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Best simultaneous diophantine approximations of Pisot numbers and Rauzy fractals

by

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1. Introduction. Let v be an element of $\mathbb{R}^2$, $\mathcal{N}$ be a norm on $\mathbb{R}^2$ and q an integer. We define

$$\mathcal{N}_0(qv) = \min_{(p,r)\in\mathbb{Z}^2} \{\mathcal{N}(qv - (p,r))\}.$$

Let $(q_n)_{n\geq 0}$ be a nondecreasing sequence of integers. We say that $(q_n)_{n\geq 0}$ is the *sequence of best approximations* of v for the norm $\mathcal{N}$ if for all $n \in \mathbb{Z}^+$ and for all $0 < q < q_n$, $\mathcal{N}_0(q_n v) < \mathcal{N}_0(qv)$.

This sequence heavily depends on the norm $\mathcal{N}$ and there is no efficient algorithm which provides the sequence of best approximations for all elements of $\mathbb{R}^2$. To be more precise, in the general case the sequence of best approximations is not given by any multidimensional continued fractions algorithm. It is hopeless to obtain the sequence of best approximations by such an algorithm for all real numbers (see [La82]). Nevertheless one can investigate this question for specific examples.

The situation is better understood if the coordinates of v belong to a Pisot cubic field. There are fundamental tools which are available in this setting: numeration systems and fractal geometry.

The first result in this direction was obtained in [CHM01]. Let β be the dominant root of the polynomial $X^3 - X^2 - X - 1$. There exists a norm (called the *Rauzy norm*) such that the Tribonacci sequence given by

$$F_0 = 1, \quad F_1 = 1, \quad F_2 = 2,$$
$$F_{n+3} = F_{n+2} + F_{n+1} + F_n \quad \forall n \geq 0$$

is the sequence of best approximations for the vector $(1/\beta, 1/\beta^2)$.

Chevallier (see [Ch99]) obtained generalizations of the previous result applying a very different method. His method works for a certain class of Pisot numbers of degree 3 with complex conjugates.

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In the present paper, we also generalize results of [CHM01] to a class of Pisot units of degree 3 with complex conjugates.

To each real number $\beta > 1$ is naturally associated a numeration system, the β-expansion, which has been extensively studied (see [Pa60]). This β-expansion induces a β-numeration system (a sequence of integer numbers). Any integer has a unique expansion in this numeration system.

The Rauzy fractal is defined by means of a β-expansion (see [Ra82], [Ak99]) and the understanding of the geometry of this fractal is closely related to the arithmetic properties of Pisot numbers.

A Pisot number is said to have *Property* (F) if the β-expansion of every nonnegative element of $\mathbb{Z}[\beta]$ is finite. Pisot units of degree 3 satisfying Property (F) are classified by Akiyama (see [Ak00]). These numbers are exactly the dominant roots of the polynomials with integer coefficients

$$x^3 - ax^2 - bx - 1, \qquad a \geq 0,\ -1 \leq b \leq a+1,$$

where for $b = -1$ we add the restriction $a \geq 2$.

We prove the following result of best simultaneous diophantine approximation using Rauzy fractals:

THEOREM 1. *Let β be a non-totally real cubic Pisot unit satisfying the equation $X^3 = aX^2 + bX + 1$; let $(R_n)_{n\geq 0}$ be the sequence of integers defined by*

$$R_0 = 1, \qquad R_1 = a, \qquad R_2 = a^2 + b,$$
$$R_{n+3} = aR_{n+2} + bR_{n+1} + R_n \qquad \forall n \geq 0.$$

If β has Property (F), *then there exists a norm $\mathcal{N}$ on $\mathbb{R}^2$* (*called a Rauzy norm*) *and an integer n_0 such that $(R_n)_{n\geq n_0}$ is the sequence of best approximations of the vector $(1/\beta, 1/\beta^2)$ for the norm $\mathcal{N}$.*

This result is an extension of a result of Chevallier [Ch99]. Our assumption is much weaker and has an arithmetic meaning.

We also prove a negative result when the conjugates of β are real numbers. The geometric situation is more complicated: the Rauzy fractal is self-affine but not self similar. Theorem 2 (see Section 2) states that the behaviors of the sequences of best approximations are also very different: there is no simple generalization of Theorem 1.

2. Background

2.1. *Numeration systems*

β-expansion. Let $\beta > 1$ be a real number. A *β-representation* of a nonnegative real number x is an infinite sequence $(x_i)_{i\leq k}$, $x_i \in \mathbb{Z}^+$, such that

$$x = x_k\beta^k + x_{k-1}\beta^{k-1} + \cdots + x_1\beta + x_0 + x_{-1}\beta^{-1} + x_{-2}\beta^{-2} + \cdots$$

where k is a nonnegative integer. It is denoted by

$$x = x_k x_{k-1} \dots x_1 x_0 . x_{-1} x_{-2} \dots .$$

A particular β-representation, called the β-*expansion*, can be computed by the "greedy algorithm" (see [Pa60] and [F92]). Denote by $\lfloor y \rfloor$ and $\{y\}$ respectively the integer and fractional parts of a number y. For every real $x \geq 0$ there exists $k \in \mathbb{Z}$ such that $\beta^k \leq x < \beta^{k+1}$. Let $x_k = \lfloor x/\beta^k \rfloor$ and $r_k = \{x/\beta^k\}$. Then for $i < k$, put $x_i = \lfloor \beta r_{i+1} \rfloor$ and $r_i = \{\beta r_{i+1}\}$. We get

$$x = x_k \beta^k + x_{k-1} \beta^{k-1} + \cdots .$$

If $k < 0$ ($x < 1$), we put $x_0 = x_{-1} = \cdots = x_{k+1} = 0$. If an expansion ends with infinitely many zeros, it is said to be finite, and the final zeros are omitted.

The digits x_i belong to the set $A = \{0, \dots, \beta - 1\}$ if β is an integer, or to $A = \{0, \dots, \lfloor \beta \rfloor\}$ if β is not an integer. The β-expansion of every positive real x is the lexicographically greatest among all β-representations of x. We will sometimes omit the splitting point between the integer part and the fractional part of the β-expansion; then the infinite sequence is just an element of $A^{\mathbb{N}}$.

For numbers $0 \leq x < 1$, the expansion defined above coincides with the β-expansion of Rényi [Re57], which can be defined by means of the β transformation of the unit interval

$$T_\beta(x) = \{\beta x\}, \qquad x \in [0, 1].$$

For $x \in [0, 1[$, we have $x_k = \lfloor \beta T_\beta^{k-1}(x) \rfloor$. However, for $x = 1$ the two algorithms differ. The greedy algorithm expansion of 1 is just $1 = 1.0000\dots$, while the Rényi expansion of 1 is

$$d(1, \beta) = .t_{-1} t_{-2} \dots, \quad \text{where} \quad t_{-k} = \lfloor \beta T_\beta^{k-1}(1) \rfloor, \quad \forall k \geq 1.$$

Pisot numbers and Property (F). We denote by $\mathrm{Fin}(\beta)$ the set of numbers which have finite greedy β-expansion. For $N \in \mathbb{Z}$, we denote by $\mathrm{Fin}_N(\beta)$ the set of numbers x such that in their β-expansion, $x_k = 0$ for all $k < N$. We will sometimes denote a β-expansion $x_n \dots x_k$, $n \geq k$, by $(x_i)_{n \geq i \geq k}$. We put

$$E_\beta = \{(x_i)_{i \geq k} \mid k \in \mathbb{Z} \text{ and } (x_i)_{n \geq i \geq k} \text{ is a finite } \beta\text{-expansion for all } n \geq k\}.$$

We say that β has *Property* (F) if

$$\mathbb{Z}[\beta] \cap [0, \infty] \subset \mathrm{Fin}(\beta).$$

In what follows, we will assume that β is a *Pisot number*. This means that β is a real algebraic integer larger than 1 with its Galois conjugates of modulus strictly less than 1.

It is known (see [Ak00]) that the class of Pisot cubic units which have Property (F) is equal to the class of real numbers that are exactly the dominant roots of the following polynomials with integer coefficients:

$$x^3 - ax^2 - bx - 1, \qquad a \geq 0,\ -1 \leq b \leq a+1.$$

(If $b = -1$ we add the restriction $a \geq 2$.)

The expansion of 1 with respect to these numbers is given by the following proposition given by Akiyama (see [Ak00]).

PROPOSITION 1. *Let β be a Pisot cubic unit number which has Property* (F) *and with minimal polynomial $x^3 - ax^2 - bx - 1$. Then*

$$d(1,\beta) = \begin{cases} .ab1 & \text{if } 0 \leq b \leq a, \\ .(a-1)(a-1)01 & \text{if } b = -1 \text{ and } a \geq 2, \\ .(a+1)00a1 & \text{if } b = a+1. \end{cases}$$

LEMMA 1. *Let $x_n \dots x_0$ and $y_m \dots y_0$ be two β-expansions. Then*

$$\sum_{i=0}^{n} x_i \beta^i < \sum_{i=0}^{m} y_i \beta^i \ \Leftrightarrow\ x_n \dots x_0 <_{\mathrm{lex}} y_m \dots y_0$$

where $<_{\mathrm{lex}}$ is the lexicographical order.

Proof. The proof is classical (see [Pa60]).

In the following, we assume that β is a Pisot cubic unit which has Property (F) and denote by α, γ its Galois conjugates. Let $P(x) = x^3 - ax^2 - bx - 1$ be the minimal polynomial of β.

Numeration systems on integers. We describe the numeration systems on integers induced by β-expansions.

Assume that $d(1,\beta) = .a_{-1} \dots a_{-t}$ where $a_{-t} \neq 0$. Let $(R_n)_{n \geq 0}$ be the integer sequence defined by

$$R_0 = 1, \qquad R_1 = a, \qquad R_2 = a^2 + b,$$
$$R_{n+3} = aR_{n+2} + bR_{n+1} + R_n \qquad \forall n \geq 0.$$

Since the polynomial $Q(x) = x^t - a_{-1}x^{t-1} - \dots - a_{-t}$ is a multiple of the polynomial $x^3 - ax^2 - bx - 1$, the sequence $(R_n)_{n \geq 0}$ satisfies

$$R_{n+t} = a_{-1}R_{n+t-1} + a_{-2}R_{n+t-2} + \dots + a_{-t}R_n \qquad \forall n \geq 0.$$

PROPOSITION 2. *Every nonnegative integer n has a unique R-ary digital expansion $(\varepsilon_j)_{k(n) \geq j \geq 0}$ such that $n = \sum_{j=0}^{k(n)} \varepsilon_j R_j$ with $\varepsilon_j \geq 0$. In particular*:

1. *If $a \geq b \geq 0$ and $a \neq 0$ then $\varepsilon_i \varepsilon_{i-1} \varepsilon_{i-2} <_{\mathrm{lex}} ab1$ for all $i \geq 2$, $\varepsilon_1 \varepsilon_0 <_{\mathrm{lex}} ab$, $\varepsilon_0 < a$.*
2. *If $b = -1$ and $a \geq 2$ then $\varepsilon_i \varepsilon_{i-1} \varepsilon_{i-2} \varepsilon_{i-3} <_{\mathrm{lex}} (a-1)(a-1)01$ for all $i \geq 3$, $\varepsilon_2 \varepsilon_1 \varepsilon_0 <_{\mathrm{lex}} (a-1)(a-1)0$, $\varepsilon_1 \varepsilon_0 <_{\mathrm{lex}} (a-1)(a-1)$, $\varepsilon_0 < a$.*
3. *If $b = a+1$ and $a \geq 1$ then $\varepsilon_i \varepsilon_{i-1} \varepsilon_{i-2} \varepsilon_{i-3} \varepsilon_{i-4} <_{\mathrm{lex}} (a+1)00a1$ for all $i \geq 4$, $\varepsilon_3 \varepsilon_2 \varepsilon_1 \varepsilon_0 <_{\mathrm{lex}} (a+1)00a$, $\varepsilon_2 \varepsilon_1 \varepsilon_0 <_{\mathrm{lex}} (a+1)00$, $\varepsilon_1 \varepsilon_0 <_{\mathrm{lex}} (a+1)1$, $\varepsilon_0 < a$.*

4. *If* $a = 0$ *and* $b = 1$ *then* $\varepsilon_i\varepsilon_{i-1}\varepsilon_{i-2}\varepsilon_{i-3}\varepsilon_{i-4} <_{\text{lex}} 10001$ *for all* $i \geq 4$ *and* $\varepsilon_0 = \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_5 = 0$.

Proof. The digits $(\varepsilon_j)_{k(n)\geq j\geq 0}$ are obtained by using the greedy algorithm for integers.

1. If $a \geq b \geq 0$ and $a \neq 0$, then $(R_n)_{n\geq 0}$ is a nonincreasing sequence of positive integers. Hence by the definition of the greedy algorithm it is easy to see ([Pa60]) that $\sum_{i=0}^{j} \varepsilon_i R_i < R_{j+1}$ for $0 \leq j \leq k(n)$. We deduce that

$$\varepsilon_j\varepsilon_{j-1}\ldots\varepsilon_{j-t+1} <_{\text{lex}} a_{-1}a_{-2}\ldots a_{-t} \qquad \forall j \geq t-1. \tag{1}$$

By (1) and Proposition 1, we have $\varepsilon_i\varepsilon_{i-1}\varepsilon_{i-2} <_{\text{lex}} ab1$. Since $n-\sum_{i=1}^{k(n)} \varepsilon_i R_i = \varepsilon_0 R_0 = \varepsilon_0 < R_1 = a$ and $n - \sum_{i=2}^{k(n)} \varepsilon_i R_i = \varepsilon_1 R_1 + \varepsilon_0 R_0 < R_2 = a^2 + b = aR_1 + bR_0$, we have $\varepsilon_0 < a$ and $\varepsilon_1\varepsilon_0 <_{\text{lex}} ab$.

2. If $b = -1$ and $a \geq 2$, then by using the same argument as in case 1, we find that $\varepsilon_i\varepsilon_{i-1}\varepsilon_{i-2}\varepsilon_{i-3} <_{\text{lex}} (a-1)(a-1)01$ for all $i \geq 3$. The other conditions follow from $\varepsilon_0 R_0 = \varepsilon_0 < R_1 = a$, $\varepsilon_1 R_1 + \varepsilon_0 R_0 < R_2 = (a-1)R_1 + (a-1)R_0$ and $\varepsilon_2 R_2 + \varepsilon_1 R_1 + \varepsilon_0 R_0 < R_3 = (a-1)R_2 + (a-1)R_1$.

3. If $b = a + 1$ and $a \geq 1$ then by the same argument as in case 1, we deduce that $\varepsilon_i\varepsilon_{i-1}\varepsilon_{i-2}\varepsilon_{i-3}\varepsilon_{i-4} <_{\text{lex}} (a+1)00a1$ for all $i \geq 4$. The other conditions follow from $R_1 = aR_0$, $R_2 = (a+1)R_1 + R_0$, $R_3 = (a+1)R_2$ and $R_4 = (a+1)R_3 + aR_0$.

4. If $a = 0$ and $b = 1$ then relation (1) and Proposition 1 imply that $\varepsilon_i\varepsilon_{i-1}\varepsilon_{i-2}\varepsilon_{i-3}\varepsilon_{i-4} <_{\text{lex}} 10001$ for all $i \geq 4$. On the other hand, $R_0 = R_2 = R_3 = R_4 = 1$, $R_1 = 0$, $R_5 = R_6 = 2$. Hence $\varepsilon_0 = \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_5 = 0$.

LEMMA 2. *For all* $n \in \mathbb{Z}^+$ *we have*

$$\beta^n = R_n + (bR_{n-1} + R_{n-2})/\beta + R_{n-1}/\beta^2.$$

Proof. The proof, left to the reader, is by induction on n.

Let $(\varepsilon_j)_{k(n)\geq j\geq 0}$ be an R-ary digital expansion of an integer $n > 0$. Let $Z = \{j \mid \varepsilon_j \neq 0\}$, $m = \min(Z)$ and $M = \max(Z)$. The number $M - m + 1$ is called the *length* of the R-ary digital expansion $(\varepsilon_j)_{k(n)\geq j\geq 0}$.

Now, we are able to state Theorem 2.

THEOREM 2. *Let* β *be a totally real cubic Pisot unit with Property* (F); *let* $(G_n)_{n\geq 0}$ *be the sequence of best approximations of the vector* $(1/\beta, 1/\beta^2)$ *for the Rauzy norm and let* $t(n)$ *be the length of the* R*-ary digital expansion of* G_n. *Then* $t(n) \to \infty$ *as* $n \to \infty$.

2.2. *Rauzy fractal.* Let $N \in \mathbb{Z}^+$ and $(\varepsilon_j)_{k(N)\geq j\geq 0}$ be its R-ary expansion. We put

$$\delta(N) = N(1/\beta, 1/\beta^2) - (P_N, Q_N),$$

where

$$P_N = \sum_{j=1}^{k(N)} \varepsilon_j R_{j-1}, \qquad Q_N = \sum_{j=2}^{k(N)} \varepsilon_j R_{j-2}.$$

Let

$$B = \begin{pmatrix} -b/\beta & -1/\beta \\ 1 - b/\beta^2 & -1/\beta^2 \end{pmatrix}.$$

LEMMA 3.

$$B \begin{pmatrix} R_n/\beta - R_{n-1} \\ R_n/\beta^2 - R_{n-2} \end{pmatrix} = \begin{pmatrix} R_{n+1}/\beta - R_n \\ R_{n+1}/\beta^2 - R_{n-1} \end{pmatrix} \qquad \forall n \geq 0,$$

where $(R_n)_{n\geq 0}$ *is the sequence defined before and* $R_{-1} = R_{-2} = 0$.

Proof. By induction on n.

COROLLARY 1. *If* $N = \sum_{j=0}^{k(N)} \varepsilon_j R_j$ *then* $\delta(N) = \sum_{j=0}^{k(N)} \varepsilon_j B^j \delta(1)$.

Consider the set

$$\mathcal{E}' = \overline{\{\delta(N) \mid N \in \mathbb{Z}^+\}}.$$

It is a subset of $\mathbb{R}^2$ called the *Rauzy fractal*. It was introduced by G. Rauzy in 1982 (see [Ra82]) in the case of the polynomial $x^3 - x^2 - x - 1$. The Rauzy fractal has been studied by many mathematicians and is related to many mathematical areas, including dynamical systems and number theory.

Another way to define the Rauzy fractal is as follows. We recall that

$$.a_{-1}a_{-2}\ldots a_{-t}$$

is the Rényi expansion of 1. We denote by E'_β the set of $(\varepsilon_j)_{j\geq 0}$ such that for all $k \in \mathbb{Z}^+$, $(\varepsilon_j)_{k\geq j\geq 0}$ is the R-ary expansion of the number $\sum_{j=0}^{k} \varepsilon_j R_j$.

We recall that in this case the set E'_β is contained in the set E_β of β-expansions given by the following conditions on $(\varepsilon_j)_{j\geq 0}$ (see [Pa60]):

$$\varepsilon_j\varepsilon_{j-1}\ldots\varepsilon_{j-t+1} <_{\text{lex}} a_{-1}a_{-2}\ldots a_{-t} \qquad \forall j \geq t-1$$

and

$$\varepsilon_j\ldots\varepsilon_0 \leq_{\text{lex}} a_{-1}a_{-2}\ldots a_{-j-1} \qquad \forall 0 \leq j < t-1.$$

Applying Corollary 1, it is easy to show that

$$\mathcal{E}' = \Big\{ \sum_{j=0}^{\infty} \varepsilon_j B^j \delta(1) \;\Big|\; (\varepsilon_j)_{j\geq 0} \in E'_\beta \Big\}.$$

Now, set $\mathcal{E} = \{\sum_{j=0}^{\infty} \varepsilon_j B^j \delta(1) \mid (\varepsilon_j)_{j\geq 0} \in E_\beta\}$. Since $E'_\beta \subset E_\beta$ and $\ldots\varepsilon_2\varepsilon_1\varepsilon_0 000000 \in E'_\beta$ for all $(\varepsilon_j)_{j\geq 0} \in E_\beta$ (see Proposition 2), we have

$$B^6 \mathcal{E} \subset \mathcal{E}' \subset \mathcal{E}.$$

Independently, Akiyama and Praggastis studied some topological properties of the set $\mathcal{E}$. In particular they proved that 0 is an interior point of $\mathcal{E}$ (see [AS98], [Ak02], [Pr92]). Since $B^6\mathcal{E} \subset \mathcal{E}'$, we deduce:

PROPOSITION 3. *Let β be a Pisot unit of degree 3 and $\mathcal{E}'$ its Rauzy fractal. Then 0 is an interior point of $\mathcal{E}'$.*

Proposition 3 is important for the proof of Theorem 1.

REMARK 2.1. In the present paper, the natural object is $\mathcal{E}'$ and not $\mathcal{E}$.

2.3. *Rauzy norm.* Let

$$M = \begin{pmatrix} \gamma + b/\beta & 1/\beta \\ -\alpha - b/\beta & -1/\beta \end{pmatrix}.$$

One can check that the matrix B (defined in 2.2) is similar to the matrix $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \gamma \end{smallmatrix}\right)$ and satisfies

$$MB = \begin{pmatrix} \alpha & 0 \\ 0 & \gamma \end{pmatrix} M. \tag{2}$$

CASE 1: *β is a non-totally real Pisot number.* The Rauzy norm $\mathcal{N}$ is defined by

$$\mathcal{N}(x) = |(\alpha + b/\beta)x_1 + x_2/\beta|, \qquad \forall x = (x_1, x_2) \in \mathbb{R}^2. \tag{3}$$

It is easy to check by using (2) that $\mathcal{N}(Bx) = |\alpha|\mathcal{N}(x)$ for all $x \in \mathbb{R}^2$.

LEMMA 4. *Let $(\varepsilon_n)_{n\geq 0}$ be a β-expansion. Then*

$$\mathcal{N}\Big(\sum_{n=0}^{\infty} \varepsilon_n B^n \delta(1)\Big) = \mathcal{N}(\delta(1))\Big|\sum_{n=0}^{\infty} \varepsilon_n \alpha^n\Big|.$$

In particular, $\mathcal{N}(\delta(R_n)) = \mathcal{N}(\delta(1))|\alpha^n|$ for all $n \geq 0$.

Proof. Immediate, using (2) and (3).

CASE 2: *β is a totally real Pisot number.* The Rauzy norm $\mathcal{N}$ is defined as the euclidean norm associated to the vector Mx for all x in $\mathbb{R}^2$.

LEMMA 5. *Let $(\varepsilon_n)_{n\geq 0}$ be a β-expansion. Then*

$$\mathcal{N}\Big(\sum_{n\geq 0} \varepsilon_n B^n \delta(1)\Big) = \sqrt{x^2\Big(\sum_{n=0}^{\infty} \varepsilon_n \alpha^n\Big)^2 + y^2\Big(\sum_{n=0}^{\infty} \varepsilon_n \gamma^n\Big)^2}$$

where x, y are the coordinates of the vector $M\delta(1)$.

Proof. The proof is a simple calculation.

3. Proof of Theorem 1. We give the proof of Theorem 1 in the case $a \geq b \geq 0$. It is straightforward to adapt it to the other two cases.

THEOREM 3. *There exists a real $c > 0$ such that for all $q \in \mathbb{Z}^+$ and all $g \in \mathbb{Z}^2$, $\mathcal{N}(q(1/\beta, 1/\beta^2) - g) < c$ implies $q(1/\beta, 1/\beta^2) - g = \delta(q)$.*

As a corollary, we have:

COROLLARY 2. *There exists a real $c > 0$ such that for all $q \in \mathbb{Z}^+$,*

$$\text{if} \quad \mathcal{N}_0(q(1/\beta, 1/\beta^2)) < c \quad \text{then} \quad \mathcal{N}_0(q(1/\beta, 1/\beta^2)) = \mathcal{N}(\delta(q)).$$

REMARK 3.1. Theorem 3 is also true in the case where β is totally real. It is of independent interest and an important step in the proof of Theorem 1.

For the proof of Theorem 3, we need some propositions and lemmas. We put

$$\mathcal{K}' = \Big\{ \sum_{i=0}^{\infty} \varepsilon_i \alpha^i \ \Big|\ (\varepsilon_i)_{i\geq 0} \in E'_\beta \Big\}, \qquad \mathcal{K} = \Big\{ \sum_{i=0}^{\infty} \varepsilon_i \alpha^i \ \Big|\ (\varepsilon_i)_{i\geq 0} \in E_\beta \Big\}.$$

It is clear that $\mathcal{K}' \subset \mathcal{K}$.

PROPOSITION 4. *There exists a linear bijection f from $\mathbb{R}^2$ to $\mathbb{C}$ such that $f(\mathcal{E}) = \mathcal{K}$, $f(\mathcal{E}') = \mathcal{K}'$ and $f(\mathbb{Z}^2) = \alpha^{-1}\mathbb{Z} + \alpha^{-2}\mathbb{Z}$.*

Proof. There exist $h, l \in \mathbb{C}$ such that $R_n = h\beta^n + l\alpha^n + \overline{l}\,\overline{\alpha}^n$ for all $n \in \mathbb{Z}^+$. Then, for all $n \in \mathbb{N}$,

$$\delta(R_n) = \begin{pmatrix} R_n/\beta - R_{n-1} \\ R_n/\beta^2 - R_{n-2} \end{pmatrix} = \begin{pmatrix} c\alpha^n + \overline{c}\,\overline{\alpha^n} \\ d\alpha^n + \overline{d}\,\overline{\alpha^n} \end{pmatrix} \tag{4}$$

where $c = l(1/\beta - 1/\alpha)$ and $d = l(1/\beta^2 - 1/\alpha^2)$. One can show by solving the system given by the initial values of R_0, R_1 and R_2 that

$$l = \frac{\alpha^2}{(\alpha - \overline{\alpha})(\alpha - \beta)}.$$

Thus

$$c\overline{d} - d\overline{c} = \frac{\alpha^2 \overline{\alpha^2}}{\overline{\alpha} - \alpha} \neq 0.$$

Therefore if we put

$$g(z) = \begin{pmatrix} cz + \overline{c}\,\overline{z} \\ dz + \overline{d}\,\overline{z} \end{pmatrix} \quad \text{for all } z \in \mathbb{C} \quad \text{and} \quad f = g^{-1},$$

we obtain $f(\mathcal{E}) = \mathcal{K}$ and $f(\mathcal{E}') = \mathcal{K}'$. On the other hand we can prove that $g(\alpha^{-1}) = \binom{0}{-1}$ and $g(\alpha^{-2}) = \binom{-1}{b}$. Thus $\alpha^{-1}\mathbb{Z} + \alpha^{-2}\mathbb{Z} = f(\mathbb{Z}^2)$.

REMARK 3.2. When β is a totally real Pisot number, we have a linear bijection f from $\mathbb{R}^2$ to $\mathbb{R}^2$ such that $f(\mathcal{E}) = \mathcal{K}$, $f(\mathcal{E}') = \mathcal{K}'$ and

$f(\mathbb{Z}^2) = \theta^{-1}\mathbb{Z} + \theta^{-2}\mathbb{Z}$, where $\mathcal{K}' = \{\sum_{i=0}^{\infty} \varepsilon_i \theta^i \mid (\varepsilon_i)_{i\geq 0} \in E'_\beta\}$, $\mathcal{K} = \{\sum_{i=0}^{\infty} \varepsilon_i \theta^i \mid (\varepsilon_i)_{i\geq 0} \in E_\beta\}$, and $\theta^i = (\alpha^i, \gamma^i)$ for all $i \geq -2$.

By Proposition 4 and the fact that 0 is an interior point of $\mathcal{E}'$ (see 2.2), we deduce the following result.

PROPOSITION 5. *The point* 0 *is an interior point of* $\mathcal{K}'$.

The next step is:

PROPOSITION 6. $\mathcal{K}' \cap (\alpha^{-1}\mathbb{Z} + \alpha^{-2}\mathbb{Z}) = \{0\}$.

The proof of Proposition 6 is postponed to Section 5.

REMARK 3.3. For $\mathcal{K}$, the last statement is not true since $-\alpha^{-1} \in \mathcal{K}$.

Proof of Theorem 3. By Propositions 4 and 6, we deduce that $\mathcal{E}' \cap \mathbb{Z}^2 = \{0\}$. Therefore there exists a real number $c > 0$ such that for all $g \in \mathbb{Z}^2$, $\inf_{x\in\mathcal{E}'} \mathcal{N}(g-x) < c \Rightarrow g = 0$. Assume that $\mathcal{N}(q(1/\beta, 1/\beta^2) - g) < c$. Since $\delta(q) - q(1/\beta, 1/\beta^2) \in \mathbb{Z}^2$ and $\delta(q) \in \mathcal{E}'$, it follows that $\mathcal{N}(\delta(q)) - (\delta(q) - q(1/\beta, 1/\beta^2) - g) < c$. Hence $\delta(q) - q(1/\beta, 1/\beta^2) + g = 0$, which completes the proof of Theorem 3.

PROPOSITION 7. *Let* $n \in \mathbb{N}$ *and* $0 < q < R_n$. *Then* $\mathcal{N}(\delta(R_n)) < \mathcal{N}(\delta(q))$.

Proof. It is sufficient to prove the assertion for $R_{n-1} \leq q < R_n$. Assume that $q = \sum_{k=0}^{n-1} \varepsilon_k R_k$ where $(\varepsilon_k)_{n-1\geq k\geq 0} \in E'_\beta$ and $\varepsilon_{n-1} \neq 0$. By Proposition 4, $\mathcal{N}(\delta(R_n)) < \mathcal{N}(\delta(q))$ is equivalent to $|\alpha^n| < |\sum_{k=0}^{n-1} \varepsilon_k \alpha^k|$.

On the other hand, let $x_{n-1} = \sum_{k=0}^{n-1} \varepsilon_k \alpha^k$. The number x_{n-1} is an algebraic integer. Then if we put $z_k = \sum_{k=0}^{n-1} \varepsilon_k \beta^k$, we have $|x_k|^2 z_k \in \mathbb{Z}$. Since z_k and x_k are not 0, we have $|x_k|^2 \geq 1/z_k$. Since $(\varepsilon_k)_{n-1\geq k\geq 0}$ is a β-expansion, by Lemma 1 we have $0 < z_k < \beta^n$. Therefore

$$|x_k|^2 > 1/\beta^n = |\alpha|^{2n}.$$

Hence $|\alpha^n| < |\sum_{k=0}^{n-1} \varepsilon_k \alpha^k|$.

End of proof of Theorem 1 in the case $a \geq b \geq 0$. Now, we will prove that there exists $n_0 \in \mathbb{N}$ such that $(R_n)_{n\geq n_0}$ is the best approximation of the vector $(1/\beta, 1/\beta^2)$ for the norm $\mathcal{N}$.

Let c be the real number defined in Theorem 3. Let $n_0 \in \mathbb{N}$ be such that $\mathcal{N}(\delta(R_{n_0})) = |\alpha^{n_0}| < c$. Let $n \geq n_0$ and $0 < q < R_n$. Assume

$$\mathcal{N}_0(q(1/\beta, 1/\beta^2)) \leq \mathcal{N}_0(R_n(1/\beta, 1/\beta^2)) < c. \tag{5}$$

By Corollary 2, $\mathcal{N}_0(q(1/\beta, 1/\beta^2)) = \mathcal{N}(\delta(q))$ and $\mathcal{N}_0(R_n(1/\beta, 1/\beta^2)) = \mathcal{N}(\delta(R_n))$. Thus $\mathcal{N}(\delta(q)) \leq \mathcal{N}(\delta(R_n))$. This contradicts Proposition 7.

REMARK 3.4. The proofs in the cases $b = a + 1$ and $b = -1$, $a \geq 2$ are mutatis mutandis analogous to the previous one.

4. Proof of Theorem 2. We give a proof of Theorem 2 in the case $a \geq b \geq 0$. It is straightforward to adapt it to the other two cases.

Let β be a Pisot unit of degree 3, and let α, γ be its *real* Galois conjugates. Let $(G_n)_{n\geq 0}$ be the sequence of best approximations of the vector $(1/\beta, 1/\beta^2)$ for the norm $\mathcal{N}$. Assume that $\alpha > 0$. Since $1 = a_{-1}/\beta + \cdots + a_{-t}/\beta^t$ where $.a_{-1} \ldots a_{-t} = d(1,\beta)$, and α is a Galois conjugate of γ, we have $1 = a_{-1}/\alpha + \cdots + a_{-t}/\alpha^t$, which is impossible because $\alpha < \beta$. Hence α and γ belong to $]-1, 0[$. Without loss of generality, we assume that $-1 < \alpha < \gamma < 0$. This means that $0 < |\gamma| < |\alpha| < 1$.

By contradiction, if the sequence $(t(n))_{n\geq 0}$ does not tend to infinity, there exists an integer $M \in \mathbb{N}$ and a subsequence $(t(n_u))_{u\geq 0}$ such that $t(n_u) \leq M$ for all u. So, for all u, there exist k_u and a β-expansion $\varepsilon_{k_u+M-1} \ldots \varepsilon_{k_u}$ such that $\varepsilon_{k_u} \geq 1$ and

$$G_{n_u} = \sum_{i=k_u}^{k_u+M-1} \varepsilon_i R_i.$$

By Theorem 3, we deduce that for n large enough,

$$\mathcal{N}_0(G_n(1/\beta, 1/\beta^2)) = \mathcal{N}(\delta(G_n)). \tag{6}$$

In what follows, we will assume that u is so large that G_{n_u} satisfies (6).

For all $d \in \mathbb{N}$,

$$\mathcal{N}(\delta(R_d)) = \sqrt{x^2\alpha^{2d} + y^2\gamma^{2d}},$$

where x, y are the coordinates of the vector $\delta(1)$. As all norms are equivalent on $\mathbb{R}^2$, there exists a constant $C > 1$ such that, for all $v = (v_1, v_2) \in \mathbb{R}^2$,

$$\frac{1}{C}\max(|v_1|, |v_2|) \leq \mathcal{N}(v) \leq C\max(|v_1|, |v_2|). \tag{7}$$

So there is $T > 0$ such that $\mathcal{N}(\delta(R_d)) \leq T|\alpha^d|$ for all $d \in \mathbb{N}$. Moreover, for all u, $G_{n_u} \geq R_{k_u}$. As G_{n_u} belongs to the sequence of best approximations, we have

$$\mathcal{N}(\delta(G_{n_u})) \leq \mathcal{N}(\delta(R_{k_u})) \leq T|\alpha^{k_u}|.$$

On the other hand, by (7),

$$\mathcal{N}(\delta(G_{n_u})) \geq \frac{1}{C}\max\Big(\Big|\sum_{i=k_u}^{k_u+M-1} \varepsilon_i\alpha^i\Big|, \Big|\sum_{i=k_u}^{k_u+M-1} \varepsilon_i\gamma^i\Big|\Big) \geq \frac{1}{C}\Big|\sum_{i=k_u}^{k_u+M-1} \varepsilon_i\alpha^i\Big|.$$

Set $K(M) = \min(|\sum_{i=0}^{M-1} \varepsilon_i\alpha^i|)$ where the minimum is taken over all β-expansions $\varepsilon_{M-1} \ldots \varepsilon_0$ of length M with $\varepsilon_0 \geq 1$. Then

$$\mathcal{N}(\delta(G_{n_u})) \geq \frac{K(M)}{C}\,|\alpha^{k_u}|.$$

We have proven that

$$\frac{K(M)}{C}\,|\alpha^{k_u}| \le \mathcal{N}(\delta(G_{n_u})) \le T|\alpha^{k_u}|. \tag{8}$$

Now we give an estimate of k_u in terms of G_{n_u}. As β is a Pisot number, there exists a constant κ such that for all u,

$$\frac{1}{\kappa}\,\beta^{k_u} \le R_{k_u} \le G_{n_u} \le R_{k_u+M} \le \kappa\beta^{k_u+M}.$$

So there are two positive constants C_1, C_2 such that

$$\frac{\log(G_{n_u})}{\log(\beta)} - C_1 \le k_u \le \frac{\log(G_{n_u})}{\log(\beta)} + C_2. \tag{9}$$

Combining (8) and (9), we obtain

$$\begin{aligned}\frac{K(M)}{C}\,|\alpha|^{C_2}\left(\frac{1}{G_{n_u}}\right)^{\log(1/|\alpha|)/\log(\beta)} \\ \le \mathcal{N}(\delta(G_{n_u})) \le T|1/\alpha|^{C_1}\left(\frac{1}{G_{n_u}}\right)^{\log(1/|\alpha|)/\log(\beta)}.\end{aligned} \tag{10}$$

As $(1/|\alpha|)(1/|\gamma|) = \beta$ and $1/|\alpha| < 1/|\gamma|$, we have

$$\frac{\log(1/|\alpha|)}{\log(\beta)} < 1/2. \tag{11}$$

But $(G_n)_{n\ge 0}$ is the sequence of best approximations of $(1/\beta, 1/\beta^2)$. So, by the Dirichlet theorem (see, for instance, [Ca57, p. 13]), there exists a constant C_3 such that, for all $n \in \mathbb{N}$,

$$\mathcal{N}_0(G_n(1/\beta, 1/\beta^2)) \le \frac{C_3}{G_n^{1/2}}.$$

Consequently, for u large enough,

$$\mathcal{N}_0(G_{n_u}(1/\beta, 1/\beta^2)) \le \frac{C_3}{G_{n_u}^{1/2}}. \tag{12}$$

Inequality (12) contradicts (10) and (11). So Theorem 2 is proven.

Remark 4.1. Theorem 2 is also true if β is a Pisot unit of degree $d \ge 4$ which has Property (F).

5. Proof of Proposition 6. The proof of Proposition 6 requires several propositions and lemmas.

Proposition 8. *Let $x = \sum_{i=0}^{\infty} \varepsilon_i\alpha^i$ and $y = \sum_{i=0}^{\infty} \varepsilon'_i\alpha^i$ where $(\varepsilon_i)_{i\ge 0}$, $(\varepsilon'_i)_{i\ge 0} \in E_\beta$. Then $x = y$ if and only if the set $\{\alpha^{-k}\sum_{i=0}^{k}(\varepsilon_i - \varepsilon'_i)\alpha^i \mid k \ge 0\}$ is finite.*

A proof of this proposition in the case of Tribonacci numbers can be found in [Me00]. Here, we apply the tools developed by Thurston [Th90].

Proof of Proposition 8. The "if" part is obvious because if there exists a fixed number M such that $|\alpha^{-k}\sum_{i=0}^{k}(\varepsilon_i - \varepsilon_i')\alpha^i| \leq M$ for all $k \geq 0$, then $x = y$ since $|\alpha| < 1$. Now assume that $x = y$ and for an integer k, define $A_k = \sum_{i=0}^{k}(\varepsilon_i - \varepsilon_i')\alpha^{i-k}$. Since β has Property (F), there exists a finite β-expansion $(c_i)_{M\geq i\geq L}$ such that $A_k = \pm\sum_{i=L}^{M} c_i\alpha^i$. Assume without loss of generality that $A_k = \sum_{i=L}^{M} c_i\alpha^i$. Then $\sum_{i=0}^{k}\varepsilon_i\beta^{i-k} = \sum_{i=0}^{k}\varepsilon_i'\beta^{i-k} + \sum_{i=L}^{M} c_i\beta^i$. Therefore $M \leq 0$, since otherwise, by Lemma 1, $\sum_{i=0}^{k}\varepsilon_i\beta^{i-k} < \sum_{i=L}^{M} c_i\beta^i$. On the other hand, since $x = y$,

$$A_k = \sum_{i=k+1}^{\infty}(\varepsilon_i' - \varepsilon_i)\alpha^{i-k} = \sum_{i=1}^{\infty}\varepsilon_{k+i}'\alpha^i - \sum_{i=1}^{\infty}\varepsilon_{k+i}\alpha^i.$$

The hypothesis implies that there exists a fixed constant $d(\beta) = d > 0$ such that

$$|A_k| < d, \quad \forall k \geq 0. \tag{13}$$

Now put $z_k = \beta^{-k}\sum_{i=0}^{k}(\varepsilon_i - \varepsilon_i')\beta^i$ for all $k \geq 0$. Since $1/\beta$ is an algebraic integer, for all $k \geq 0$, z_k is also an algebraic integer in $\mathbb{Q}[\beta]$. The Galois conjugates of z_k are contained in the set $\{\beta_j^{-k}\sum_{i=0}^{k}(\varepsilon_i - \varepsilon_i')\beta_j^i \mid j = 1, 2, 3\}$, where $\beta_1 = \beta$, $\beta_2 = \alpha$, $\beta_3 = \overline{\alpha}$.

By (13) and the fact that z_k is bounded by a constant independent of k, we deduce that the set $\{A_k \mid k \geq 0\}$ is finite.

Lemma 6 ([FS92]). *Let β be a Pisot real number. There exists $S = S(\beta)$ with the following property. Let $x, y \in \mathrm{Fin}_N(\beta)$, $x > y$. If $x + y \in \mathrm{Fin}(\beta)$ then $x + y \in \mathrm{Fin}_{N+S}(\beta)$, and if $x - y \in \mathrm{Fin}(\beta)$ then $x - y \in \mathrm{Fin}_{N+S}(\beta)$.*

Lemma 7. *Let $(c_i)_{i\geq 0}$ be an element of E_β and $(d_i)_{P\geq i\geq 0}$ be a finite β-expansion. If $\sum_{i=0}^{\infty} c_i\alpha^i = \sum_{i=0}^{P} d_i\alpha^i$ then $c_i = d_i$ for all $0 \leq i \leq P$, and $c_i = 0$ for all $i > P$.*

Proof. Put $X = \{i \in \mathbb{N} \mid c_i \neq 0\}$. Assume that X is a finite set; then there exists $N \in \mathbb{N}$ such that $c_i = 0$ for all $i > N$. Since β is a Galois conjugate of α, we deduce that $\sum_{i=0}^{N} c_i\beta^i = \sum_{i=0}^{P} d_i\beta^i$. Hence, by Lemma 1, we obtain $N = P$ and $c_i = d_i$ for all $0 \leq i \leq P$.

Assume that X is an infinite set and put $d_i = 0$ for all $i > P$. For all $k \in X$, put $A_k = \sum_{i=0}^{k} c_i\alpha^i$ and $B = \sum_{i=0}^{P} d_i\alpha^i$. By Proposition 8, the set $V = \{\alpha^{-k}\sum_{i=0}^{k}(c_i - d_i)\alpha^i \mid k \geq 0\}$ is finite, so there exists an integer L, as large as we want, such that for some $k \geq P$, $\alpha^{-k}(A_k - B) =$

$\alpha^{-k-L}(A_{k+L} - B)$. Then

$$\sum_{i=0}^{k+L} c_i\alpha^i = \Big(\sum_{i=L}^{k+L} c_{i-L}\alpha^i - \sum_{i=L}^{L+P} d_{i-L}\alpha^i\Big) + \sum_{i=0}^{P} d_i\alpha^i. \tag{14}$$

By Lemma 6 and the fact that $\sum_{i=L}^{k+L} c_{i-L}\beta^i - \sum_{i=L}^{L+P} d_{i-l}\beta^i > 0$, there exists an integer $S = S(\beta) > 0$ such that

$$\sum_{i=L}^{k+L} c_{i-L}\alpha^i - \sum_{i=L}^{L+P} d_{i-L}\alpha^i = \sum_{i=L-S}^{R} e_i\alpha^i,$$

where $(e_i)_{R\geq i\geq L-S} \in E_\beta$. We deduce that

$$\sum_{i=0}^{P} d_i\alpha^i + \sum_{i=L-S}^{R} e_i\alpha^i = \sum_{i=0}^{k+L} c_i\alpha^i.$$

If we choose L such that $L-S-P$ is large enough to guarantee that the word $e_R \dots e_{L-S}0\dots 0d_P\dots d_0$ is a β-expansion, we infer by Lemma 1 that $c_i = d_i$ for all $0 \leq i \leq P$. Hence $\sum_{i=P+1}^{\infty} c_i\alpha^i = 0$ and then by applying the same argument we conclude that $c_i = 0$ for all $i > P$.

Remark 5.1. The previous lemma also follows from Theorem 2 in [Ak99].

Proof of Proposition 6. Let $z = p_1\alpha^{-1} + p_2\alpha^{-2}$, $p_1, p_2 \in \mathbb{Z}$. Assume that

$$z = \sum_{i=0}^{\infty} \varepsilon_i\alpha^i, \tag{15}$$

where $(\varepsilon_i)_{i\geq 0} \in E'_\beta$. We divide the proof into two cases.

Case 1: $p_1\beta^{-1} + p_2\beta^{-2} > 0$. Since β has Property (F), there exists a β-expansion $(a_i)_{N\geq i\geq M}$ with $a_M \neq 0$ such that

$$p_1\beta^{-1} + p_2\beta^{-2} = \sum_{i=M}^{N} a_i\beta^i. \tag{16}$$

Therefore

$$\sum_{i=0}^{\infty} \varepsilon_i\alpha^i = \sum_{i=M}^{N} a_i\alpha^i. \tag{17}$$

Multiplying both sides of (17) by α^{-M} and using Lemma 7, we deduce that $M \geq 0$ and $a_i = \varepsilon_i$ for all $M \leq i \leq N$. By (16) and Lemma 2, we have

$$p_1\beta^{-1} + p_2\beta^{-2} = \sum_{i=M}^{N} a_iR_i + \sum_{i=M}^{N} a_i(bR_{i-1} + R_{i-2})/\beta + \sum_{i=M}^{N} a_iR_{i-1}/\beta^2.$$

Therefore $a_M = 0$. This leads to a contradiction.

Case 2: $p_1\beta^{-1} + p_2\beta^{-2} < 0$. By (15), $\sum_{i=0}^{k} \varepsilon_i\alpha^i - p_1\alpha^{-1} - p_2\alpha^{-2}$ converges to 0 as $k \to \infty$. Since $0 \in \mathrm{int}(\mathcal{K}')$, for k large enough

$$(*) \qquad \sum_{i=0}^{k} \varepsilon_i\alpha^i - p_1\alpha^{-1} - p_2\alpha^{-2} \in \mathcal{K}'.$$

We fix k satisfying $(*)$; then there exist $(d_i)_{i\geq 0} \in E'_\beta$ such that

$$\sum_{i=0}^{k} \varepsilon_i\alpha^i - p_1\alpha^{-1} - p_2\alpha^{-2} = \sum_{i=0}^{\infty} d_i\alpha^i.$$

On the other hand, since $\sum_{i=0}^{k} \varepsilon_i\beta^i - p_1\beta^{-1} - p_2\beta^{-2} > 0$ and β has Property (F), there exists a β-expansion $(b_i)_{M\geq i\geq L}$ such that $\sum_{i=L}^{M} b_i\alpha^i = \sum_{i=0}^{\infty} d_i\alpha^i$. This implies by Lemma 7 that $d_i = 0$ for all $i > M$. Thus

$$\sum_{i=0}^{k} \varepsilon_i\alpha^i - p_1\alpha^{-1} - p_2\alpha^{-2} = \sum_{i=0}^{M} d_i\alpha^i.$$

Using Lemma 2, we deduce that

$$\sum_{i=0}^{k} \varepsilon_i R_i + \Big(\sum_{i=0}^{k} \varepsilon_i(bR_{i-1} + R_{i-2}) - p_1\Big)/\alpha + \sum_{i=0}^{k}(\varepsilon_i R_{i-1} - p_2)/\alpha^2$$
$$= \sum_{i=0}^{M} d_i R_i + \sum_{i=0}^{M} d_i(bR_{i-1} + R_{i-2})/\alpha + \sum_{i=0}^{k} d_i R_{i-1}/\alpha^2.$$

Since α is an algebraic integer of degree 3, we find that $\sum_{i=0}^{k} \varepsilon_i R_i = \sum_{i=0}^{M} d_i R_i$. We deduce that $\varepsilon_i = d_i$ for all i. Hence $p_1 = p_2 = 0$, which leads to a contradiction.

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