ACTA ARITHMETICA
120.1 (2005)

A note on the zeros of the derivative of the Riemann zeta function near the critical line

by

SHAOJI FENG (Beijing)

1. Introduction. Let $s = \sigma + it$ be a complex variable and $\zeta(s)$ the Riemann zeta function. Throughout this paper $\varrho = \beta + i\gamma$ denotes the zeros of $\zeta(s)$, and $\varrho' = \beta' + i\gamma'$ the zeros of $\zeta'(s)$, the first derivative of $\zeta(s)$.

The distribution of the zeros of $\zeta'(s)$ is important in the theory of the Riemann zeta function, being closely related to the distribution of the zeros of $\zeta(s)$. This intimate relationship can be best illustrated by the following three results. The first is by Levinson and Montgomery [6]. They have proved that

$$N_1^-(T) = N^-(T) + O(\log T), \tag{1.1}$$

and, unless $N^-(T) > T/2$ for all large T, there exists a sequence $\{T_j\}$ with $T_j \to \infty$ as $j \to \infty$ such that

$$N_1^-(T_j) = N^-(T_j), \tag{1.2}$$

where $N_1^-(T)$ and $N^-(T)$ are the numbers of zeros of $\zeta'(\sigma+it)$ and $\zeta(\sigma+it)$, respectively, in the rectangle $0 < t \le T$, $0 < \sigma < 1/2$, counted according to multiplicity. This is a quantitative version of an earlier result of Speiser [9], who shows that the Riemann Hypothesis (RH) that

all the complex zeros of $\zeta(s)$ lie on the critical line $\sigma = 1/2$

is equivalent to the non-vanishing of $\zeta'(s)$ in $0 < \sigma < 1/2$. The second result comes from Levinson's famous work [5]. He has further exploited this close relationship and showed that more than one third of the zeros of $\zeta(s)$ lie on the critical line. The third result is due to Guo [3], who has showed that there is also a close connection between the vertical distribution of the zeros of $\zeta'(s)$ and that of $\zeta(s)$.

The distribution of the zeros of $\zeta'(s)$, as well as its relationship with that of the zeros of $\zeta(s)$, has been investigated by many authors (see [1–4, 6, 8,

2000 *Mathematics Subject Classification*: Primary 11M06.

9, 11]). In [8], Soundararajan introduced the following functions (for $a \in \mathbb{R}$):

$$m^-(a) = \liminf_{T\to\infty} N_1(T)^{-1} \sum_{\substack{\beta' \le 1/2 + a/\log T \\ 0<\gamma' \le T}} 1, \tag{1.3}$$

$$m^+(a) = \limsup_{T\to\infty} N_1(T)^{-1} \sum_{\substack{\beta' \le 1/2 + a/\log T \\ 0<\gamma' \le T}} 1, \tag{1.4}$$

where $N_1(T)$ is the number of zeros of $\zeta'(s)$ in $0 < t \le T$, counted according to multiplicity. The behavior of these functions determines the horizontal distribution of the zeros of $\zeta'(s)$ near the critical line. Soundararajan [8] proved that RH implies $m^-(a) > 0$ for $a > 2.6$. He conjectured that $m^-(a) \equiv m^+(a)$ $(= m(a))$, $m(a)$ is continuous, $m(a) > 0$ for all $a > 0$, and $m(a) \to 1$ as $a \to \infty$.

To state Zhang's important results on $m^-(a)$, we need to explain a conjecture on small gaps between the zeros of $\zeta(s)$. Let the zeros of $\zeta(s)$ in the upper half-plane be arranged as $\varrho_1, \varrho_2, \ldots$ with $\varrho_n = \beta_n + i\gamma_n$ and

$$0 < \gamma_1 \le \gamma_2 \le \cdots.$$

If a zero is multiple with multiplicity m, then it appears m times consecutively in the above sequence. If two zeros $\varrho_{n_1} \ne \varrho_{n_2}$ have the same imaginary part (this will happen if RH is not true), then $n_1 < n_2$ implies $\beta_{n_1} < \beta_{n_2}$. For $a > 0$, define

$$D^-(a) = \liminf_{T\to\infty} N(T)^{-1} \sum_{\substack{\gamma_n \le T \\ \gamma_{n+1} - \gamma_n < a/\log T}} 1, \tag{1.5}$$

where $N(T)$ denotes the number of zeros of $\zeta(s)$ in $0 < t \le T$, counted according to multiplicity. The following conjecture is well known and is denoted by SGZ for short.

Conjecture. *For any $a > 0$,*

$$D^-(a) > 0. \tag{1.6}$$

The statement of SGZ is independent of RH.

In [11], Zhang proved the following.

Theorem A. *If a is sufficiently large, then $m^-(a) > 0$.*

Theorem B. (*Assume RH and SGZ.*) *For any $a > 0$, $m^-(a) > 0$.*

In this note, we will show $m^-(a) > 0$ for any $a > 0$ without the assumption of RH. Namely, we prove the following.

Theorem 1. (*Assume SGZ.*) *For any $a > 0$, $m^-(a) > 0$.*

2. Proof of Theorem 1. Let

$$h(s) = \pi^{-s/2}\Gamma(s/2), \qquad \eta(s) = h(s)\zeta'(s).$$

By applying Hadamard's Theorem to $(s-1)^2\zeta'(s)$, Zhang [11] proved the following.

Lemma 1 ([11]). *Let*

$$F(t) = -\operatorname{Re}\frac{\eta'}{\eta}\left(\frac{1}{2}+it\right) \tag{2.1}$$

(*here and hereafter* $\operatorname{Re} z$ *denotes the real part of* z) *if* $\eta(1/2+it) \neq 0$, *and*

$$F(t) = \lim_{\tau\to T} F(\tau) \tag{2.2}$$

if $\eta(1/2+it) = 0$. *Then* $F(t)$ *is continuous for all* t,

$$F(t) = F_1(t) - F_2(t) + O(1), \tag{2.3}$$

where

$$F_1(t) = -\sum_{\beta'>1/2} \operatorname{Re}\frac{1}{1/2+it-\varrho'}, \tag{2.4}$$

$$F_2(t) = \sum_{0<\beta'<1/2} \operatorname{Re}\frac{1}{1/2+it-\varrho'}, \tag{2.5}$$

and the implied constant is absolute.

Now given T large and $a>0$ arbitrary, let

$$F_1^*(t) = -\sum_{\substack{\beta'>1/2+2a/\log T\\ 0<\gamma'\le T}} \operatorname{Re}\frac{1}{1/2+it-\varrho'}. \tag{2.6}$$

Then

$$0 \le F_1^*(t) \le F_1(t). \tag{2.7}$$

We also need the following lemmas.

Lemma 2 ([11]). *We have*

$$\int_0^T F_1^*(t)\,dt \ge \pi \sum_{\substack{\beta'>1/2+2a/\log T\\ 0<\gamma'\le T}} 1 + O(T). \tag{2.8}$$

Lemma 3. *Let the zeros of* $\zeta(s)$ *in the upper half-plane be arranged as* $\varrho_1, \varrho_2, \ldots$ *with* $\varrho_n = \beta_n + i\gamma_n$ (*we do not assume* $\beta_n \equiv 1/2$ *here*) *and*

$$0 < \gamma_1 \le \gamma_2 \le \cdots.$$

If a zero is multiple with multiplicity m, *then it appears* m *times consecutively in the above sequence. If two zeros* $\varrho_{n_1} \neq \varrho_{n_2}$ *have the same imaginary* *part, then $n_1 < n_2$ implies $\beta_{n_1} < \beta_{n_2}$. For $n > 1$, define*

$$I_n = \int\limits_{\gamma_{n-1}}^{\gamma_{n+1}} F_1^*(t)\,dt.$$

(i) *For any n,*

$$I_n \le 2\pi + \int\limits_{\gamma_{n-1}}^{\gamma_{n+1}} F_2(t)\,dt + O(\gamma_{n+1}-\gamma_{n-1}). \tag{2.9}$$

(ii) *If $\beta_n = 1/2$, $\gamma_{n+1} \le T$ and $\gamma_{n+1}-\gamma_n < a/\log T$, then*

$$I_n \le \pi + 2a + \int\limits_{\gamma_{n-1}}^{\gamma_{n+1}} F_2(t)\,dt + O(\gamma_{n+1}-\gamma_{n-1}). \tag{2.10}$$

Proof. (i) This follows from Lemma 1, (2.7) and [11, Lemma 4].

(ii) This follows from Lemma 1, (2.7) and a slightly modified version of the proof of [11, Lemma 8]. ∎

LEMMA 4 ([1]). *We have*

$$N_1(T) = \frac{T}{2\pi}\left(\log\frac{T}{4\pi} - 1\right) + O(\log T). \tag{2.11}$$

LEMMA 5. *We have*

$$\int\limits_0^T F_2(t)\,dt \le \pi \sum_{\substack{0<\beta'<1/2\\ 0<\gamma'\le T}} 1 + O(T). \tag{2.12}$$

Proof. By Lemma 4, for $0 \le t \le T$ and $n \ge 1$,

$$\begin{aligned}\sum_{\substack{0<\beta'<1/2\\ n\log T\le|\gamma'-t|<(n+1)\log T}} 1 &\le 2\log T\log(T+(n+1)\log T) + O(\log(T+(n+1)\log T))\\ &\le 2\log^2 T + 2\log T\log n + O(\log n) + o(\log^2 T),\end{aligned} \tag{2.13}$$

where the implied constants are independent of n. Then

$$\sum_{\substack{0<\beta'<1/2\\ \gamma'\ge T+\log T \text{ or } \gamma'\le -\log T}} \operatorname{Re}\frac{1}{1/2+it-\varrho'} \tag{2.14}$$

$$\le \sum_{n=1}^{\infty} \sum_{\substack{0<\beta'<1/2\\ n\log T\le|\gamma'-t|<(n+1)\log T}} \frac{1}{(\gamma'-t)^2}$$
$$\le \sum_{n=1}^{\infty}(2\log^2 T + 2\log T\log n + O(\log n) + o(\log^2 T))\,\frac{1}{(n\log T)^2}$$
$$= O(1).$$

Thus

$$F_2(t) = \sum_{\substack{0<\beta'<1/2\\ -\log T<\gamma'<T+\log T}} \operatorname{Re}\frac{1}{1/2+it-\varrho'} + O(1). \tag{2.15}$$

For $\beta' < 1/2$,

$$\int_0^T \operatorname{Re}\frac{1}{1/2+it-\varrho'}\,dt = \int_0^T \operatorname{Re}\frac{1/2-\beta'}{|1/2+it-\varrho'|^2}\,dt = \arctan\biggl(\frac{T-\gamma'}{1/2-\beta'}\biggr) + \arctan\biggl(\frac{\gamma'}{1/2-\beta'}\biggr) < \pi. \tag{2.16}$$

By (2.15) and (2.16),

$$\int_0^T F_2(t)\,dt \le \pi \sum_{\substack{0<\beta'<1/2\\ -\log T<\gamma'<T+\log T}} 1 + O(T). \tag{2.17}$$

By Lemma 4,

$$\sum_{\substack{0<\beta'<1/2\\ -\log T<\gamma'\le 0}} 1 = O(\log T\log\log T), \qquad \sum_{\substack{0<\beta'<1/2\\ T<\gamma'<T+\log T}} 1 = O(\log^2 T). \tag{2.18}$$

Combining (2.17) and (2.18), we get (2.12). ■

Lemma 6 ([10, Theorem 9.4]). *We have*

$$N(T) = \frac{T}{2\pi}\biggl(\log\frac{T}{2\pi} - 1\biggr) + O(\log T). \tag{2.19}$$

Proof of Theorem 1. It is no restriction to assume $a < \pi/2$. Let

$$N = \max\{n : \gamma_{n+1} \le T\},$$
$$S_1 = \{n : 1 < n \le N,\ \gamma_{n+1} - \gamma_n \ge a/\log T\},$$
$$S_2 = \{n : 1 < n \le N,\ \beta_n = 1/2,\ \gamma_{n+1} - \gamma_n < a/\log T\},$$
$$S_3 = \{n : 1 < n \le N,\ \beta_n \ne 1/2,\ \gamma_{n+1} - \gamma_n < a/\log T\}.$$

On the one hand, by Lemma 3,

$$\begin{aligned}(2.20)\qquad \sum_{n=2}^{N} I_n &= \sum_{n\in S_1\cup S_3} I_n + \sum_{n\in S_2} I_n \\ &\le 2\pi \sum_{n\in S_1\cup S_3} 1 + \sum_{n\in S_1\cup S_3} \int_{\gamma_{n-1}}^{\gamma_{n+1}} F_2(t)\,dt + O\Big(\sum_{n\in S_1\cup S_3} (\gamma_{n+1}-\gamma_{n-1})\Big) \\ &\quad + (\pi+2a) \sum_{n\in S_2} 1 + \sum_{n\in S_2} \int_{\gamma_{n-1}}^{\gamma_{n+1}} F_2(t)\,dt + O\Big(\sum_{n\in S_2} (\gamma_{n+1}-\gamma_{n-1})\Big) \\ &\le 2\pi N - (\pi-2a) \sum_{n\in S_2} 1 + \int_0^{\gamma_N} F_2(t)\,dt + \int_0^{\gamma_{N+1}} F_2(t)\,dt + O(T),\end{aligned}$$

where the fact that $F_2(t)\ge 0$ is used.

On the other hand, by Lemma 2,

$$\begin{aligned}(2.21)\qquad \sum_{n=2}^{N} I_n + \int_0^{\gamma_1} F_1^*(t)\,dt + \int_0^{\gamma_2} F_1^*(t)\,dt + \int_{\gamma_N}^{T} F_1^*(t)\,dt + \int_{\gamma_{N+1}}^{T} F_1^*(t)\,dt \\ = 2\int_0^T F_1^*(t)\,dt \ge 2\pi \sum_{\substack{\beta'>1/2+2a/\log T\\ 0<\gamma'\le T}} 1 + O(T).\end{aligned}$$

Lemma 6 implies

$$\gamma_{n+1}-\gamma_n = O(1);$$

then by (2.7) and Lemma 3, we have

$$\begin{aligned}(2.22)\qquad \int_{\gamma_N}^{T} F_1^*(t)\,dt &\le \int_{\gamma_N}^{\gamma_{N+2}} F_1^*(t)\,dt = I_{N+1} \\ &\le 2\pi + \int_{\gamma_N}^{\gamma_{N+2}} F_2(t)\,dt + O(\gamma_{N+2}-\gamma_N) = \int_{\gamma_N}^{\gamma_{N+2}} F_2(t)\,dt + O(1).\end{aligned}$$

By Lemma 1, (2.7) and [11, Lemma 4],

$$\begin{aligned}(2.23)\qquad \int_{\gamma_{N+1}}^{T} F_1^*(t)\,dt &\le \int_{\gamma_{N+1}}^{\gamma_{N+2}} F_1^*(t)\,dt \le \int_{\gamma_{N+1}}^{\gamma_{N+2}} F_1(t)\,dt \\ &\le \pi + \int_{\gamma_{N+1}}^{\gamma_{N+2}} F_2(t)\,dt + O(\gamma_{N+2}-\gamma_{N+1}) = \int_{\gamma_{N+1}}^{\gamma_{N+2}} F_2(t)\,dt + O(1).\end{aligned}$$

By (2.7),

$$\int_0^{\gamma_1} F_1^*(t)\,dt \le \int_0^{\gamma_1} F_1(t)\,dt = O(1), \tag{2.24}$$

$$\int_0^{\gamma_2} F_1^*(t)\,dt \le \int_0^{\gamma_2} F_1(t)\,dt = O(1), \tag{2.25}$$

where the implied constant is absolute. Combining (2.21)–(2.25), we have

$$\sum_{n=2}^{N} I_n \ge 2\pi \sum_{\substack{\beta'>1/2+2a/\log T\\ 0<\gamma'\le T}} 1 - \int_{\gamma_N}^{\gamma_{N+2}} F_2(t)\,dt - \int_{\gamma_{N+1}}^{\gamma_{N+2}} F_2(t)\,dt + O(T). \tag{2.26}$$

It follows from Lemmas 4 and 6 that

$$N = N_1(T) + O(T).$$

Then combining (2.20) and (2.26) we obtain

$$\sum_{\substack{\beta'\le 1/2+2a/\log T\\ 0<\gamma'\le T}} 1 \ge \left(\frac{1}{2} - \frac{a}{\pi}\right) \sum_{n\in S_2} 1 - \frac{1}{\pi}\int_0^{\gamma_{N+2}} F_2(t)\,dt + O(T). \tag{2.27}$$

By Lemma 5, Lemma 4 and the fact that $\gamma_{N+2} - T = O(1)$,

$$\begin{aligned}\int_0^{\gamma_{N+2}} F_2(t)\,dt &\le \pi \sum_{\substack{0<\beta'<1/2\\ 0<\gamma'\le \gamma_{N+2}}} 1 + O(\gamma_{N+2})\\ &= \pi \sum_{\substack{0<\beta'<1/2\\ 0<\gamma'\le T}} 1 + \pi \sum_{\substack{0<\beta'<1/2\\ T<\gamma'\le \gamma_{N+2}}} 1 + O(T)\\ &= \pi \sum_{\substack{0<\beta'<1/2\\ 0<\gamma'\le T}} 1 + O(T).\end{aligned} \tag{2.28}$$

Now (2.27) and (2.28) imply

$$\begin{aligned} 2 \sum_{\substack{\beta'\le 1/2+2a/\log T\\ 0<\gamma'\le T}} 1 &\ge \sum_{\substack{\beta'\le 1/2+2a/\log T\\ 0<\gamma'\le T}} 1 + \sum_{\substack{0<\beta'<1/2\\ 0<\gamma'\le T}} 1\\ &\ge \left(\frac{1}{2} - \frac{a}{\pi}\right) \sum_{n\in S_2} 1 + O(T).\end{aligned} \tag{2.29}$$

By (1.5) and Lemma 6,

$$\sum_{n\in S_2} 1 + \sum_{n\in S_3} 1 \ge D^-(a)\,\frac{T\log T}{2\pi} + o(T\log T). \tag{2.30}$$

By (1.1),

$$\sum_{n\in S_3} 1 \le 2 \sum_{\substack{0<\beta<1/2\\ 0<\gamma\le T}} 1 \le 2 \sum_{\substack{0<\beta'<1/2\\ 0<\gamma'\le T}} 1 + O(\log T). \tag{2.31}$$

Therefore

$$\begin{aligned} \sum_{n\in S_2} 1 &\ge D^-(a)\,\frac{T\log T}{2\pi} - 2 \sum_{\substack{0<\beta'<1/2\\ 0<\gamma'\le T}} 1 + o(T\log T) \\ &\ge D^-(a)\,\frac{T\log T}{2\pi} - 2 \sum_{\substack{\beta'\le 1/2+2a/\log T\\ 0<\gamma'\le T}} 1 + o(T\log T). \end{aligned} \tag{2.32}$$

Combining (2.29) and (2.32), we have

$$\sum_{\substack{\beta'\le 1/2+2a/\log T\\ 0<\gamma'\le T}} 1 \ge \frac{\pi-2a}{6\pi+4a}\,D^-(a)\,\frac{T\log T}{2\pi} + o(T\log T). \tag{2.33}$$

That is,

$$m^-(2a) \ge \frac{\pi-2a}{6\pi+4a}\,D^-(a) > 0. \tag{2.34}$$

The proof is complete. ∎

3. Remark. In Levinson's well known work [5], he proved that more than one third of the zeros of $\zeta(s)$ lie on the critical line by using the relationship between $N^-(T)$ and $N_1^-(T)$. We outline the principle behind his proof here. Let $g(s)$ be the function $\zeta'(1-s)$ and σ be $1/2 - a/\log T$, where $a > 0$ is a small positive real number. Note that $\varrho^* = \beta^* + i\gamma^*$ is a zero of $g(s)$ if and only if $\varrho' = 1-\beta^* + i\gamma^*$ is a zero of $\zeta'(s)$. If one can show that

$$\sum_{\substack{\sigma<\beta^*\le 2\\ 0<\gamma^*\le T}} (\beta^*-\sigma) \le C_a T + o(T), \tag{3.1}$$

where C_a is a constant depending on a, then since

$$\sum_{\substack{\sigma<\beta^*\le 2\\ 0<\gamma^*\le T}} (\beta^*-\sigma) \ge \left(\frac{1}{2}-\sigma\right) \sum_{\substack{1/2<\beta^*\le 2\\ 0<\gamma^*\le T}} 1, \tag{3.2}$$

one has

$$\sum_{\substack{1/2<\beta^*\le 2\\ 0<\gamma^*\le T}} 1 \le \frac{C_a}{a}\,T\log T + o(T\log T). \tag{3.3}$$

Then by (1.1),

$$\sum_{\substack{0<\beta<1/2\\0<\gamma\le T}} 1 \le \frac{C_a}{a} T\log T + o(T\log T). \tag{3.4}$$

Therefore the proportion of the zeros of $\zeta(s)$ on the critical line is more than

$$1 - \frac{4\pi C_a}{a}.$$

By estimating C_a carefully (applying the Littlewood Theorem [7, 10]) and choosing a suitably, Levinson proved the proportion is more than 1/3. This result can be improved slightly by having a better estimate for C_a.

But we can see in (3.2) there is some loss in the argument. In the process of obtaining an upper bound for the number of zeros of $\zeta'(s)$ with $\beta' < 1/2$, one has also counted those zeros satisfying

$$\frac{1}{2} \le \beta' < \frac{1}{2} + \frac{a}{\log T}, \qquad 0 < \gamma' \le T \tag{3.5}$$

(with the weight $1/2 + a/\log T - \beta'$). Theorem 1 shows that on the SGZ, there is a positive proportion of the zeros of $\zeta'(s)$ satisfying (3.5) or

$$\beta' < 1/2, \qquad 0 < \gamma' \le T.$$

Thus, if SGZ is valid, no matter how precisely C_a are estimated and what a is chosen, the framework of [5] cannot prove that 100% of the zeros of $\zeta(s)$ lie on the critical line (although it is likely true).

However, once a good (large) lower estimate of $m^-(a)$ is obtained, the result of [5] can be significantly improved by combining this estimate and (3.1). We can see in the statement and proof of Theorem 1 that the lower bound for $m^-(a)$ is closely connected with the vertical distribution of the zeros of the Riemann zeta function.

References

[1] B. C. Berndt, *The number of zeros for* $\zeta^{(k)}(s)$, J. London Math. Soc. (2) 2 (1970), 577–580.

[2] J. B. Conrey and A. Ghosh, *Zeros of derivatives of the Riemann zeta-function near the critical line*, in: Analytic Number Theory (Allerton Park, IL, 1989), Progr. Math. 85, Birkhäuser, Boston, 1990, 95–110.

[3] C. R. Guo, *On the zeros of* $\zeta(s)$ *and* $\zeta'(s)$, J. Number Theory 54 (1995), 206–210.

[4] —, *On the zeros of the derivative of the Riemann zeta function*, Proc. London Math. Soc. (3) 72 (1996), 28–62.

[5] N. Levinson, *More than one third of zeros of Riemann's zeta-function are on* $\sigma = \frac{1}{2}$, Adv. Math. 13 (1974), 383–436.

[6] N. Levinson and H. L. Montgomery, *Zeros of the derivatives of the Riemann zeta-function*, Acta Math. 133 (1974), 49–65.

[7] J. E. Littlewood, *On the zeros of the Riemann zeta function*, Proc. Cambridge Philos. Soc. 22 (1924), 295–318.
[8] K. Soundararajan, *The horizontal distribution of zeros of* $\zeta'(s)$, Duke Math. J. 91 (1998), 33–59.
[9] A. Speiser, *Geometrisches zur Riemannschen Zetafunktion*, Math. Ann. 110 (1934), 514–521.
[10] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed., revised by D. R. Heath-Brown, Clarendon Press, Oxford, 1986.
[11] Y. Zhang, *On the zeros of* $\zeta'(s)$ *near the critical line*, Duke Math. J. 110 (2001), 555–571.

Academy of Mathematics and Systems Science
Chinese Academy of Sciences
Beijing 100080, P.R. China
E-mail: jxfsj@hotmail.com

Received on 13.12.2004 (4900)