A note on the zeros of the derivative of the Riemann zeta function near the critical line

by

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1. Introduction. Let \( s = \sigma + it \) be a complex variable and \( \zeta(s) \) the Riemann zeta function. Throughout this paper \( \varrho = \beta + i\gamma \) denotes the zeros of \( \zeta(s) \), and \( \varrho' = \beta' + i\gamma' \) the zeros of \( \zeta'(s) \), the first derivative of \( \zeta(s) \).

The distribution of the zeros of \( \zeta'(s) \) is important in the theory of the Riemann zeta function, being closely related to the distribution of the zeros of \( \zeta(s) \). This intimate relationship can be best illustrated by the following three results. The first is by Levinson and Montgomery [6]. They have proved that

\[
N_1^-(T) = N^-(T) + O(\log T),
\]

(1.1) and, unless \( N^-(T) > T/2 \) for all large \( T \), there exists a sequence \( \{T_j\} \) with \( T_j \to \infty \) as \( j \to \infty \) such that

\[
N_1^-(T_j) = N^-(T_j),
\]

(1.2) where \( N_1^-(T) \) and \( N^-(T) \) are the numbers of zeros of \( \zeta'(\sigma + it) \) and \( \zeta(\sigma + it) \), respectively, in the rectangle \( 0 < t \leq T, 0 < \sigma < 1/2 \), counted according to multiplicity. This is a quantitative version of an earlier result of Speiser [9], who shows that the Riemann Hypothesis (RH) that

all the complex zeros of \( \zeta(s) \) lie on the critical line \( \sigma = 1/2 \)

is equivalent to the non-vanishing of \( \zeta'(s) \) in \( 0 < \sigma < 1/2 \). The second result comes from Levinson’s famous work [5]. He has further exploited this close relationship and showed that more than one third of the zeros of \( \zeta(s) \) lie on the critical line. The third result is due to Guo [3], who has showed that there is also a close connection between the vertical distribution of the zeros of \( \zeta'(s) \) and that of \( \zeta(s) \).

The distribution of the zeros of \( \zeta'(s) \), as well as its relationship with that of the zeros of \( \zeta(s) \), has been investigated by many authors (see [1–4, 6, 8, 2000 Mathematics Subject Classification: Primary 11M06.]

[59]
In [8], Soundararajan introduced the following functions (for $a \in \mathbb{R}$):

$$m^-(a) = \liminf_{T \to \infty} \frac{N_1(T)}{T} \sum_{\beta' \leq 1/2 + a/\log T, 0 < \gamma' \leq T} 1,$$

$$m^+(a) = \limsup_{T \to \infty} \frac{N_1(T)}{T} \sum_{\beta' \leq 1/2 + a/\log T, 0 < \gamma' \leq T} 1,$$

where $N_1(T)$ is the number of zeros of $\zeta'(s)$ in $0 < t \leq T$, counted according to multiplicity. The behavior of these functions determines the horizontal distribution of the zeros of $\zeta'(s)$ near the critical line. Soundararajan [8] proved that RH implies $m^-(a) > 0$ for $a > 2.6$. He conjectured that $m^-(a) \equiv m^+(a) (= m(a))$, $m(a)$ is continuous, $m(a) > 0$ for all $a > 0$, and $m(a) \to 1$ as $a \to \infty$.

To state Zhang’s important results on $m^-(a)$, we need to explain a conjecture on small gaps between the zeros of $\zeta(s)$. Let the zeros of $\zeta(s)$ in the upper half-plane be arranged as $\rho_1, \rho_2, \ldots$ with $\rho_n = \beta_n + i\gamma_n$ and $0 < \gamma_1 \leq \gamma_2 \leq \ldots$.

If a zero is multiple with multiplicity $m$, then it appears $m$ times consecutively in the above sequence. If two zeros $\rho_{n_1} \neq \rho_{n_2}$ have the same imaginary part (this will happen if RH is not true), then $n_1 < n_2$ implies $\beta_{n_1} < \beta_{n_2}$.

For $a > 0$, define

$$D^-(a) = \liminf_{T \to \infty} N(T) \sum_{\gamma_n \leq T \gamma_n+1 - \gamma_n < a/\log T} 1,$$

where $N(T)$ denotes the number of zeros of $\zeta(s)$ in $0 < t \leq T$, counted according to multiplicity. The following conjecture is well known and is denoted by SGZ for short.

**Conjecture.** For any $a > 0$,

$$D^-(a) > 0.$$

The statement of SGZ is independent of RH.

In [11], Zhang proved the following.

**Theorem A.** If $a$ is sufficiently large, then $m^-(a) > 0$.

**Theorem B.** (Assume RH and SGZ.) For any $a > 0$, $m^-(a) > 0$.

In this note, we will show $m^-(a) > 0$ for any $a > 0$ without the assumption of RH. Namely, we prove the following.

**Theorem 1.** (Assume SGZ.) For any $a > 0$, $m^-(a) > 0$. 
2. Proof of Theorem 1. Let

\[ h(s) = \pi^{-s/2} \Gamma(s/2), \quad \eta(s) = h(s)\zeta'(s). \]

By applying Hadamard’s Theorem to \((s - 1)^2\zeta'(s)\), Zhang [11] proved the following.

**Lemma 1** ([11]). Let

\[ F(t) = -\Re \frac{\eta'}{\eta} \left( \frac{1}{2} + it \right) \]  
(2.1)

(here and hereafter \(\Re z\) denotes the real part of \(z\)) if \(\eta(1/2 + it) \neq 0\), and

\[ F(t) = \lim_{\tau \to T} F(\tau) \]  
(2.2)

if \(\eta(1/2 + it) = 0\). Then \(F(t)\) is continuous for all \(t\),

\[ F(t) = F_1(t) - F_2(t) + O(1), \]  
(2.3)

where

\[ F_1(t) = -\sum_{\beta' > 1/2} \Re \frac{1}{1/2 + it - \rho'}, \]  
(2.4)

\[ F_2(t) = \sum_{0 < \beta' < 1/2} \Re \frac{1}{1/2 + it - \rho'}, \]  
(2.5)

and the implied constant is absolute.

Now given \(T\) large and \(a > 0\) arbitrary, let

\[ F_1^*(t) = -\sum_{\beta' > 1/2 + 2a/\log T} \Re \frac{1}{1/2 + it - \rho'} \]  
(2.6)

Then

\[ 0 \leq F_1^*(t) \leq F_1(t). \]  
(2.7)

We also need the following lemmas.

**Lemma 2** ([11]). We have

\[ \int_0^T F_1^*(t) \, dt \geq \pi \sum_{\beta' > 1/2 + 2a/\log T} 1 + O(T). \]  
(2.8)

**Lemma 3.** Let the zeros of \(\zeta(s)\) in the upper half-plane be arranged as \(\rho_1, \rho_2, \ldots\) with \(\rho_n = \beta_n + i\gamma_n\) (we do not assume \(\beta_n \equiv 1/2\) here) and

\[ 0 < \gamma_1 \leq \gamma_2 \leq \cdots. \]

If a zero is multiple with multiplicity \(m\), then it appears \(m\) times consecutively in the above sequence. If two zeros \(\rho_{n_1} \neq \rho_{n_2}\) have the same imaginary
part, then \( n_1 < n_2 \) implies \( \beta_{n_1} < \beta_{n_2} \). For \( n > 1 \), define
\[
I_n = \int_{\gamma_n}^{\gamma_{n+1}} F_1^*(t) \, dt.
\]

(i) For any \( n \),
\[
I_n \leq 2\pi + \int_{\gamma_{n-1}}^{\gamma_n} F_2(t) \, dt + O(\gamma_{n+1} - \gamma_{n-1}).
\]

(ii) If \( \beta_n = 1/2 \), \( \gamma_{n+1} \leq T \) and \( \gamma_{n+1} - \gamma_n < a/\log T \), then
\[
I_n \leq \pi + 2a + \int_{\gamma_{n-1}}^{\gamma_n} F_2(t) \, dt + O(\gamma_{n+1} - \gamma_{n-1}).
\]

Proof. (i) This follows from Lemma 1, (2.7) and [11, Lemma 4].

(ii) This follows from Lemma 1, (2.7) and a slightly modified version of the proof of [11, Lemma 8].

Lemma 4 ([1]). We have
\[
N_1(T) = \frac{T}{2\pi} \left( \log \frac{T}{4\pi} - 1 \right) + O(\log T).
\]

Lemma 5. We have
\[
\int_0^T F_2(t) \, dt \leq \pi \sum_{0 < \gamma' < 1/2 \atop 0 < \beta' < 1} 1 + O(T).
\]

Proof. By Lemma 4, for \( 0 \leq t \leq T \) and \( n \geq 1 \),
\[
\sum_{0 < \gamma' < 1/2 \atop n \log T \leq |\gamma' - t| < (n+1) \log T} 1 \\
\leq 2 \log T \log(T + (n + 1) \log T) + O(\log(T + (n + 1) \log T)) \\
\leq 2 \log^2 T + 2 \log T \log n + O(\log n) + o(\log^2 T),
\]
where the implied constants are independent of \( n \). Then
\[
\sum_{0 < \gamma' < 1/2 \atop \gamma' \geq T + \log T \text{ or } \gamma' \leq - \log T} \frac{1}{1/2 + it - \gamma'}
\]
\[ \sum_{n=1}^{\infty} \sum_{n \log T \leq |\gamma' - t| < (n+1) \log T} \frac{1}{(\gamma' - t)^2} \leq \sum_{n=1}^{\infty} \frac{(2 \log^2 T + 2 \log T \log n + O(\log n) + o(\log^2 T))}{(n \log T)^2} \]

Thus

\[ F_2(t) = \sum_{0<\beta' < 1/2 \atop -\log T < \gamma' < T + \log T} \text{Re} \frac{1}{1/2 + it - \beta'} + O(1). \quad \text{(2.15)} \]

For \( \beta' < 1/2 \),

\[ \int_0^T \text{Re} \frac{1}{1/2 + it - \beta'} \, dt = \int_0^T \text{Re} \frac{1/2 - \beta'}{|1/2 + it - \beta'|^2} \, dt = \arctan \left( \frac{T - \gamma'}{1/2 - \beta'} \right) + \arctan \left( \frac{\gamma'}{1/2 - \beta'} \right) < \pi. \quad \text{(2.16)} \]

By (2.15) and (2.16),

\[ \int_0^T F_2(t) \, dt \leq \pi \sum_{0<\beta' < 1/2 \atop -\log T < \gamma' < T + \log T} 1 + O(T). \quad \text{(2.17)} \]

By Lemma 4,

\[ \sum_{0<\beta' < 1/2 \atop -\log T < \gamma' \leq 0} 1 = O(\log T \log \log T), \quad \sum_{0<\beta' < 1/2 \atop T < \gamma' < T + \log T} 1 = O(\log^2 T). \quad \text{(2.18)} \]

Combining (2.17) and (2.18), we get (2.12).

**Lemma 6 ([10, Theorem 9.4]).** We have

\[ N(T) = \frac{T}{2\pi} \left( \log \frac{T}{2\pi} - 1 \right) + O(\log T). \quad \text{(2.19)} \]

**Proof of Theorem 1.** It is no restriction to assume \( a < \pi/2 \). Let

- \( N = \max\{n : \gamma_{n+1} \leq T\} \),
- \( S_1 = \{n : 1 < n \leq N, \gamma_{n+1} - \gamma_n \geq a/\log T\} \),
- \( S_2 = \{n : 1 < n \leq N, \beta_n = 1/2, \gamma_{n+1} - \gamma_n < a/\log T\} \),
- \( S_3 = \{n : 1 < n \leq N, \beta_n \neq 1/2, \gamma_{n+1} - \gamma_n < a/\log T\} \).
On the one hand, by Lemma 3,

\begin{equation}
\sum_{n=2}^{N} I_n = \sum_{n \in S_1 \cup S_3} I_n + \sum_{n \in S_2} I_n \\
\leq 2\pi \sum_{n \in S_1 \cup S_3} 1 + \sum_{n \in S_1 \cup S_3} \int_{\gamma_n}^{\gamma_{n+1}} F_2(t) \, dt + O\left( \sum_{n \in S_1 \cup S_3} (\gamma_{n+1} - \gamma_n) \right) \\
+ (\pi + 2a) \sum_{n \in S_2} 1 + \sum_{n \in S_2} \int_{\gamma_n}^{\gamma_{n+1}} F_2(t) \, dt + O\left( \sum_{n \in S_2} (\gamma_{n+1} - \gamma_n) \right) \\
\leq 2\pi N - (\pi - 2a) \sum_{n \in S_2} 1 + \int_{0}^{\gamma_N} F_2(t) \, dt + \int_{0}^{\gamma_{N+1}} F_2(t) \, dt + O(T),
\end{equation}

where the fact that $F_2(t) \geq 0$ is used.

On the other hand, by Lemma 2,

\begin{equation}
\sum_{n=2}^{N} I_n + \int_{0}^{\gamma_1} F_1^*(t) \, dt + \int_{\gamma_N}^{\gamma_2} F_1^*(t) \, dt + \int_{\gamma_N}^{T} F_1^*(t) \, dt + \int_{\gamma_N}^{T} F_1^*(t) \, dt = 2 \int_{0}^{T} F_1^*(t) \, dt \geq 2\pi \sum_{\beta' > 1/2 + 2a/\log T} 1 + O(T).
\end{equation}

Lemma 6 implies

$$\gamma_{n+1} - \gamma_n = O(1);$$

then by (2.7) and Lemma 3, we have

\begin{equation}
\int_{\gamma_N}^{\gamma_{N+2}} F_1^*(t) \, dt \leq \int_{\gamma_N}^{\gamma_{N+2}} F_1^*(t) \, dt = I_{N+1} \\
\leq 2\pi + \int_{\gamma_N}^{\gamma_{N+2}} F_2(t) \, dt + O(\gamma_{N+2} - \gamma_N) = \int_{\gamma_N}^{\gamma_{N+2}} F_2(t) \, dt + O(1).
\end{equation}

By Lemma 1, (2.7) and [11, Lemma 4],

\begin{equation}
\int_{\gamma_{N+1}}^{T} F_1^*(t) \, dt \leq \int_{\gamma_{N+1}}^{T} F_1^*(t) \, dt \leq \int_{\gamma_{N+1}}^{T} F_1(t) \, dt \\
\leq \pi + \int_{\gamma_{N+1}}^{\gamma_{N+2}} F_2(t) \, dt + O(\gamma_{N+2} - \gamma_{N+1}) = \int_{\gamma_{N+1}}^{\gamma_{N+2}} F_2(t) \, dt + O(1).
\end{equation}
By (2.7),
\[ (2.24) \quad \gamma_1 \int_0^{\gamma_1} F_1^*(t) \, dt \leq \int_0^{\gamma_1} F_1(t) \, dt = O(1), \]
\[ (2.25) \quad \gamma_2 \int_0^{\gamma_2} F_1^*(t) \, dt \leq \int_0^{\gamma_2} F_1(t) \, dt = O(1), \]
where the implied constant is absolute. Combining (2.21)–(2.25), we have
\[ (2.26) \quad \sum_{n=2}^{N} I_n \geq 2\pi \sum_{\beta' > 1/2 + 2a/\log T}^{\gamma_{N+2}} 1 - \int_{\gamma_N}^{\gamma_{N+2}} F_2(t) \, dt - \int_{\gamma_{N+1}}^{\gamma_{N+2}} F_2(t) \, dt + O(T). \]

It follows from Lemmas 4 and 6 that
\[ N = N_1(T) + O(T). \]

Then combining (2.20) and (2.26) we obtain
\[ (2.27) \quad \sum_{\beta' \leq 1/2 + 2a/\log T}^{\gamma_{N+2}} 1 \geq \left( \frac{1}{2} - \frac{a}{\pi} \right) \sum_{n \in S_2} 1 - \frac{1}{\pi} \int_0^{\gamma_{N+2}} F_2(t) \, dt + O(T). \]

By Lemma 5, Lemma 4 and the fact that \( \gamma_{N+2} - T = O(1) \),
\[ (2.28) \quad \int_0^{\gamma_{N+2}} F_2(t) \, dt \leq \pi \sum_{0 < \beta' < 1/2} \sum_{0 < \gamma' \leq \gamma_{N+2}} 1 + O(\gamma_{N+2}) \]
\[ = \pi \sum_{0 < \beta' < 1/2} \sum_{0 < \gamma' \leq T} 1 + \pi \sum_{0 < \beta' < 1/2} \sum_{T < \gamma' \leq \gamma_{N+2}} 1 + O(T) \]
\[ = \pi \sum_{0 < \beta' < 1/2} \sum_{0 < \gamma' \leq T} 1 + O(T). \]

Now (2.27) and (2.28) imply
\[ (2.29) \quad 2 \sum_{\beta' \leq 1/2 + 2a/\log T}^{\gamma_{N+2}} 1 \geq \sum_{\beta' \leq 1/2 + 2a/\log T}^{\gamma_{N+2}} 1 + \sum_{0 < \beta' < 1/2} \sum_{0 < \gamma' \leq T} 1 + O(T). \]

By (1.5) and Lemma 6,
\[ (2.30) \quad \sum_{n \in S_2} 1 + \sum_{n \in S_3} 1 \geq D^-(a) \frac{T \log T}{2\pi} + o(T \log T). \]
By (1.1),
\[ (2.31) \sum_{n \in S_3} 1 \leq 2 \sum_{0 < \beta < 1/2} \sum_{0 < \gamma \leq T} 1 + O(\log T). \]
Therefore
\[ (2.32) \sum_{n \in S_2} 1 \geq D-a \frac{T \log T}{2\pi} - 2 \sum_{0 < \beta' < 1/2} \sum_{0 < \gamma' \leq T} 1 + o(T \log T) \]
\[ \geq D-a \frac{T \log T}{2\pi} - 2 \sum_{\beta' \leq 1/2 + o T} \sum_{0 < \gamma' \leq T} 1 + o(T \log T). \]
Combining (2.29) and (2.32), we have
\[ (2.33) \sum_{\beta' \leq 1/2 + 2a/\log T} \sum_{0 < \gamma' \leq T} 1 \geq \pi - 2a \frac{T \log T}{6\pi + 4a} + o(T \log T). \]
That is,
\[ (2.34) m-a \geq \pi - 2a \frac{T \log T}{6\pi + 4a} > 0. \]
The proof is complete. 

3. Remark. In Levinson’s well known work [5], he proved that more than one third of the zeros of \( \zeta(s) \) lie on the critical line by using the relationship between \( N^{-1}(T) \) and \( N^{-1}_{\text{I}}(T) \). We outline the principle behind his proof here. Let \( g(s) \) be the function \( \zeta'(1-s) \) and \( \sigma = 1/2 - a/\log T \), where \( a > 0 \) is a small positive real number. Note that \( g^* = \beta^* + i\gamma^* \) is a zero of \( g(s) \) if and only if \( g' = 1 - \beta^* + i\gamma^* \) is a zero of \( \zeta'(s) \). If one can show that
\[ (3.1) \sum_{\sigma < \beta^* \leq 2/\log T} \sum_{0 < \gamma^* \leq T} (\beta^* - \sigma) \leq C\sigma T + o(T), \]
where \( C\sigma \) is a constant depending on \( a \), then since
\[ (3.2) \sum_{\sigma < \beta^* \leq 2/\log T} \sum_{0 < \gamma^* \leq T} 1 \geq \left( \frac{1}{2} - \sigma \right) \sum_{\sigma < \beta^* \leq 2/\log T} \sum_{0 < \gamma^* \leq T} 1, \]
one has
\[ (3.3) \sum_{1/2 < \beta^* \leq 2/\log T} \sum_{0 < \gamma^* \leq T} 1 \leq \frac{C\sigma}{a} T \log T + o(T \log T). \]
Then by (1.1),

\[
\sum_{0 < \beta < 1/2} 1 \leq \frac{C_a}{a} T \log T + o(T \log T).
\]

Therefore the proportion of the zeros of \( \zeta(s) \) on the critical line is more than

\[
1 - \frac{4\pi C_a}{a}.
\]

By estimating \( C_a \) carefully (applying the Littlewood Theorem [7, 10]) and choosing \( a \) suitably, Levinson proved the proportion is more than 1/3. This result can be improved slightly by having a better estimate for \( C_a \).

But we can see in (3.2) there is some loss in the argument. In the process of obtaining an upper bound for the number of zeros of \( \zeta'(s) \) with \( \beta' < 1/2 \), one has also counted those zeros satisfying

\[
\frac{1}{2} \leq \beta' < 1/2 + \frac{a}{\log T}, \quad 0 < \gamma' \leq T
\]

(with the weight \( 1/2 + a/\log T - \beta' \)). Theorem 1 shows that on the SGZ, there is a positive proportion of the zeros of \( \zeta'(s) \) satisfying (3.5) or

\[
\beta' < 1/2, \quad 0 < \gamma' \leq T.
\]

Thus, if SGZ is valid, no matter how precisely \( C_a \) are estimated and what \( a \) is chosen, the framework of [5] cannot prove that 100\% of the zeros of \( \zeta(s) \) lie on the critical line (although it is likely true).

However, once a good (large) lower estimate of \( m^-(a) \) is obtained, the result of [5] can be significantly improved by combining this estimate and (3.1). We can see in the statement and proof of Theorem 1 that the lower bound for \( m^-(a) \) is closely connected with the vertical distribution of the zeros of the Riemann zeta function.

References


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