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Periods of β-expansions and linear recurrent sequences

by

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1. Introduction

1.1. *β-numeration system.* Let $\beta > 1$ be a real number. The *β-transformation* is a piecewise linear transformation on $[0,1)$ defined by

$$T_\beta : x \mapsto \beta x - \lfloor \beta x \rfloor,$$

where $\lfloor \alpha \rfloor$ is the largest integer not exceeding α. By iterating this map and considering its trajectory

$$x \xrightarrow{x_1} T_\beta(x) \xrightarrow{x_2} T_\beta^2(x) \xrightarrow{x_3} \cdots$$

with $x_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor$, we obtain the *$\beta$-expansion* of x,

$$x = \frac{x_1}{\beta} + \frac{x_2}{\beta^2} + \frac{x_3}{\beta^3} + \cdots = 0.x_1x_2x_3\ldots.$$

An expansion is *finite* if $(x_i)_{i\ge 1}$ is eventually 0. A β-expansion is *periodic* if there exists $p \ge 1$ and $M \ge 1$ such that $x_k = x_{k+p}$ for all $k \ge M$; if $x_k = x_{k+p}$ for all $k \ge 1$, then it is *strictly periodic* (or purely periodic). When the β-expansion of x is periodic, we denote by $L_\beta(x)$ its minimal period.

When $\beta = b \ge 2$ is an integer, the β-expansion is the b-adic expansion. In this case, it is well known that:

(i) *A number $x \in [0,1)$ has periodic expansion if and only if x is a rational number.*

(ii) *The expansion of $x = p/q$ is strictly periodic if and only if* $\gcd\{q,b\} = 1$.

(iii) *The minimum period of the b-adic expansion of p/q coincides with the minimum period of the sequence* $\{b^n \pmod q\}_{n\ge 0}$.

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It is natural to ask the same questions for β-expansions when β is not an integer.

(Q1) *For which* $x \in [0,1)$, *is the* β*-expansion periodic*?
(Q2) *For which* $x \in [0,1)$, *is the* β*-expansion strictly periodic*?
(Q3) *If the* β*-expansion of* x *is periodic*, *what is the minimum period*?

1.2. *Periodic* β*-expansions and Pisot numbers.* Question (Q1) has been settled for all Pisot numbers (see Schmidt [Sch], 1980). An algebraic integer strictly greater than 1 is a *Pisot number* if all its algebraic conjugates have modulus strictly less than 1. A number is called a *Pisot unit* if it is a Pisot number and an algebraic unit. Let $\mathbb{Q}(\beta)$ be the smallest field containing the rational numbers and β. The following is well known:

Theorem A (Schmidt [Sch]). *Let* β *be a Pisot number and* $x \in [0,1)$. *Then* x *has periodic* β*-expansion if and only if* $x \in \mathbb{Q}(\beta)$.

The β-expansion with a Pisot base has drawn attention of many mathematicians. For a Pisot unit β, Rauzy ([Rau], 1982) and Thurston ([Thu], 1989) constructed a self-similar tiling system. The tiles (usually with fractal boundaries) are called *Rauzy fractals* or *atomic surfaces*. Several interesting dynamical system are related to Rauzy fractals (cf. [AI, IR2]).

According to the behavior of the β-expansion, several algebraic properties of β have been defined, for example, the (F)-property introduced by Frougny and Solomyak ([FS]) and the (W)-property introduced by Akiyama [Aki]. Further investigation of these properties can be found in [Hol] (for the (F)-property) and [ARS] (for the (W)-property).

Akiyama [Aki] showed that the algebraic properties of β characterize the tiling and dynamical properties of the associated Rauzy fractals.

On the other hand, Ito and his coauthors ([HI, IS, IR1]) employed the Rauzy fractals to study the β-expansions. They answered Question (Q2) when β is a Pisot unit (see Section 1.3). The goal of this paper is to answer Question (Q3) when β is a quadratic Pisot unit.

1.3. *Strictly periodic* β*-expansions with Pisot unit base.* Let β be a Pisot unit of degree d. By using *Rauzy fractals*, a region K has been constructed to serve as a Markov partition of a group automorphism on d-dimensional torus ([Pra, IR2]). It has been proved that the region K completely characterizes the strictly periodic β-expansions in base β. The result was first obtained for quadratic Pisot units in [HI], generalized to a family of Pisot units in [IS], and to all Pisot units in [IR2].

Here we state only the result for quadratic Pisot units, which is needed in the present paper. A number $\beta > 1$ is a *quadratic Pisot unit* if either $\beta^2 = n\beta + 1$ ($n \geq 1$) or $\beta^2 = n\beta - 1$ ($n \geq 3$). Since $1, \beta$ is a basis of the field $\mathbb{Q}(\beta)$, each $x \in \mathbb{Q}(\beta)$ can be uniquely expressed as $x = x_1 + x_2\beta$ with

$x_1, x_2 \in \mathbb{Q}$. The number $x' = x_1 + x_2\beta'$ is the *conjugate* of x in $\mathbb{Q}(\beta)$ by definition, where β' is the algebraic conjugate of β.

THEOREM B ([HI, IS, IR1]). (i) *If $\beta > 1$ satisfies $\beta^2 = n\beta + 1$ and $x \in [0,1]$ then x has strictly periodic β-expansion if and only if $x \in \mathbb{Q}(\beta)$ and (x, x') belongs to the closed set K in Figure* 1.a *minus the lines $x = 1/\beta$ and $x = 1$.*

(ii) *If $\beta > 1$ satisfies $\beta^2 = n\beta - 1$ and $x \in [0,1]$ then x has strictly periodic β-expansion if and only if $x \in \mathbb{Q}(\beta)$ and (x, x') belongs to the closed set K in Figure* 1.b.

If we use the concept of "weakly admissible", then we have a uniform statement of Theorem B, which is discussed in full detail in Section 2.

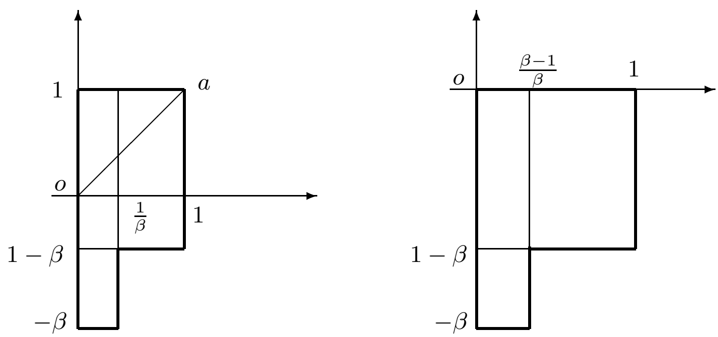

Fig. 1.a Fig. 1.b

As a consequence, if $\beta^2 = n\beta + 1$, then every rational number in $[0,1)$ has strictly periodic β-expansion since the segment oa in Figure 1.a is contained in K (this result was first proved by Schmidt [Sch]); if $\beta^2 = n\beta - 1$, then none of the β-expansions of the rational numbers $\neq 0$ is strictly periodic.

1.4. *Periods of β-expansions with quadratic Pisot unit base.* The main purpose of this paper is to investigate the periods of β-expansions.

When $\beta^2 = n\beta + 1$, the length of the period of the β-expansion of p/q has been studied by Schmidt [Sch]. He characterized the function $L_\beta(p/q)$ by a certain dynamical system.

Using the dynamical and tiling properties of the Rauzy fractal, we obtain a satisfactory answer to Question (Q3) when β is a quadratic Pisot unit. Precisely, we show that $L_\beta(x)$ is determined by a linear recurrent sequence related to β and x.

Let β be a quadratic Pisot unit with minimal polynomial $P(x) = x^2 - a_1x - a_0$ with $a_0 = \pm 1$. We set

$$\mathbb{Z}[\beta] := \{c_0 + c_1\beta;\ c_0, c_1 \in \mathbb{Z}\}.$$

For $x \in \mathbb{Q}(\beta)$, $x \neq 0$, let $q \geq 1$ be the smallest integer such that $qx \in \mathbb{Z}[\beta]$. Notice that 1, $a_0\beta^{-1}$ is a basis of $\mathbb{Z}[\beta]$, hence x can be uniquely written as

$$x = \frac{u_0(a_0\beta^{-1}) + u_1}{q}$$

with u_0, u_1 integers and $\gcd\{u_0, u_1, q\} = 1$.

Let $\{u_k\}$ be the sequence of integers defined by the initial set u_0, u_1 and the recurrence relation

$$u_{k+1} = a_1 u_k + a_0 u_{k-1} \quad (k \geq 1). \tag{1.1}$$

We denote this sequence by $u_k = u_k(u_0, u_1)$. It is easy to show that $\{u_k \pmod q\}_{k\geq 0}$ is always strictly periodic ([Eng, W]). It is interesting that this sequence characterizes the periods of β-expansions.

THEOREM 1.1. *Let β be a quadratic Pisot unit. Suppose $x \neq 0$ and $x = (u_0(a_0\beta^{-1}) + u_1)/q$ with u_0, u_1, q integers and* $\gcd\{u_0, u_1, q\} = 1$. *If the β-expansion of x is strictly periodic, then its periods coincide with the periods of the sequence* $\{u_k(u_0, u_1) \pmod q\}_{k\geq 0}$.

Theorem 1.1 is the main result of this paper. It is proved in Section 2. The fact that the region K translationally tiles $\mathbb{R}^2$ plays a crucial role in our proof (see Figure 2).

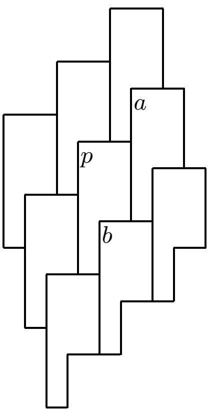

Fig. 2. A tiling by K

In the following, we give a second characterization of the function $L_\beta(x)$. Denote by $D = a_1^2 + 4a_0$ the discriminant of the polynomial $P(x)$. Let d_0 be the maximal square-free factor of D; clearly $\mathbb{Q}(\beta) = \mathbb{Q}(\sqrt{d_0})$. Let D_0 be the discriminant of the field $\mathbb{Q}(\beta)$; then $D_0 = d_0$ when $d_0 \equiv 1 \pmod 4$ and $D_0 = 4d_0$ otherwise. We note that $I = \sqrt{D/D_0}$, the *index* of the polynomial $P(x)$ in K, is an integer. (See for example Hecke [Hec].)

Let $q \geq 1$ be an integer. Then the sequence β^k modulo q is strictly periodic since β is an algebraic unit. We will prove in Section 3 that this sequence also characterizes the periods of β-expansions.

THEOREM 1.2. *Let β be a quadratic Pisot unit. Suppose the β-expansion of x is strictly periodic and q is the smallest integer such that $qx \in \mathbb{Z}[\beta]$. If q is prime to D/D_0, then the periods of the β-expansion of x coincide with the periods of the sequence $\{qx\beta^k \pmod{q}\}_{k\geq 0}$.*

Let $N(\beta) = \beta\beta' = -a_0$ denote the norm of β. When $N(\beta) = -1$, every rational number p/q has strictly periodic β-expansion. If q is prime to D/D_0, then the periods coincide with the periods of the sequence $\beta^k \pmod{q}$, which is an analogue of the result for b-adic expansions.

If q is not prime to D/D_0, then $L_\beta(x)$ is usually a multiple of the minimum period of the sequence $\{qx\beta^k \pmod{q}\}_{k\geq 0}$.

EXAMPLE 1.3. Let $\beta = (1+\sqrt{5})/2$ (which satisfies $\beta^2 = \beta + 1$) be the golden number. Then every rational number p/q in $[0,1)$ has strictly periodic β-expansion by Theorem B. According to Theorem 1.1, the minimum period $L_\beta(p/q)$ coincides with the minimum period of the sequence f_k modulo q, where f_k is the famous *Fibonacci sequence* which is defined by

$$f_0 = 0, \qquad f_1 = 1, \qquad f_{k+1} = f_k + f_{k-1} \qquad (k \geq 1).$$

The first few terms of the sequence are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, \ldots.$$

For example, since the sequence f_k modulo 7 is

$$0, 1, 1, 2, 3, 5, 1, 6, 0, 6, 6, 5, 4, 2, 6, 1, 0, 1, \ldots$$

and has minimum period 16, we conclude that $L_\beta(1/7) = 16$. Actually the β-expansion of $1/7$ is $1/7 = 0.\dot{0}00010101010000\dot{0}$. It is easy to show that $L_\beta(p/7) = 16$ for all $1 \leq p \leq 6$.

1.5. *Linear recurrent sequences modulo q.* The periods of linear recurrent sequences modulo q have been studied as early as 1920 by Carmichael [Car] and 1931 by Engstrom [Eng]. By using Dedekind's theorem on decomposition of primes in a number field, Engstrom obtained very general results. Wall [W] (1960) studied the Fibonacci sequence modulo q. He obtained many precise and interesting results by a simple method. A complete investigation of linear recurrent sequences of degree 2 modulo q is done in [WY].

2. Proof of Theorem 1.1. In this section, first we investigate the weakly admissible β-expansions, which is closely related to the boundary of the region K. Then we show that K can tile $\mathbb{R}^2$. Thanks to this tiling property, finally we prove Theorem 1.1.

2.1. *Admissible and weakly admissible.* Let

$$\Omega = \{x_1x_2\ldots;\ 0.x_1x_2\ldots \text{ is a } \beta\text{-expansion}\}.$$

We may endow the space Ω with the discrete metric ([Wal]). The space Ω is not complete under this metric. Let $\overline{\Omega}$ be the completion of Ω. We will see that $\overline{\Omega}$ consists of weakly admissible sequences.

Formally we may consider the trajectory of 1:

$$1 \xrightarrow{b_1} T_\beta(1) \xrightarrow{b_2} T_\beta^2(1) \xrightarrow{b_3} \cdots .$$

We call $b_1b_2b_3\ldots$ the *expansion of* 1 and denote it by $d_\beta(1)$. Define

$$d_\beta^*(1) = \begin{cases} d_\beta(1) & \text{if } d_\beta(1) \text{ is infinite,} \\ (b_1\ldots b_{d-1}(b_d-1))^\infty & \\ & \text{if } d_\beta(1) = b_1\ldots b_{d-1}b_d \text{ is finite with } b_d \neq 0. \end{cases}$$

A sequence over the alphabet $\{0, 1, \ldots, \lfloor\beta\rfloor\}$ is *admissible* if starting from any place of the sequence, the right side truncation is lexicographically strictly less than $d_\beta^*(1)$. If the right side truncations are less than or equal to $d_\beta^*(1)$, then the sequence is called *weakly admissible*.

A sequence is a β-expansion of a certain number if and only if this sequence is admissible. If a β-expansion is infinite, then it is admissible and weakly admissible. If a β-expansion of x is finite, then x has another infinite expansion in base β which is weakly admissible but not admissible. One can show that $\overline{\Omega}$ is the set of all weakly admissible sequences (cf. [P, Thu]).

Each weakly admissible sequence $(x_i)_{i\geq 1}$ determines a real number x, and we call (x_i) the *weakly admissible β-expansion* of x. Let $\mathrm{Pur}(\beta)$ be the set of numbers in $[0,1)$ with strictly periodic β-expansions, and let $\mathrm{Pur}'(\beta)$ be the set of numbers in $[0,1]$ with strictly periodic, weakly admissible β-expansions. The difference between $\mathrm{Pur}'(\beta)$ and $\mathrm{Pur}(\beta)$ is very small.

THEOREM 2.1. *Let $\beta > 1$ be a real number. If $d_\beta(1)$ is infinite, then* $\mathrm{Pur}(\beta) = \mathrm{Pur}'(\beta)$; *if $d_\beta(1) = b_1b_2\ldots b_d$ is finite, then* $\mathrm{Pur}'(\beta) = \mathrm{Pur}(\beta) \cup \{r_1, \ldots, r_d\}$, *where $r_i = 0.b_i\ldots b_d$.*

Proof. Suppose $x \in \mathrm{Pur}(\beta)$. Then the β-expansion of x is infinite and thus it is also admissible. This proves that $\mathrm{Pur}(\beta) \subset \mathrm{Pur}'(\beta)$.

Suppose $\mathrm{Pur}(\beta) \neq \mathrm{Pur}'(\beta)$. Take any $x \in \mathrm{Pur}'(\beta) \setminus \mathrm{Pur}(\beta)$. Then the β-expansion of x is finite, say $x = 0.x_1\ldots x_k$. The weakly admissible β-expansion of x is

$$x = 0.x_1x_2\ldots(x_k-1)x_{k+1}x_{k+2}\ldots$$

where $x_{k+1}x_{k+2}\ldots = d^*_\beta(1)$. Since the above expansion is strictly periodic, we see that $d^*_\beta(1)$ is strictly periodic. It follows that $d_\beta(1)$ is finite, and $x_1x_2\ldots(x_k-1)x_{k+1}x_{k+2}\ldots$ coincides with a right side truncation of $d^*_\beta(1)$. These together imply that $x = r_i$ for some i. ∎

The following theorem is a special case of a result of [IR1].

THEOREM B′. *Let β be a quadratic Pisot unit. Then $x \in \mathrm{Pur}'(\beta)$ if and only if $x \in \mathbb{Q}(\beta) \cap [0,1]$ and $(x, x') \in K$.*

If $\beta^2 = n\beta+1$, then $d_\beta(1) = n1$; if $\beta^2 = n\beta-1$, then $d_\beta(1) = (n-1)(n-2)^\infty$. We see that Theorem B in Section 1 follows directly from Theorem B′ and Theorem 2.1.

2.2. *A tiling by K.* Let β be a quadratic Pisot unit with minimal polynomial $P(x) = x^2 - a_1x - a_0$ where $a_0 = \pm 1$. Set

$$\mathcal{J} = \{(x, x');\ x \in \mathbb{Z}[\beta]\}.$$

Then $\mathcal{J}$ is a lattice of $\mathbb{R}^2$ with a basis $(1,1)$, (β, β'). Denote by E° the interior of a set E.

LEMMA 2.2. *Let β be a quadratic Pisot unit, and let K be the region in Figure 1 associated with β. Then the collection $\{K + v;\ v \in \mathcal{J}\}$ is a translation tiling of $\mathbb{R}^2$. That is, $\mathbb{R}^2 = \bigcup_{v\in\mathcal{J}}(K + v)$ and $(K + v_1)^\circ \cap (K + v_2)^\circ = \emptyset$ whenever $v_1 \neq v_2$, $v_1, v_2 \in \mathcal{J}$.*

Proof. First we note that $1, 1/\beta$ is a basis of $\mathbb{Z}[\beta]$ and thus $(1,1)$, $(1/\beta, 1/\beta')$ is a basis of $\mathcal{J}$ when $N(\beta) = -1$. Likewise $1, 1-\beta^{-1}$ is a basis of $\mathbb{Z}[\beta]$ and thus $(1,1), (1-\beta^{-1}, 1-(\beta')^{-1})$ is a basis of $\mathcal{J}$ when $N(\beta) = 1$.

From Figure 2 we see clearly that K tiles the plane by translation, and the translation set is a lattice with a basis $\vec{pa}$, $\vec{pb}$. Since $\vec{pa} = (1,1)$, $\vec{pb} = (1/\beta, 1/\beta')$ when $N(\beta) = -1$, and $\vec{pa} = (1,1)$, $\vec{pb} = (1-\beta^{-1}, 1-(\beta')^{-1})$ when $N(\beta) = 1$, we conclude that the translation set is $\mathcal{J}$ and the lemma is proved. ∎

LEMMA 2.3. *Let β be a quadratic Pisot unit. Let $x, y \in \mathbb{Q}(\beta) \cap [0,1)$, $x, y \neq 0$ and $x \neq y$. If $x - y \in \mathbb{Z}[\beta]$, then at most one of x, y has strictly periodic β-expansion.*

Proof. Suppose one of $(x,x'), (y,y')$, say (x,x'), is an inner point of the region K. Since $K + \mathcal{J}$ is a tiling of $\mathbb{R}^2$, we infer that (y,y') is an inner point of the tile $K + (y-x, (y-x)')$. Hence (y,y') is not in K and thus the β-expansion of y is not strictly periodic. The lemma holds in this case.

Now we consider the case that both $(x,x'), (y,y')$ are on the boundary of K. It is easy to check that

$$\{x \in \mathbb{Q}(\beta);\ (x,x') \in \partial K\} = \begin{cases} \{0, \beta^{-1}, 1\} & \text{when } N(\beta) = -1, \\ \{0, 1-\beta^{-1}\} & \text{when } N(\beta) = 1. \end{cases}$$

When $N(\beta) = -1$, only 0 belongs to $\mathrm{Pur}(\beta)$, while β^{-1} and 1 belong to $\mathrm{Pur}'(\beta)$ but not to $\mathrm{Pur}(\beta)$. When $N(\beta) = 1$, we have $0, 1 - \beta^{-1} \in \mathrm{Pur}(\beta)$. So there is at most one $x \neq 0$ such that (x, x') is on the boundary of K and $x \in \mathrm{Pur}(\beta)$. The lemma is hence proved in this case. ∎

REMARK 2.4. By the above discussion, we see that $\mathbb{Z}[\beta] \cap \mathrm{Pur}(\beta) = \{0\}$ when $N(\beta) = -1$, while $\mathbb{Z}[\beta] \cap \mathrm{Pur}(\beta) = \{0, 1 - \beta^{-1}\}$ when $N(\beta) = 1$. Since $1 - \beta^{-1} = 0.(n-2)^\infty$ when $N(\beta) = 1$, one checks directly that Theorem 1.1 holds when $q = 1$.

2.3. *Carry sequence.* Let

$$P(x) = x^2 - a_1 x - a_0 \tag{2.1}$$

be the minimal polynomial of β. Let $q \geq 1$ be the smallest integer such that $qx \in \mathbb{Z}[\beta]$. Then x can be written uniquely as

$$x = \frac{u_0(a_0\beta^{-1}) + u_1}{q} \tag{2.2}$$

with u_0, u_1 integers and $\gcd\{u_0, u_1, q\} = 1$.

We define a sequence $\widetilde{u}_k$ as follows. Set $\widetilde{u}_0 = u_0$, $\widetilde{u}_1 = u_1$. Supposing $\widetilde{u}_0, \widetilde{u}_1, \ldots, \widetilde{u}_k$ are defined, we define $\widetilde{u}_{k+1}$ to be the unique integer such that

$$\widetilde{u}_{k+1} \equiv a_1\widetilde{u}_k + a_0\widetilde{u}_{k-1} \pmod q, \qquad 0 \leq \widetilde{u}_k(a_0\beta^{-1}) + \widetilde{u}_{k+1} < q. \tag{2.3}$$

Let us call $\{\widetilde{u}_k\}_{k\geq 0}$ the *carry sequence* of x.

The carry sequence was first introduced by Hollander [Hol] in the case $q = 1$, i.e., for $x \in \mathbb{Z}[\beta]$; it has been used in [AR, ARS]. Here we generalize it to all $x \in \mathbb{Q}(\beta)$. The carry sequence is closely related to the β-expansion.

Let $x = 0.x_1x_2\ldots$ be the β-expansion of x. Then by (2.1)–(2.3), one has

$$T_\beta(x) = \beta x - x_1 = \frac{\widetilde{u}_1(a_0\beta^{-1}) + (a_1u_1 + a_0u_0 - x_1q)}{q} = \frac{\widetilde{u}_1(a_0\beta^{-1}) + \widetilde{u}_2}{q}.$$

In general, it is easy to show by induction that

$$T_\beta^k(x) = \frac{\widetilde{u}_k(a_0\beta^{-1}) + \widetilde{u}_{k+1}}{q}.$$

Therefore x has strictly periodic β-expansion if and only if its carry sequence $\widetilde{u}_k$ is strictly periodic. Moreover, $L_\beta(x)$ equals the minimal period of the sequence $\widetilde{u}_k$.

Proof of Theorem 1.1. Let $\{u_k(u_0, u_1)\}_{k\geq 0}$ be the linear recurrent sequence defined by (1.1). We claim that

$$u_k \equiv \widetilde{u}_k \pmod q \tag{2.4}$$

for all $k \geq 0$. For suppose (2.4) holds for $1, \ldots, k$; then

$$\widetilde{u}_{k+1} \equiv a_1\widetilde{u}_k + a_0\widetilde{u}_{k-1} \equiv a_1u_k + a_0u_{k-1} = u_{k+1} \pmod q.$$

It remains to show that the sequences $\widetilde{u}_k$ and $u_k \pmod q$ have the same periods, which completes the proof of the theorem.

Let h be a period of the sequence $\widetilde{u}_k$. Then $\widetilde{u}_0 = \widetilde{u}_h, \widetilde{u}_1 = \widetilde{u}_{h+1}$. Thus by (2.4), we have $u_0 \equiv u_h, u_1 \equiv u_{h+1} \pmod q$. Therefore h is a period of the sequence u_k modulo q since u_{k+1} is completely determined by u_k and u_{k-1}.

Conversely, suppose h is a period of $u_k \pmod q$. Again by (2.4), we have $\widetilde{u}_0 \equiv \widetilde{u}_h, \widetilde{u}_1 \equiv \widetilde{u}_{h+1} \pmod q$. Let $y = T_\beta^h(x)$. Then

$$x - y = \frac{(\widetilde{u}_0 - \widetilde{u}_h)(a_0\beta^{-1}) + (\widetilde{u}_1 - \widetilde{u}_{h+1})}{q} \in \mathbb{Z}[\beta].$$

Since both x and y have strictly periodic β-expansions, we have $x = y$ by Lemma 2.3. Hence h is a period of the β-expansion of x as well as a period of $\widetilde{u}_k$. ∎

3. Proof of Theorem 1.2. To prove Theorem 1.2, we need three easy lemmas. Lemma 3.1 gives a formula for the general term of the sequence u_k (see for example [Eng, W]).

LEMMA 3.1. *The general term of u_k is given by*

$$u_k = c_1\beta^k + c_2(\beta')^k \tag{3.1}$$

where

$$c_1 = \frac{u_1 + a_0\beta^{-1}u_0}{\beta - \beta'}, \qquad c_2 = (c_1)' = \frac{u_1 + a_0(\beta')^{-1}u_0}{\beta' - \beta}.$$

Proof. One can check that the initial value of (3.1) is u_0, u_1, and u_k in (3.1) satisfies the recurrence relation (1.1). ∎

Let $H(q)$ be the minimal period of the sequence u_k modulo q. Then we have

LEMMA 3.2. *If $q = p_1^{e_1} \cdots p_k^{e_k}$, then*

$$H(q) = \operatorname{lcm}\{H(p_1^{e_1}), \ldots, H(p_k^{e_k})\} \tag{3.2}$$

where lcm *denotes the least common multiple.*

Lemma 3.2 can be found in [Eng, W]. We leave the easy proof to the reader.

Let $\Re$ be an ideal in $\mathbb{Q}(\beta)$. Denote by $G(x, \Re)$ the minimal period of the sequence $x, x\beta, x\beta^2, \ldots$ modulo $\Re$. Then similar to Lemma 3.2, we have

LEMMA 3.3. *If $q = p_1^{e_1} \cdots p_k^{e_k}$, then*

$$G(x, q) = \operatorname{lcm}\{G(x, p_1^{e_1}), G(x, p_2^{e_2}), \ldots, G(x, p_k^{e_k})\}.$$

THEOREM 3.4. *Let $x_0 = u_0(a_0\beta^{-1}) + u_1$. If q is prime to D/D_0, then the periods of the sequence $\{u_k \pmod q\}$ coincide with the periods of the sequence*

$$x_0, x_0\beta, x_0\beta^2, \ldots \pmod q.$$

Proof. Since $a_0 = \pm 1$, it is easy to see that both $\{x_0\beta^k \pmod q\}$ and $\{u_k \pmod q\}$ are strictly periodic.

According to Lemmas 3.2 and 3.3, we need only show that the assertion is valid for $q = p^m$ where p is prime to D/D_0.

Set $x_k = u_k(a_0\beta^{-1}) + u_{k+1}$, $k \geq 0$. Then $x_{k+1} = \beta x_k$ and so $x_k = x_0\beta^k$. Hence $x_k \in \mathbb{Z}[\beta]$ are algebraic integers in $\mathbb{Q}(\beta)$.

Clearly if h is a period of $\{u_k \pmod q\}$, then it is a period of $\{x_k \pmod q\}$.

Conversely, suppose h is a period of $\{x_k \pmod q\}$. Then $x_0(\beta^h - 1)$ is divisible by $q = p^m$ and so the algebraic conjugate $(x_0(\beta^h - 1))'$ is also divisible by p^m. By Lemma 3.1, we have

$$u_k = \frac{(u_1 + a_0\beta^{-1}u_0)\beta^k - (u_1 + a_0\beta^{-1}u_0)'(\beta')^k}{\beta - \beta'}.$$

Hence

$$u_{k+h} - u_k = \frac{x_0(\beta^h - 1)\beta^k - x_0'((\beta')^h - 1)(\beta')^k}{\beta - \beta'}. \tag{3.3}$$

By the assumption of h, the numerator of the right side of (3.3) is divisible by p^m. Notice that $\beta - \beta' = \sqrt{D} = I\sqrt{D_0}$.

If p is prime to D, then the denominator is prime to p^m. Hence $u_{k+h} - u_k$ is divisible by p^m for all $k \geq 0$, and h is a period of $\{u_k \pmod{p^m}\}$.

If p is a factor of D_0, then $p = \Re^2$ where $\Re$ is a prime ideal in $\mathbb{Q}(\beta)$. We divide the discussion into two cases: $p \neq 2$ and $p = 2$. Recall that $D_0 = d_0$ or $4d_0$ where d_0 is square-free.

If $p \neq 2$, then $p^2 \nmid D_0$ and so the power of $\Re$ contained in the denominator of (3.3) is 1. Since the power of $\Re$ contained in the numerator of (3.3) is at least $2m$, we find that $(u_{k+h} - u_k)/p^{m-1}$ is an integer and it is divisible by p, the norm of $\Re$. Therefore $u_{k+h} - u_k$ is divisible by $q = p^m$ for all $k \geq 0$.

If $p = 2$, then D_0 is an even number and hence $D_0 = 4d_0$. Moreover $1, \sqrt{d_0}$ is a basis of $\mathbb{Q}(\beta)$. Let $x_0(\beta^h - 1)\beta^k = X + Y\sqrt{d_0}$. Then $q = 2^m$ dividing $X + Y\sqrt{d_0}$ implies that $2^m \mid Y$. Formula (3.3) becomes

$$u_{k+h} - u_k = \frac{2Y\sqrt{d_0}}{\sqrt{D}} = \frac{2Y\sqrt{d_0}}{I\sqrt{D_0}} = \frac{2Y\sqrt{d_0}}{2I\sqrt{d_0}} = \frac{Y}{I}.$$

Since $p = 2$ is coprime to I, we conclude that $u_{k+h} - u_k$ is divisible by 2^m for all $k \geq 0$. This completes the proof the theorem. ■

Theorem 1.2 follows immediately from Theorems 1.1 and 3.4.

References

[Aki] S. Akiyama, *On the boundary of self-affine tilings generated by Pisot numbers*, J. Math. Soc. Japan 54 (2002), 283–308.

[AR] S. Akiyama and H. Rao, *New criteria for canonical number systems*, Acta Arith. 111 (2004), 5–25.

[ARS] S. Akiyama, H. Rao and W. Steiner, *A certain finiteness property of Pisot number systems*, J. Number Theory 107 (2004), 135–160.
[AI] P. Arnoux and S. Ito, *Pisot substitutions and Rauzy fractals*, Bull. Belg. Math. Soc. 8 (2001), 181–207.
[Car] R. D. Carmichael, *On sequences of integers defined by recurrence relations*, Quart. J. Math. 41 (1920), 343–372.
[Eng] H. T. Engstrom, *On sequences defined by linear recurrence relations*, Trans. Amer. Math. Soc. 33 (1931), 210–218.
[FS] C. Frougny and B. Solomyak, *Finite beta-expansions*, Ergodic Theory Dynam. Systems 12 (1992), 713–723.
[HI] Y. Hara and S. Ito, *On real quadratic fields and periodic expansions*, Tokyo J. Math. 12 (1989), 357–370.
[Hec] E. Hecke, *Lectures on the Theory of Algebraic Numbers*, Grad. Texts in Math. 77, Springer, 1981.
[Hol] M. Hollander, *Linear numeration systems, finite beta expansions, and discrete spectrum of substitution dynamical systems*, Ph.D. thesis, Univ. of Washington, 1996.
[IR1] S. Ito and H. Rao, *Purely periodic β-expansions with Pisot unit base*, Proc. Amer. Math. Soc. 133 (2005), 953–964.
[IR2] —, —, *Atomic surfaces, tilings and coincidences I. Irreducible case*, Israel J. Math., to appear.
[IS] S. Ito and Y. Sano, *On periodic β-expansions of Pisot numbers and Rauzy fractals*, Osaka J. Math. 38 (2001), 349–368.
[P] W. Parry, *On the β-expansion of real numbers*, Acta Math. Acad. Sci. Hungar. 11 (1960), 401–416.
[Pra] B. Praggastis, *Numeration systems and Markov partitions from self-similar tilings*, Trans. Amer. Math. Soc. 351 (1999), 3315–3349.
[Rau] G. Rauzy, *Nombres algébriques et substitutions*, Bull. Soc. Math. France 110 (1982), 147–178.
[Sch] K. Schmidt, *On periodic expansions of Pisot numbers and Salem numbers*, Bull. London Math. Soc. 12 (1980), 269–278.
[Thu] W. Thurston, *Groups, Tilings, and Finite State Automata*, AMS Colloq. Lectures, Amer. Math. Soc., Providence, RI, 1989.
[W] D. Wall, *Fibonacci series modulo m*, Amer. Math. Monthly 67 (1960), 525–532.
[Wal] P. Walters, *An Introduction to Ergodic Theory*, Grad. Texts in Math. 79, Springer, 1982.
[WY] M. Wu and Y. M. Yang, *Linear recurrent sequence modulo q*, preprint, 2004.

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