf{Q} \in \mathbb{Z})$ *is defined by*

$$\theta = \langle a_0, s_0, s_1, \ldots, s_{m-1}, (\mathbf{P} + \sqrt{D})/\mathbf{Q} \rangle,$$

then

$$\mathbf{P} = \frac{M}{L} - \frac{H|N|}{dL}, \qquad \mathbf{Q} = \frac{|N|Q_0}{d^2},$$

where $H = 1$ if $F = E$, and $H(E/d) \equiv (-1)^{m-1} \bmod F/d$ $(0 \le H < F/d)$ *otherwise.*

Proof. Let $(P_1 + \sqrt{D})/Q_1$ be the first complete quotient of the regular continued fraction expansion of $(P_0 + \sqrt{D})/Q_0$. Then the values of $\mathbf{P}$ and $\mathbf{Q}$ are defined by

$$\frac{P_1 + \sqrt{D}}{Q_1} = \left\langle s_0, s_1, \ldots, s_{m-1}, \frac{\mathbf{P} + \sqrt{D}}{\mathbf{Q}} \right\rangle.$$

If we substitute Q_1 for Q, P_1 for P, $Q_0 = (D - P_1^2)/Q_1$ for Q', $M - LP_1$ for E and LQ_0 for F in (3), then we get

$$\begin{aligned} \mathbf{Q} &= \frac{(-1)^m}{d^2} Q_0(L^2(D - P_1^2) - (M - LP_1)^2 - 2(LP_1)(M - LP_1)) \\ &= \frac{(-1)^m}{d^2} Q_0(L^2 D - (LP_1 + M - LP_1)^2) = \frac{|N|Q_0}{d^2}. \end{aligned}$$

Similarly, by (2),

$$\begin{aligned} L\mathbf{P} &= \frac{(-1)^m}{d} (B_{m-2}LMQ_0 - A_{m-2}L(LQ_0)Q_1 + A_{m-2}(M - LP_1)LP_1) \\ &= \frac{(-1)^m}{d} (B_{m-2}LMQ_0 + A_{m-2}LMPd_1 - A_{m-2}M^2 + A_{m-2}N). \end{aligned}$$

Since $LQ_0 = A_{m-1}d$ and $M - LP_1 = B_{m-1}d$, we have

$$B_{m-2}LQ_0 - A_{m-2}(M - LP_1) = (A_{m-1}B_{m-2} - A_{m-2}B_{m-1})d = (-1)^m d.$$

Thus,

$$\mathbf{P} = \frac{(-1)^m(-1)^m dM}{dL} + \frac{(-1)^m A_{m-2}N}{dL} = \frac{M}{L} - \frac{A_{m-2}|N|}{dL}.$$

Put $H = A_{m-2}$. It follows that $H = 1$ when $F = E$, and $H(E/d) \equiv (-1)^{m-1} \bmod F/d$ $(0 \le H < F)$ otherwise. In either case, we may write

$$\mathbf{P} = \frac{M}{L} - \frac{H|N|}{dL}. \ \blacksquare$$

The above result will play a pivotal role in our approach. We will construct the regular continued fraction expansion of $\sqrt{D(X)}$ by combining segments of the expansion. These segments are found by applying the above theorem. Note that the above result gives a segment for some continued fraction, which may not be regular. Thus, we need a criterion to determine when it is regular.

THEOREM 2.3. *Let θ be an irrational number and suppose that*

$$\theta = \langle a_0, a_1, \ldots, a_{n-1}, \theta_n^* \rangle,$$

where $a_1, \ldots, a_{n-1} \in \mathbb{N}$. If $\theta^* > 1$, then

$$\theta = (a_0, a_1, \ldots, a_{n-1}, \theta_n),$$

where $\theta_n = \theta_n^*$.

Proof. Since $\theta^* > 1$, the partial quotients in the RCF expansion of θ^* are all in $\mathbb{N}$. Thus, the result of the theorem follows easily by the uniqueness of the RCF expansion of θ. ∎

REMARK 2.1. If $F > E$, i.e. $s_0 \in \mathbb{N}$, and $(\mathbf{P} + \sqrt{D})/\mathbf{Q} > 1$ in Theorem 2.2, then by Theorem 2.3, the RCF expansion of θ is given by

$$(a_0, s_0, s_1, \ldots, s_{m-1}, (\mathbf{P} + \sqrt{D})/\mathbf{Q}).$$

3. Main results. Before we get to the main results, we need to introduce some notation. Assume that

$$\varDelta = B^2 - A^2C \mid 4\gcd(A^2, B)^2 \quad \text{and} \quad |\varDelta| = \varDelta_1\varDelta_2^2\varDelta_4^4,$$

where $\varDelta_1$ and $\varDelta_2$ are squarefree. It follows that

$$\varDelta_1\varDelta_2\varDelta_4^2 \mid 2A^2, \qquad \varDelta_1\varDelta_2\varDelta_4^2 \mid 2B. \tag{4}$$

Also, since $|B^2 - A^2C| = \varDelta_1\varDelta_2^2\varDelta_4^4$, we see that if $G = \gcd(A, B)$, then

$$G = \gcd(A, \varDelta_2\varDelta_4^2). \tag{5}$$

We define

$$A' = A/G, \qquad \varDelta' = \varDelta_2\varDelta_4^2/G. \tag{6}$$

Note that $\gcd(A', \varDelta') = 1$, and by (4), we have $\varDelta_1\varDelta' \mid 2A$. Also, since $\varDelta_1\varDelta' \mid 2B$, we have $\varDelta_1\varDelta' \mid 2G$. If $A \mid B$, then $A = G$ and $A' = 1$. Define $\sigma = \varDelta/|\varDelta| = \operatorname{sgn}(\varDelta)$, and

$$\eta = \begin{cases} 1 & \text{if } A \mid B \text{ and } \sigma = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

For integers $a \ge r \ge 0$, define an ordered set

$$\mathcal{S}(a, r) = \begin{cases} \emptyset & \text{if } r = 0, \\ \{s_0, s_1, \ldots, s_{m-1}\} & \text{otherwise,} \\ & \text{where } a/r = (s_0, s_1, \ldots, s_{m-1}) \\ & \text{with } (-1)^{m-1} = \sigma. \end{cases} \tag{8}$$

Finally, we define $\varepsilon_i \in \{0, 1\}$, $\varepsilon_i \equiv i \bmod 2$. Note that $\varepsilon_{i+1} = 1 - \varepsilon_i$.

As we mentioned earlier, our result on the continued fraction expansion of $\sqrt{D(X)}$ requires X to be sufficiently large. Henceforth, we say X is *sufficiently large* if X is a non-negative integer such that

$$X > \frac{|\Delta|}{AG^2} + \frac{1}{2A} - \frac{B}{A^2} \quad \text{and} \quad X > 1 + \frac{1}{A} - \frac{B}{A^2}. \tag{9}$$

It is easy to see that when the first inequality is satisfied, X is greater than the lower bounds in Theorem 2.1. The second inequality is needed for reasons that will become clear later. We let $X \equiv K \bmod \Delta'$, where $0 \le K < \Delta'$, and write $B = Aq + r$, where $0 \le r < A$.

THEOREM 3.1 (The regular continued fraction expansion of $\sqrt{D(X)}$). *Suppose that $D(X) = A^2X^2 + 2BX + C$ satisfies the Schinzel condition, where X is sufficiently large, i.e.,* (9) *holds. Write $X = W\Delta' + K$ for some $W \ge 0$. Put $d_0 = \Delta'$ and $r_0 = (r + A\eta)/G$ and inductively define the following.*

For $i \ge 0$, define

$$\mathcal{S}_i = \mathcal{S}(A'\Delta'/d_i, r_i), \qquad d_{i+1} = \gcd(\Delta'/d_i, r_i),$$

where the parity of $|\mathcal{S}_i|$ is even if $\sigma = -1$ and odd if $\sigma = 1$. Put $g_{i+1} = 0$ or $g_{i+1} = d_i$ according as $r_i = 0$ or $r_i = A'\Delta'/d_i$. If $r_i \not\equiv 0 \bmod A'\Delta'/d_i$, then choose $g_{i+1} \in \mathbb{Z}$ so that

$$\frac{g_{i+1}}{d_i}\,\frac{r_i}{d_{i+1}} \equiv \sigma \bmod \frac{A'\Delta'}{d_i d_{i+1}} \quad \textit{and} \quad 0 < g_{i+1} < \frac{A'\Delta'}{d_{i+1}}. \tag{10}$$

Also,

$$q_{i+1}(X) = \frac{2AWd_{i+1}^2}{\Delta_1^{\varepsilon_{i+1}}\Delta} + \left\lfloor \frac{2A^2K + 2B - \Delta_1^{\varepsilon_{i+1}} g_{i+1}(A/A')(\Delta'/d_{i+1}) - A\eta - r}{A\Delta_1^{\varepsilon_{i+1}}(\Delta'/d_{i+1})^2} \right\rfloor.$$

Compute r_{i+1} such that

$$r_{i+1} \equiv \frac{d_{i+1}(2A^2K + 2B)}{\Delta_1^{\varepsilon_{i+1}}\Delta_2\Delta_4^2} - g_{i+1} \bmod \frac{A'\Delta'}{d_{i+1}}, \tag{11}$$

where $0 \le r_{i+1} < A'\Delta'/d_{i+1}$ when $\sigma = -1$ and $0 < r_{i+1} \le A'\Delta'/d_{i+1}$ when $\sigma = 1$.

Then the regular continued fraction expansion of $\sqrt{D(X)}$ is given by

$$(AX + q - \eta, \overline{\mathcal{S}_0, q_1(X), \mathcal{S}_1, q_2(X), \ldots, \mathcal{S}_{\kappa-1}, q_\kappa(X)}),$$

where η is defined by (7) *and κ is the least natural number such that*

$$d_\kappa = \Delta' \quad \textit{and} \quad \Delta_1^{\varepsilon_\kappa} = 1.$$

Proof. We prove the result by induction on the subscript i. We do this by establishing formulas for $\mathbf{P}_i$ and $\mathbf{Q}_i$ such that

$$\sqrt{D(X)} = (AX + q - \eta, \mathcal{S}_0, (\mathbf{P}_1 + \sqrt{D(X)})/\mathbf{Q}_1)$$

and

$$(\mathbf{P}_i + \sqrt{D(X)})/\mathbf{Q}_i = (q_i(X), \mathcal{S}_i, (\mathbf{P}_{i+1} + \sqrt{D(X)})/\mathbf{Q}_{i+1})$$

for $i \geq 1$.

When X is sufficiently large, it follows from Theorem 2.1 that

$$\lfloor \sqrt{D(X)} \rfloor = AX + q - \eta.$$

Initial step: Put $\mathbf{P}_0 = 0$ and $\mathbf{Q}_0 = 1$. Then

$$\sqrt{D(X)} = (\mathbf{P}_0 + \sqrt{D(X)})/\mathbf{Q}_0, \qquad P_1 = a_0\mathbf{Q}_0 - \mathbf{P}_0 = AX + q - \eta.$$

In view of Theorem 2.2, we put $L = A$, $M = AX^2 + B$ and $N = \Delta$, so that

$$E = M - LP' = r + A\eta, \qquad F = L\mathbf{Q}_0 = A.$$

Let $d = \gcd(F, E) = \gcd(A, r + A\eta)$. Then $d = \gcd(A, r) = G$ by (5). Put $r_0 = (r + A\eta)/G$. Since $A' = A/G$, we have $d_1 = (A', r_0) = 1$. By the definition of η, it is clear that $0 \leq E \leq F$, i.e., $0 \leq r_0 \leq A'$. If $E = 0$, we must have $r = 0 = \eta$, i.e., $A \mid B$ and $\sigma = -1$. If $E = F$, then $r = 0$ and $\eta = 1$, i.e., $A \mid B$ and $\sigma = 1$. Also, if $r > 0$, then $\eta = 0$ and $0 < E < F$.

Case (1). Suppose that $E = 0$, i.e., $r_0 = 0$ and $\sigma = -1$. In this case, we cannot apply Theorem 2.2. We put $\mathcal{S}_0 = \emptyset$ and compute

$$\mathbf{P}_1 = P' = AX + \frac{B}{A}, \qquad \mathbf{Q}_1 = \frac{D(X) - \mathbf{P}_1^2}{\mathbf{Q}_0} = \frac{-\Delta}{A^2} = \Delta_1\left(\frac{\Delta'}{A'}\right)^2.$$

Case (2). Assume that $E \neq 0$, i.e., $0 < r_0 \leq A'$. Let the RCF expansion of $A/(r + A\eta)$ be given by $(s_0, s_1, \ldots, s_{m-1})$, where $(-1)^{m-1} = \sigma$. By Theorem 2.2, we get integers

$$\mathbf{P}_1 = \frac{A^2X + B}{A} - \frac{H|\Delta|}{Ad}, \qquad \mathbf{Q}_1 = \frac{|\Delta|}{d^2}, \tag{12}$$

where $H = 1$ if $F = E$, and $H \cdot (r + A\eta)/d \equiv \sigma \bmod A/d$ $(0 \leq H < A/d)$ otherwise.

Note that

$$\frac{H|\Delta|}{Ad} = \Delta_1 \frac{H\Delta'^2}{A'} \quad \text{and} \quad \frac{|\Delta|}{d^2} = \Delta_1\left(\frac{\Delta'}{d_1}\right)^2.$$

Set

$$g_1 = H\Delta', \qquad d_0 = \Delta'. \tag{13}$$

If $r_0 = A'$, then $g_1 = d_0$; otherwise, g_1 satisfies

$$\frac{g_1 r_0}{d_0 d_1} \equiv \sigma \bmod \frac{A'\Delta'}{d_0 d_1} \quad \text{and} \quad 0 < g_1 < \frac{A'\Delta'}{d_1}. \tag{14}$$

Also,

$$\mathbf{P}_1 = \frac{A^2X + B}{A} - \Delta_1 \frac{g_1}{A'} \frac{\Delta'}{d_1}, \qquad \mathbf{Q}_1 = \Delta_1\left(\frac{\Delta'}{d_1}\right)^2. \tag{15}$$

Put $\mathcal{S}_0 = \mathcal{S}(A', r_0)$. We now need to establish that

$$(AX + q - \eta, \mathcal{S}_0, (\mathbf{P}_1 + \sqrt{D(X)})/\mathbf{Q}_1)$$

is indeed the regular continued fraction expansion of $\sqrt{D(X)}$. By Remark 2.1, since we know that $F > E$, we need $(\mathbf{P}_1 + \sqrt{D(X)})/\mathbf{Q}_1 > 1$. This can be easily verified under the assumption that $X > |\Delta|/(AG^2) - (2B - A)/(2A^2)$.

Now, regardless of Case (1) or (2), we have $d_0 = \Delta'$, $\mathcal{S}_0 = \mathcal{S}(\Delta'/d_0, r_0)$ and $d_1 = \gcd(A'\Delta'/d_0, r_0) = 1$. When $r_0 \equiv 0 \bmod A'$, put $g_1 = 0$ or $g_1 = d_0$ according as $r_0 = 0$ or $r_0 = A'$; otherwise, choose g_1 according to (14). Also,

$$\mathbf{P}_1 = \frac{A^2X + B}{A} - \Delta_1 \frac{g_1}{A'} \frac{\Delta'}{d_1}, \qquad \mathbf{Q}_1 = \Delta_1 \left(\frac{\Delta'}{d_1}\right)^2.$$

Since $X = W\Delta' + K$, where $0 \le K < \Delta'$, it is not difficult to deduce that

$$\begin{aligned} q_1(X) &= \left\lfloor \frac{\mathbf{P}_1 + \sqrt{D(X)}}{\mathbf{Q}_1} \right\rfloor \\ &= \frac{2AWd_1^2}{\Delta_1\Delta'} + \left\lfloor \frac{2A^2K + 2B - \Delta_1 g_1 (A/A')(\Delta'/d_1) - A\eta - r}{A\Delta_1(\Delta'/d_1)^2} \right\rfloor. \end{aligned}$$

Inductive step. For some $i \in \mathbb{N}$, suppose that $d_i > 0$, $d_i \mid \Delta'$ and $g_i \ge 0$ are integers and $(\mathbf{P}_i + \sqrt{D(X)})/\mathbf{Q}_i$ is a complete quotient of the RCF expansion of $\sqrt{D(X)}$, where

$$\mathbf{P}_i = AX + \frac{B}{A} - \Delta_1^{\varepsilon_i} \frac{g_i}{A'} \frac{\Delta'}{d_i}, \qquad \mathbf{Q}_i = \Delta_1^{\varepsilon_i} \left(\frac{\Delta'}{d_i}\right)^2 \tag{16}$$

are integers. Let $q_i(X) = \lfloor (\mathbf{P}_i + \sqrt{D(X)})/\mathbf{Q}_i \rfloor$, $\mathbf{P}' = q_i(X)\mathbf{Q}_i - \mathbf{P}_i$ and

$$\mathbf{R}_i = (\mathbf{P}_i + \lfloor \sqrt{D(X)} \rfloor) - q_i(X)\mathbf{Q}_i, \tag{17}$$

where $0 \le \mathbf{R}_i < \mathbf{Q}_i$. Since $0 \le r/A + \eta \le 1$, we have $0 \le \mathbf{R}_i + r/A + \eta \le \mathbf{Q}_i$, that is,

$$0 \le A\mathbf{R}_i + r + A\eta \le A\mathbf{Q}_i. \tag{18}$$

Note that if $r = 0$ and $\sigma = -1$, then $\eta = 0$ and the above inequalities become $0 \le A\mathbf{R}_i + A < A\mathbf{Q}_i$. If $r = 0$ and $\sigma = 1$, then $\eta = 1$ and we get $0 < A\mathbf{R}_i \le A\mathbf{Q}_i$. Also, if $r > 0$, then $0 < A\mathbf{R}_i + r < A\mathbf{Q}_i$.

In Theorem 2.2, we put

$$F = A\mathbf{Q}_i, \qquad E = A^2X + B - A\mathbf{P}' = A\mathbf{R}_i + r + A\eta.$$

Hence, $0 \le E \le F$ by (18). If $E = 0$, i.e., $A\mathbf{R}_i + r + A\eta = 0$, then $\mathbf{R}_i = 0 = r = \eta$, and hence, $\sigma = -1$. If $E = F$, then $\mathbf{R}_i + r/A + \eta = \mathbf{Q}_i$. It follows that $r = 0$ and $\mathbf{R}_i + \eta = \mathbf{Q}_i$. Since $\mathbf{R}_i < \mathbf{Q}_i$, we have $\eta = 1$, which implies $\sigma = 1$.

We now note that

$$A\mathbf{R}_i + r + A\eta \equiv 2A^2X + 2B - A\Delta_1^{\varepsilon_i}\,\frac{g_i}{A'}\,\frac{\Delta'}{d_i} \bmod A\Delta_1^{\varepsilon_i}\left(\frac{\Delta'}{d_i}\right)^2.$$

By the assumption that $X \equiv K \bmod \Delta'$, we have

$$2A^2X \equiv 2A^2K \bmod A\Delta_1^{\varepsilon_i}\left(\frac{\Delta'}{d_i}\right)^2. \tag{19}$$

Hence,

$$A\mathbf{R}_i + r + A\eta \equiv 2A^2K + 2B - A\Delta_1^{\varepsilon_i}\,\frac{g_i}{A'}\,\frac{\Delta'}{d_i} \bmod A\Delta_1^{\varepsilon_i}\left(\frac{\Delta'}{d_i}\right)^2. \tag{20}$$

Notice that $(\Delta_1^{\varepsilon_i}\Delta_2\Delta_4^2/d_i)$ divides every term on the right of the above congruence as well as the modulus, so it must also divide $A\mathbf{R}_i + r + A\eta$. Put

$$r_i = \frac{A\mathbf{R}_i + r + A\eta}{\Delta_1^{\varepsilon_i}\Delta_2\Delta_4^2/d_i}.$$

Since $0 \le E \le F$, we get $0 \le r_i \le A'\Delta'/d_i$. Also,

$$r_i \equiv \frac{d_i(2A^2K + 2B)}{\Delta_1^{\varepsilon_i}\Delta_2\Delta_4^2} - g_i \bmod \frac{A'\Delta'}{d_i}. \tag{21}$$

We note that $r_i = 0$ if and only if $E = 0$. Similarly, $r_i = A'\Delta'/d_i$ if and only if $E = F$.

Case (1). If $r_i = 0$, then $r = 0$ and $A' = 1$ and we compute

$$\mathbf{P}_{i+1} = q_i(X)\mathbf{Q}_i - \mathbf{P}_i = AX + \frac{B}{A}, \qquad \mathbf{Q}_{i+1} = \frac{D(X) - \mathbf{P}_{i+1}^2}{\mathbf{Q}_i} = \Delta_1^{\varepsilon_{i+1}}\left(\frac{d_i}{A'}\right)^2.$$

Put $\mathcal{S}_i = \mathcal{S}(A'\Delta'/d_i, r_i) = \emptyset$, $d_{i+1} = \Delta'/d_i$ and $g_{i+1} = 0$, and write

$$\mathbf{P}_{i+1} = AX + \frac{B}{A} - \Delta_1^{\varepsilon_{i+1}}\,\frac{g_{i+1}}{A'}\,\frac{\Delta'}{d_{i+1}}, \qquad \mathbf{Q}_{i+1} = \Delta_1^{\varepsilon_{i+1}}\left(\frac{\Delta'}{d_{i+1}}\right)^2.$$

Case (2). Suppose that $r_i > 0$, i.e., $0 < r_i \le A'\Delta'/d_i$. Let the RCF expansion of $F/E = (A'\Delta'/d_i)/r_i$ be given by $(s_0, s_1, \ldots, s_{m-1})$, where m is chosen so that $(-1)^{m-1} = \sigma$. Put

$$\mathcal{S}_i = \mathcal{S}\left(\frac{A'\Delta'}{d_i}, r_i\right).$$

Let $d = \gcd(E, F)$ and $d_{i+1} = \gcd(A'\Delta'/d_i, r_i)$. Then $d = (\Delta_1^{\varepsilon_i}\Delta_2\Delta_4^2/d_i)d_{i+1}$.

If $E = F$, put $H = 1$; otherwise, choose H such that

$$H\,\frac{E}{d} \equiv \sigma \bmod \frac{F}{d} \quad \text{and} \quad 0 \le H < \frac{F}{d}. \tag{22}$$

Then, by Theorem 2.2, we get

$$\mathbf{P}_{i+1} = AX + \frac{B}{A} - \frac{H|\Delta|}{Ad} \quad \text{and} \quad \mathbf{Q}_{i+1} = \frac{|\Delta|\mathbf{Q}_i}{d^2}. \tag{23}$$

Since $E/d \equiv r_i/d_{i+1} \bmod F/d$ and $F/d = (A'\Delta')/(d_i d_{i+1})$, (22) can be written as

$$H\frac{r_i}{d_{i+1}} \equiv \sigma \bmod \frac{A'\Delta'}{d_i d_{i+1}} \quad \text{and} \quad 0 \le H < \frac{A'\Delta'}{d_i d_{i+1}}.$$

Also,

$$\frac{H|\Delta|}{dA} = \frac{d_i H \Delta_1^{\varepsilon_{i+1}}\Delta'}{A'd_{i+1}} \quad \text{and} \quad \frac{|\Delta|\mathbf{Q}_i}{d^2} = \Delta_1^{\varepsilon_{i+1}}\left(\frac{\Delta'}{d_{i+1}}\right)^2.$$

Put $g_{i+1} = d_i H$; it follows that $g_{i+1} = d_i$ if $r_i = A'\Delta'/d_i$, or g_{i+1} satisfies

$$\frac{g_{i+1}r_i}{d_i d_{i+1}} \equiv \sigma \bmod \frac{A'\Delta'}{d_i d_{i+1}} \quad \text{and} \quad 0 < g_{i+1} < \frac{A'\Delta'}{d_{i+1}}.$$

Moreover,

$$\mathbf{P}_{i+1} = AX + \frac{B}{A} - \Delta_1^{\varepsilon_{i+1}}\frac{g_{i+1}}{A'}\frac{\Delta'}{d_{i+1}} \quad \text{and} \quad \mathbf{Q}_{i+1} = \Delta_1^{\varepsilon_{i+1}}\left(\frac{\Delta'}{d_{i+1}}\right)^2.$$

It is a simple matter to verify $(\mathbf{P}_{i+1} + \sqrt{D(X)})/\mathbf{Q}_{i+1} > 1$ using the assumption that X satisfies the first inequality in (9). Hence, by Remark 2.1, the expansion of $(\mathbf{P}_i + \sqrt{D(X)})/\mathbf{Q}_i$ is given by

$$\left(q_i(X), \mathcal{S}_i, \frac{\mathbf{P}_{i+1} + \sqrt{D(X)}}{\mathbf{Q}_{i+1}}\right).$$

Also, it is easy to check that $q_{i+1}(X) = \lfloor(\mathbf{P}_{i+1} + \sqrt{D(X)})/\mathbf{Q}_{i+1}\rfloor$ is given by

$$\frac{2AWd_{i+1}^2}{\Delta_1^{\varepsilon_{i+1}}\Delta'} + \left\lfloor\frac{2A^2K + 2B - \Delta_1^{\varepsilon_{i+1}}g_{i+1}(A/A')(\Delta'/d_{i+1}) - A\eta - r}{A\Delta_1^{\varepsilon_{i+1}}(\Delta'/d_{i+1})^2}\right\rfloor.$$

By induction (on the subscript i), the regular continued fraction expansion of $\sqrt{D(X)}$ is given by

$$(AX + q - \eta, \mathcal{S}_0, q_1(X), \mathcal{S}_1, q_2(X), \ldots).$$

Since $d_0 = \Delta'$ and $\mathbf{Q}_i = \Delta_1^{\varepsilon_i}(\Delta'/d_i)^2 \in \mathbb{N}$ for all $i \in \mathbb{N}$, it follows that $d_i \mid \Delta'$ for all $i \ge 0$. Moreover, since $\gcd(A', \Delta') = 1$, we get $d_{i+1} = \gcd(\Delta'/d_i, r_i)$ for all $i \ge 0$. Also, by the definition of g_{i+1}, we have $d_i \mid g_{i+1}$ for all $i \ge 0$.

Since $\sqrt{D(X)}$ is a quadratic irrational, its continued fraction expansion is periodic. The end of the period is signaled by $Q_\ell = 1$ for some minimal $\ell \in \mathbb{N}$ and the corresponding partial quotient takes the form $2AX + 2q - 2\eta$, twice the initial partial quotient. Since the partial quotients in the sequences $\mathcal{S}_i$ are all less than $A'\Delta'$, by the second inequality of (9) and the fact that $2G \ge \Delta_1\Delta'$, the only partial quotient that can be as large as $2AX + 2q - 2\eta$ must be $q_\kappa(X)$ for some κ. Hence, if ℓ is the period length, then $Q_\ell = 1$ and

$P_\ell = AX + q - \eta$; since

$$Q_\ell = \mathbf{Q}_\kappa = \Delta_1^{\varepsilon_\kappa}\left(\frac{\Delta'}{d_\kappa}\right)^2 \quad \text{and} \quad P_\ell = \mathbf{P}_\kappa = AX + \frac{B}{A} - \Delta_1^{\varepsilon_\kappa}\frac{g_\kappa}{A'}\frac{\Delta'}{d_\kappa},$$

we must have $\Delta_1^{\varepsilon_\kappa} = 1$ and $d_\kappa = \Delta'$. In other words, our computation for the regular continued fraction expansion of $\sqrt{D(X)}$ is complete when we get the minimal $\kappa > 0$ such that $\Delta_1^{\varepsilon_\kappa} = 1$ and $d_\kappa = \Delta'$. ■

Remarks 3.1. (1) It is clear that $\Delta_1 > 1$ implies that κ must be even.

(2) We see that $\mathcal{S}_0$ is independent of the residue classes of X modulo Δ' and $\mathcal{S}_i$ with $i \geq 1$ depends on the residue classes. In particular, if $r = 0$, i.e., $A \mid B$, then $\mathcal{S}_0 = \emptyset$ or $\{1\}$ according as $\sigma = -1$ or 1. Also, when $r_i \equiv 0 \bmod A'\Delta'/d_i$ for $i \geq 1$, $\mathcal{S}_i = \emptyset$ or $\{1\}$ according as $\sigma = -1$ or 1. Further, if $r > 0$, then $r_i > 0$ for $i \geq 0$.

Example 3.1. Consider $D(X) = 119^2X^2 + 2(2205)X + 343$. We first look at $X \equiv 1 \bmod 7$, where $X \geq 1$, and list $q_i(X)$, $\mathcal{S}_i$, g_i, d_i and r_i:

$q_0(1) = 137,$	$\mathcal{S}_0 = \{1,1,8\},$	$d_0 = \Delta' = 7,$		$r_0 = 9,$
$q_1(1) = 2,$	$\mathcal{S}_1 = \{1,2,4,1,1,1,2\},$	$d_1 = 1,$	$g_1 = 14,$	$r_1 = 82,$
$q_2(1) = 5,$	$\mathcal{S}_2 = \{4,3,1\},$	$d_2 = 1,$	$g_2 = 45,$	$r_2 = 28,$
$q_3(1) = 136,$	$\mathcal{S}_3 = \{1,3,4\},$	$d_3 = 7,$	$g_3 = 13,$	$r_3 = 13,$
$q_4(1) = 5,$	$\mathcal{S}_4 = \{2,1,1,1,4,2,1\},$	$d_4 = 1,$	$g_4 = 28,$	$r_4 = 45,$
$q_5(1) = 2,$	$\mathcal{S}_5 = \{8,1,1\},$	$d_5 = 1,$	$g_5 = 82,$	$r_5 = 14,$
$q_6(1) = 274,$	$\mathcal{S}_6 = \{1,1,8\},$	$d_6 = \Delta' = 7,$		$r_6 = 9.$

Since $\varepsilon_6 = 0$ by the definition of ε_i, we have $\Delta_1^{\varepsilon_6} = 1$. Also, since $d_6 = \Delta'$, the computation of $\mathcal{S}_i$ of the continued fraction expansion of $\sqrt{D(1)}$ is complete and $\kappa = 6$. Hence, when $X = 7W + 1$ for $W \geq 0$, we have

$$\sqrt{D(7W+1)} = (833W + 137, \overline{1,1,8,q_1(W),1,2,4,1,1,1,2,q_2(W),4,3,1,q_3(W),}$$
$$\overline{1,3,4,q_4(W),2,1,1,1,4,2,1,q_5(W),8,1,1,2(833W+137)}),$$

where $q_1(W) = q_5(W) = 17W + 2$, $q_2(W) = q_4(W) = 34W + 5$ and $q_3(W) = 833W + 136$.

Similarly, when $X = 7W + 2$ and $W \geq 0$,

$$\sqrt{D(7W+2)} = (833W + 256, \overline{1,1,8,17W+5,8,1,1,2(833W+256)}).$$

When $X = 7W + 3$ and $W \geq 0$,

$$\sqrt{D(7W+3)} = (833W + 375, \overline{1,1,8,q_1(W),1,1,4,1,10,q_2(W),4,3,1,q_3(W),}$$
$$\overline{1,3,4,q_4(W),10,1,4,1,1,q_5(W),8,1,1,2(833W+375)}),$$

where $q_1(W) = q_5(W) = 17W + 7$, $q_2(W) = q_4(W) = 34W + 15$ and $q_3(W) = 833W + 374$.

When $X = 7W + 4$ and $W \geq 0$, the continued fraction expansion of $\sqrt{D(7W+4)}$ is given by

$$(833W + 494, \overline{1, 1, 8, q_1(W), 1, 38, 1, 1, 1, q_2(W), 1, 1, 11, 2, 2, q_3(W), 1, 2, 4, 1, 1, 1, 2, q_4(W),}$$
$$\overline{1, 4, 5, 1, 3, q_5(W), 1, 4, 1, 18, 1, q_6(W), 4, 3, 1, q_7(W), 1, 3, 4, q_8(W), 1, 18, 1, 4, 1, q_9(W),}$$
$$\overline{3, 1, 5, 4, 1, q_{10}(W), 2, 1, 1, 1, 1, 4, 2, 1, q_{11}(W), 2, 2, 11, 1, 1, q_{12}(W), 1, 1, 1, 38, 1, q_{13}(W),}$$
$$\overline{8, 1, 1, 2(833W + 494)}),$$

where $q_1(W) = q_3(W) = q_5(W) = q_9(W) = q_{11}(W) = q_{13}(W) = 17W + 9$, $q_2(W) = q_4(W) = q_6(W) = q_8(W) = q_{10}(W) = q_{12}(W) = 34W + 19$ and $q_7(W) = 833W + 493$.

When $X = 7W + 5$ and $W \geq 0$,

$$\sqrt{D(7W+5)} = (833W + 613, \overline{1, 1, 8, q_1(W), 2, 2, 11, 1, 1, q_2(W),}$$
$$\overline{2, 2, 11, 1, 1, q_3(W), 8, 1, 1, 2(833W + 613)}),$$

where $q_1(W) = q_3(W) = 17W + 12$ and $q_2(W) = 34W + 24$.

When $X = 7W + 6$ and $W \geq 0$,

$$\sqrt{D(7W+6)} = (833W + 732, \overline{1, 1, 8, q_1(W), 1, 4, 1, 18, 1, q_2(W),}$$
$$\overline{1, 18, 1, 4, 1, q_3(W), 8, 1, 1, 2(833W + 732)}),$$

where $q_1(W) = q_3(W) = 17W + 14$ and $q_2(W) = 34W + 28$.

When $X = 7W$ and $W \geq 1$, the continued fraction expansion of $\sqrt{D(7W)}$ is given by

$$(833W + 18, \overline{1, 1, 8, q_1(W), 3, 1, 5, 4, 1, q_2(W), 1, 18, 1, 4, 1, q_3(W), 1, 1, 4, 1, 10, q_4(W),}$$
$$\overline{1, 1, 3, 8, 1, q_5(W), 2, 2, 11, 1, 1, q_6(W), 4, 3, 1, q_7(W), 1, 3, 4, q_8(W), 1, 1, 11, 2, 2, q_9(W),}$$
$$\overline{1, 38, 1, 1, 1, q_{10}(W), 10, 1, 4, 1, 1, q_{11}(W), 1, 4, 1, 18, 1, q_{12}(W), 1, 4, 5, 1, 3, q_{13}(W),}$$
$$\overline{8, 1, 1, 2(833W + 18)}),$$

where $q_1(W) = q_{13}(W) = 17W$, $q_2(W) = q_{12}(W) = 34W - 1$, $q_3(W) = q_5(W) = q_9(W) = q_{11}(W) = 17W - 1$, $q_4(W) = q_6(W) = q_8(W) = q_{10}(W) = 34W$ and $q_7(W) = 833W + 17$. ∎

4. Upper bound for $\mathrm{lp}(\sqrt{D(X)})$**.** In this section, we give a simple presentation concerning an explicit upper bound for the period length of the RCF expansion of $\sqrt{D(X)}$, denoted by $\mathrm{lp}(\sqrt{D(X)})$. The details of the results here can be found in Cheng [1, Chapter 5].

By Theorem 3.1, we know that the calculation of the RCF expansion of $\sqrt{D(X)}$ ends when $d_\kappa = \Delta'$ and $\Delta_1^{\varepsilon_\kappa} = 1$ for some minimal κ. Hence, if we write

$$|\mathcal{S}_i| = \left|\mathcal{S}\left(\frac{A'\Delta'}{d_i}, r_i\right)\right| = |\mathcal{S}(A'\Delta, d_i r_i)|$$

as the cardinality of the set $\mathcal{S}_i$, then

$$\mathrm{lp}(\sqrt{D(X)}) = \sum_{i=0}^{\kappa-1}(1 + |\mathcal{S}_i|). \tag{24}$$

Recall that $|\mathcal{S}_i|$ is the length of the RCF expansion of $A'\Delta'/d_i r_i$ when $r_i > 0$, and $|\mathcal{S}_i| = 0$ when $r_i = 0$. We henceforth consider $r_i > 0$.

It is well known (see, for example, Knuth [6, p. 343]) that the maximum length for the RCF expansion of a/r is at most $\lfloor \log_\varphi(\sqrt{5}\cdot a)\rfloor - 1$. Hence, $|\mathcal{S}_i| \le \lfloor \log_\varphi(\sqrt{5}\cdot A'\Delta')\rfloor - 1$ and

$$\mathrm{lp}(\sqrt{D(X)}) \le \sum_{i=0}^{\kappa-1}(1 + \lfloor \log_\varphi(\sqrt{5}\cdot A'\Delta')\rfloor - 1) = \kappa \cdot \lfloor \log_\varphi(\sqrt{5}\cdot A'\Delta')\rfloor.$$

It remains to determine an upper bound on the value of κ. Define

$$T = \frac{2A^2K + 2B}{\Delta_2\Delta_4^2}, \qquad P = \frac{T^2}{\Delta_1} - 2\sigma. \tag{25}$$

Let α and β be roots of $x^2 - Px + 1 = 0$. Then $\alpha + \beta = P$ and $\alpha\beta = 1$. Define

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{26}$$

and let $\omega(m)$ be the rank of apparition of m in the Lucas function U_n.

It was shown in [1, Chapter 5] that $\kappa \mid 2\omega(\Delta')$ and that $\omega(m) \le m$ if m is odd and $\omega(m) \le (3/2)m$ if m is even. Hence, we have the following result.

THEOREM 4.1.

$$\mathrm{lp}(\sqrt{D(X)}) \le \begin{cases} 3\Delta'\lfloor \log_\varphi(\sqrt{5}\cdot A'\Delta')\rfloor & \text{if } \Delta' \text{ is even},\\ 2\Delta'\lfloor \log_\varphi(\sqrt{5}\cdot A'\Delta')\rfloor & \text{if } \Delta' \text{ is odd}, \end{cases}$$

where $\varphi = (1+\sqrt{5})/2$ *and* $\log_\varphi(x)$ *is the logarithm of* x *with base* φ.

The result $\kappa \mid 2\omega(\Delta')$ can be improved to

THEOREM 4.2. *The value of* κ *is either* $2\omega(\Delta'/\delta)$ *or* $\omega(\Delta')$, *where* $\delta = \gcd(\Delta', T/\Delta_1)$. *The latter case can occur only if* $\omega(\Delta')$ *is odd and* $\Delta_1 = 1$.

EXAMPLE 4.1. Consider $D(X) = 119^2X^2 + 2(833)X + 245$. Then $A = 119$, $B = 833$, $C = 245$, $\Delta = -2^2\cdot 7^4\cdot 17^2$ and $\sigma = -1$. Also, $\Delta_1 = 1$, $\Delta_2 = 34$, $\Delta_4 = 7$ and $\Delta' = 14$. As K varies from 0 to 13, we first compute κ and then either $\omega(\Delta')$ or $\omega(\Delta'/\delta)$ and arrive at Table 1:

Table 1. Values of κ and $\omega(\Delta'/\delta)$ of $\sqrt{119^2X^2+2(833)X+245}$

K	0	1	2	3	4	5	6	7	8	9	10	11	12	13
κ	24	8	6	8	24	6	6	8	24	2	24	8	6	6
$\omega(\Delta'/\delta)$	12	4	3	4	12	6	3	4	12	1	12	4	3	3

REMARK 4.1. The results in [9] assume that $\gcd(A^2, 2B, C)$ is squarefree. This assumption leads to $\Delta' \mid 2$. It was shown in [1, Chapter 5] that κ in this case is 1, 2, 3, 4 or 6.

5. Fundamental unit of $[1, \sqrt{D(X)}]$**.** Stender [13] studied the fundamental unit of the order $[1, \sqrt{D(X)}]$, where the radicand $D(X)$ is assumed to be squarefree for some sufficiently large integer X. In this section, we make use of Theorem 3.1 to get the fundamental unit of the real quadratic order $[1, \sqrt{D(X)}]$ for sufficiently large X.

It is well known that if ε is the fundamental unit of $[1, \sqrt{D}]$ for some nonsquare natural number D, $\ell = \operatorname{lp}(\sqrt{D})$ and θ_i is the ith complete quotient in the RCF expansion of $\sqrt{D}$, then

$$\varepsilon = \prod_{i=1}^{\ell} \theta_i.$$

LEMMA 5.1. *Let* $\theta(X) = (P + \sqrt{D(X)})/Q$ *be a complete quotient of* $\sqrt{D(X)}$, $q(X) = \lfloor (P+\sqrt{D(X)})/Q \rfloor$ *and* E *and* F *as defined in Theorem* 2.2 *so that* $F/E = (s_0, s_1, \ldots, s_{m-1})$ *and*

$$\frac{P+\sqrt{D(X)}}{Q} = (q(X), s_0, s_1, \ldots, s_{m-1}, \theta^*(X)),$$

where the parity of m *is chosen so that* $(-1)^{m-1} = \Delta/|\Delta|$. *If* $\theta_j(X)$ *is the* j*th complete quotient of* $(P + \sqrt{D(X)})/Q$, *then*

$$\prod_{j=1}^{m+1} \theta_j(X) = \frac{d(A^2X + B + A\sqrt{D(X)})}{|\Delta|},$$

where $d = \gcd(E, F)$.

Proof. Let A_j/B_j be the jth convergent of $\theta(X)$. Then by (1), we have

$$\theta_j(X) = -\frac{B_{j-2}\theta(X) - A_{j-2}}{B_{j-1}\theta(X) - A_{j-1}}$$

for $j \geq 0$. Thus,

$$\prod_{j=1}^{m+1} \theta_j(X) = \frac{(-1)^{m+1}}{B_m\theta(X) - A_m}.$$

Since $A_m/B_m = (q(X), s_0, s_1, \ldots, s_{m-1})$, we have

$$\frac{A_m}{B_m} = q(X) + \frac{1}{(s_0, s_1, \ldots, s_{m-1})} = q(X) + \frac{E}{F} = \frac{q(X)F + E}{F}.$$

Recall from Theorem 2.2 that $E = A^2X + B - AP'$ and $F = AQ$, where $P' = q(X)Q - P$. If $d = \gcd(E, F)$, then $d = \gcd(q(X)F + E, F)$. Hence,

$$dA_m = q(X)F + E = q(X)AQ + A^2X + B - AP', \qquad dB_m = AQ.$$

So,

$$B_m\theta(X) - A_m = \frac{1}{d}\,(AQ\theta(X) - q(X)AQ - A^2X - B + AP').$$

Since

$$\begin{aligned} &AQ\theta(X) - q(X)AQ + AP' \\ &\quad = AQ\left(\frac{P + \sqrt{D(X)}}{Q}\right) - q(X)AQ + A(q(X)Q - P) = A\sqrt{D(X)}, \end{aligned}$$

we have

$$B_m\theta(X) - A_m = -\frac{A^2X + B - A\sqrt{D(X)}}{d}.$$

Hence,

$$\prod_{j=1}^{m+1} \theta_j(X) = \frac{(-1)^{m+1}}{B_m\theta(X) - A_m} = \frac{d(A^2X + B + A\sqrt{D(X)})}{|\Delta|}. \ \blacksquare$$

THEOREM 5.1. *The fundamental unit* ε *of* $[1, \sqrt{D(X)}]$ *for* X *satisfying* (9) *is given by*

$$\varepsilon = |\Delta|^{\kappa/2}\left(\frac{A^2X + B + A\sqrt{D(X)}}{|\Delta|}\right)^{\kappa} = \left(\frac{A^2X + B + A\sqrt{D(X)}}{\sqrt{|\Delta|}}\right)^{\kappa}.$$

Moreover, the norm of ε *is* σ^κ. *By Remarks* 3.1, κ *is even when* $\Delta_1 > 1$, *so* $|\Delta|^{\kappa/2} \in \mathbb{N}$.

Proof. By Theorem 3.1,

$$\sqrt{D(X)} = (q_0(X), \overline{\mathcal{S}_0, q_1(X), \mathcal{S}_1, \ldots, \mathcal{S}_{\kappa-1}, q_\kappa(X)}).$$

For $i \geq 0$, let $\theta_i(X) = (q_i(X), \mathcal{S}_i, \theta_{i+1}(X))$, $\theta'_j(X)$ be the jth complete quotient of $\theta_i(X)$ and

$$\vartheta_i(X) = \prod_{j=1}^{1+|\mathcal{S}_i|} \theta'_j(X).$$

Further, let $\theta_k(X)$ be the kth complete quotient of $\sqrt{D(X)}$ and $\ell = \sum_{i=0}^{\kappa-1}(1+|\mathcal{S}_i|)$. Then by the remark at the beginning of this section, the fundamental unit ε of $[1, \sqrt{D(X)}]$ is given by

$$\varepsilon = \prod_{i=1}^{\ell} \theta_i(X) = \prod_{i=0}^{\kappa-1} \vartheta_i(X).$$

By Lemma 5.1, we have

$$\vartheta_i(X) = \frac{\delta_{i+1}(A^2X + B + A\sqrt{D(X)})}{|\Delta|},$$

where $\delta_1 = Gd_1$ and $\delta_i = (\Delta_1^{\varepsilon_{i-1}}\Delta_2\Delta_4^2/d_{i-1})d_i$ for $i \geq 2$. Thus,

$$\begin{aligned}\varepsilon = \prod_{i=0}^{\kappa-1} \vartheta_i = \prod_{i=0}^{\kappa-1} \frac{\delta_{i+1}(A^2X + B + A\sqrt{D(X)})}{|\Delta|} \\ = \left(\frac{A^2X + B + A\sqrt{D(X)}}{|\Delta|}\right)^{\kappa} \prod_{i=1}^{\kappa} \delta_i.\end{aligned}$$

Now,

$$\prod_{i=1}^{\kappa} \delta_i = Gd_1 \cdot \frac{\Delta_1^{\varepsilon_1}\Delta_2\Delta_4^2}{d_1}\, d_2 \cdot \frac{\Delta_1^{\varepsilon_2}\Delta_2\Delta_4^2}{d_2}\, d_3 \cdots \frac{\Delta_1^{\varepsilon_{\kappa-1}}\Delta_2\Delta_4^2}{d_{\kappa-1}}\, d_\kappa.$$

Since ε_i is either 0 or 1 according as i is even or odd, and since κ is even if $\Delta_1 > 1$ by Remarks 3.1, we may write

$$\prod_{i=1}^{\kappa} \delta_i = G\Delta_1^{\kappa/2}(\Delta_2\Delta_4^2)^{\kappa-1} d_\kappa.$$

Since $d_\kappa = \Delta'$ by Theorem 3.1, we have $Gd_\kappa = \Delta_2\Delta_4^2$. Thus,

$$\prod_{i=1}^{\kappa} \delta_i = |\Delta|^{\kappa/2} \quad \text{and} \quad \varepsilon = |\Delta|^{\kappa/2}\left(\frac{A^2X + B + A\sqrt{D(X)}}{|\Delta|}\right)^{\kappa}.$$

Now, the norm of ε is

$$\frac{|\Delta|^\kappa}{|\Delta|^{2\kappa}} \cdot \mathcal{N}(A^2X + B + A\sqrt{D(X)})^\kappa = \frac{|\Delta|^\kappa}{|\Delta|^{2\kappa}}\, \Delta^\kappa = \frac{\Delta^\kappa}{|\Delta|^\kappa} = \sigma^\kappa. \ \blacksquare$$

COROLLARY 5.1. *If $\Delta_1 > 1$, then the norm of ε is* 1.

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