ACTA ARITHMETICA
124.1 (2006)

Uniformly thin bases of order two

by

CHRISTOPH SCHMITT (Hattersheim)

1. Introduction

DEFINITION 1. Let $\mathrm{A}, \mathrm{B} \subseteq \mathbb{N}_0$.

(a) $\mathrm{A} + \mathrm{B} = \{a + b \mid a \in \mathrm{A},\ b \in \mathrm{B}\}$.
(b) $\mathrm{A}(n) = \#\{a \in \mathrm{A} \mid 1 \le a \le n\}$.
(c) If $\mathrm{A} + \mathrm{A} = \mathbb{N}_0$, then A is called a *basis* (of order two).
(d) We call $\overline{\mathrm{d}}_2(\mathrm{A}) = \limsup_{n\to\infty} \mathrm{A}(n)/\sqrt{n}$ the *upper asymptotic density* of A. If $\overline{\mathrm{d}}_2(\mathrm{A}) < \infty$, then A is called *thin*.
(e) If $\lim_{n\to\infty} \mathrm{A}(n)/\sqrt{n}$ exists, then we call $\mathrm{d}_2(\mathrm{A}) = \lim_{n\to\infty} \mathrm{A}(n)/\sqrt{n}$ the *asymptotic density* of A. If $\mathrm{d}_2(\mathrm{A}) < \infty$, then A is called *uniformly thin*.
(f) We write $a_n \uparrow \infty$ for a sequence $(a_n)_{n\in\mathbb{N}_0}$ if $a_0 < a_1 < a_2 < \cdots$ and $a_n \to \infty$.

REMARK 1. Here we only consider bases of order two. So we will say "basis" instead of "basis of order two". The existence of uniformly thin bases was proved by Cassels [2] in 1957. He found a uniformly thin basis C with asymptotic density $\mathrm{d}_2(\mathrm{C}) = 3\sqrt{3} = 5.19615\ldots$. In 2001 Hofmeister [3] showed the existence of a basis H with $\mathrm{d}_2(\mathrm{H}) = (10/\sqrt{6})\sqrt[4]{5/3} = 4.63859\ldots$. Here we produce a uniformly thin basis B with $\mathrm{d}_2(\mathrm{B}) = 2\sqrt{3} = 3.46410\ldots$. Cassels and Hofmeister used the following lemma, which we also apply.

LEMMA 1 (Cassels, $k = 2$). *Let* $\mathrm{A} = \{a_0 < a_1 < a_2 < \cdots\} \subseteq \mathbb{N}_0$ *and* $\mu > 0$. *If*

$$\limsup_{n\to\infty} \frac{\sqrt{a_n}}{a_{n+1} - a_n} < \frac{\mu}{2},$$

then there is a set $\mathrm{B} = \{b_0 < b_1 < b_2 < \cdots\} \subseteq \mathbb{N}_0$ *and an* $n_0 \in \mathbb{N}$ *such that*

(a) $\mathrm{A} \subseteq \mathrm{B}$,
(b) $\mathrm{d}_2(\mathrm{B}) = \mu$,
(c) $b_n = (1/\mu^2)n^2 + r_n,\ 0 \le r_n < (1/\mu^2)(2n+1)\ \forall n \ge n_0$.

2000 *Mathematics Subject Classification*: Primary 11B13.

Remark 2. Note that this is not exactly Cassels' lemma of [2]. We have translated (a) into the language of our asymptotic density. But our result is equivalent to Cassels' for $k = 2$. Further we do not mention case "=", as we do not need it here. The small improvement in (c) follows from Cassels' proof in [2].

Definition 2. Let $a, b, m \in \mathbb{N}_0$, $a \leq b$, and $m \mid (b - a)$.

(a) $[\,a, (m)\,, b\,] = \{a + km \mid 0 \leq k \leq (b - a)/m,\ k \in \mathbb{N}_0\}$.
(b) $[\,a, b\,] = [\,a, (1)\,, b\,]$.
(c) $[\,a, +\infty\,] = \{n \in \mathbb{N}_0 \mid n \geq a\}$.
(d) R is called a *complete set of residues modulo* m if any congruence class modulo m has exactly one residue in R. Then $\#\mathrm{R} = m$.

Lemma 2 (without proof). *Let* $a, b, m \in \mathbb{N}_0$, $a \leq b$, $m > 1$, $m \mid (b - a)$, *and* $(b_n)_{n \in \mathbb{N}}$ *with* $m^n \mid b_n$ *and* $b_n \uparrow \infty$.

(a) *If* R *is a complete set of residues modulo* m, *then for* $\mathrm{P} = [\,a, (m)\,, b\,]$, $r^- = \min \mathrm{R}$, *and* $r^+ = \max \mathrm{R}$ *we obtain* $[\,a + r^+, b + r^-\,] \subseteq \mathrm{P} + \mathrm{R}$.
(b) *If* $\mathrm{R}_n, n \in \mathbb{N}$, *are complete sets of residues modulo* m^n, *then any* R_k *is a set of residues of pairwise distinct congruence classes modulo* m^n *for all* $n \geq k$.
(c) *Let* $\mathrm{P}_n = [\,0, (m^n)\,, b_n\,]$, $n \in \mathbb{N}$. *Then* $\mathrm{P}_n \cap [\,0, b_k\,] \subseteq \mathrm{P}_k$ *for all* $k \leq n$.

Remark 3. Lemma 2 is clear and will be often used.

2. PR-bases

Theorem 1 (PR-bases). *Let* $m \in \mathbb{N} \setminus \{1\}$, $b_0 = 0$, $(b_n)_{n \in \mathbb{N}}$ *with* $m^n \mid b_n$ *and* $b_n \uparrow \infty$. *Let* $\mathrm{P}_n = [\,0, (m^n)\,, b_n\,]$ *and* R_n *be a complete set of residues modulo* m^n *for any* $n \in \mathbb{N}$. *Define* P *to be the set of all those arithmetic progressions and* R *to be the set of all the above residues*:

$$\mathrm{P} = \bigcup_{n=1}^{\infty} \mathrm{P}_n, \qquad \mathrm{R} = \bigcup_{n=1}^{\infty} \mathrm{R}_n.$$

Suppose that there is an $n_0 \in \mathbb{N}$, $n_0 \geq 2$, *such that for all* $n \geq n_0$,

(a) $r_n^+ = \max \mathrm{R}_n < b_{n-1}$,
(b) $\mathrm{R}_{n-1} \subseteq \mathrm{R}_n$,
(c) $\mathrm{R}_n \setminus \mathrm{R}_{n-1} \subseteq [\,b_{n-2}, b_{n-1}\,]$.

Then

$$\mathrm{A} = \mathrm{P} \cup \mathrm{R} \cup [\,0, m^{n_0 - 1} - 1\,]$$

is a basis. We call any basis constructed in this way a PR-basis.

Proof. Set $r_n^- = \min \mathrm{R}_n$ for all $n \geq n_0$. Then $[\,r_n^+, b_n + r_n^-\,] \subseteq \mathrm{P}_n + \mathrm{R}_n$ by Lemma 2(a), so $[\,b_{n-1}, b_n\,] \subseteq \mathrm{P}_n + \mathrm{R}_n$, and hence $[\,b_{n_0 - 1}, +\infty\,] \subseteq \mathrm{P} + \mathrm{R}$.

For any $1 \le n \le n_0 - 1$ the set $[0, m^n - 1]$ is a complete set of residues modulo m^n. Then $[0, b_n] \subseteq \mathrm{P}_n + [0, m^n - 1]$ for any $1 \le n \le n_0 - 1$. So $[0, b_{n_0-1}] \subseteq \mathrm{A} + \mathrm{A}$ and A is a basis.

Remark 4. We say "PR-bases" as this type of basis mainly consists of a set P of arithmetic **p**rogressions and a set R of **r**esidues. Note that (a) is sufficient to obtain a basis. But condition (b) reduces the number of elements in A and the further requirement (c) ensures that we introduce the residues when they are needed. Note that (b) is possible by Lemma 2(b).

3. Upper asymptotic density of PR-bases

Lemma 3. *For any thin PR-basis there are $n_0 \in \mathbb{N}$ and $s, S \in \mathbb{R}$, $0 < s \le S$, such that*

$$s \le c_n = \frac{b_n}{m^{2n}} \le S \qquad \forall n \ge n_0.$$

Proof. For thin bases $\mathrm{A}(n)/\sqrt{n}$ is bounded, so in particular for thin PR-bases we have $\mathrm{A}(b_n) = \mathrm{O}(\sqrt{b_n})$. Moreover, $\mathrm{A}(b_n) \ge m^{n+1} - 1$, as $r^+_{n+1} \le b_n$; thus $m^{2n} = \mathrm{O}(b_n)$. From $\mathrm{A}(b_n) \ge \#\mathrm{P}_n = b_n/m^n$ we obtain $b_n = \mathrm{O}(m^{2n})$.

Theorem 2. *Let* A *be a thin PR-basis. Let*

$$s = \liminf_{n\to\infty} \frac{b_n}{m^{2n}}.$$

Then the upper asymptotic density of A *satisfies*

$$\overline{\mathrm{d}}_2(\mathrm{A}) \ge \left(\frac{m}{s} + \frac{1}{m} + 1\right)\sqrt{s}.$$

It follows that

$$\overline{\mathrm{d}}_2(\mathrm{A}) \ge 2\sqrt{3}$$

for any PR-basis, where $\overline{\mathrm{d}}_2(\mathrm{A}) = 2\sqrt{3}$ is only possible if $m = 2$ and $s = 4/3$.

Proof. To estimate the upper asymptotic density, we can estimate the counting function at the points b_n. For n_0 and b_0 of Theorem 1, Lemma 2(c) shows that for all $n \ge n_0$,

$$\begin{aligned}
\mathrm{A}(b_n) &\ge \#\mathrm{R}_{n+1} + \#\bigcup_{l=1}^{n} \mathrm{P}_l - \#\Big(\mathrm{R}_{n+1} \cap \bigcup_{k=1}^{n} \mathrm{P}_k\Big) - 1 \\
&= \#\mathrm{R}_{n+1} + \sum_{l=1}^{n} \frac{b_l - b_{l-1}}{m^l} - \#\Big(\mathrm{R}_{n+1} \cap \bigcup_{k=1}^{n} \mathrm{P}_k\Big) \\
&= m^{n+1} + \frac{b_n}{m^n} + (m-1)\sum_{l=1}^{n-1} \frac{b_l}{m^{l+1}} - \#\Big(\mathrm{R}_{n+1} \cap \bigcup_{k=1}^{n} \mathrm{P}_k\Big).
\end{aligned}$$

Note that we do not count 0: If $0 \in \mathrm{R}_{n+1}$ then $0 \in \mathrm{R}_{n+1} \cap \bigcup_{k=1}^{n} \mathrm{P}_k$. First we estimate the last term. Since A is a PR-basis, we have

$$\mathrm{R}_l \setminus \mathrm{R}_{l-1} \subseteq [\, b_{l-2}, b_{l-1} \,] \quad \text{and} \quad \Big(\bigcup_{k=1}^{n} \mathrm{P}_k \Big) \cap [\, b_{l-2}, b_{l-1} \,] = [\, b_{l-2}, (m^{l-1})\, , b_{l-1} \,]$$

for any l with $n_0 \le l \le n+1$. As $[\, b_{l-2}, (m^{l-1})\, , b_{l-1} \,]$ contains residues of at most m distinct congruence classes modulo m^l, we get

$$\#\Big((\mathrm{R}_l \setminus \mathrm{R}_{l-1}) \cap \bigcup_{k=1}^{n} \mathrm{P}_k \Big) \le m.$$

Then from $\mathrm{R}_{n+1} = \mathrm{R}_{n_0-1} \cup \bigcup_{l=n_0-1}^{n} (\mathrm{R}_{l+1} \setminus \mathrm{R}_l)$ we deduce that

$$\begin{aligned} \#\Big(\mathrm{R}_{n+1} \cap \bigcup_{k=1}^{n} \mathrm{P}_k \Big) &\le \#\mathrm{R}_{n_0-1} + \sum_{l=n_0-1}^{n} \#\Big(\big(\mathrm{R}_{l+1} \setminus \mathrm{R}_l \big) \cap \bigcup_{k=1}^{n} \mathrm{P}_k \Big) \\ &\le m^{n_0-1} + (n - n_0 + 2)m. \end{aligned}$$

Hence

$$(*) \qquad \frac{\mathrm{A}(b_n)}{\sqrt{b_n}} \ge \frac{m^{n+1}}{\sqrt{b_n}} + \frac{\sqrt{b_n}}{m^n} + \frac{m-1}{\sqrt{b_n}} \sum_{l=1}^{n-1} \frac{b_l}{m^{l+1}} - \frac{m^{n_0-1} + (n - n_0 + 2)m}{\sqrt{b_n}}.$$

To get a lower estimate of the upper asymptotic density it will suffice to examine a sequence $(n_k)_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \frac{b_{n_k}}{m^{2n_k}} = s.$$

Further we consider each term on the right hand side of $(*)$. The sum of the lower limits of these terms is less than or equal to the lower limit of their sum. Note that $s > 0$ by Lemma 3. Thus Theorem 2 will be proved if we can show the following limits and estimates:

$$\begin{aligned} &\lim_{k\to\infty} \frac{m^{n_k+1}}{\sqrt{b_{n_k}}} = \frac{m}{\sqrt{s}}, \\ &\lim_{k\to\infty} \frac{\sqrt{b_{n_k}}}{m^{n_k}} = \sqrt{s}, \\ &\lim_{k\to\infty} \frac{m^{n_0-1} + (n_k - n_0 + 2)m}{\sqrt{b_{n_k}}} = 0, \\ &\liminf_{k\to\infty} \frac{m-1}{\sqrt{b_{n_k}}} \sum_{l=1}^{n_k-1} \frac{b_l}{m^{l+1}} \ge \frac{\sqrt{s}}{m}. \end{aligned}$$

The three limits follow from $b_{n_k} \cong s m^{2n_k}$. For any $\varepsilon > 0$ there is a positive integer $l_0 = l_0(\varepsilon)$ such that $b_l \ge (1-\varepsilon) s m^{2l}$ for all $l \ge l_0$. Then for all n_k

with $n_k - 2 \ge l_0$ we obtain

$$(m-1)\sum_{l=1}^{n_k-1}\frac{b_l}{m^{l+1}} \ge (1-\varepsilon)(m-1)\sum_{l=l_0}^{n_k-1}\frac{sm^{2l}}{m^{l+1}} = (1-\varepsilon)(m^{n_k-1}-m^{l_0-1})s.$$

This is equivalent to

$$\liminf_{k\to\infty}\frac{m-1}{\sqrt{b_{n_k}}}\sum_{l=1}^{n_k-1}\frac{b_l}{m^{l+1}} \ge (1-\varepsilon)\frac{\sqrt{s}}{m}.$$

Letting $\varepsilon \to 0$ completes the proof.

Remark 5. The second part of Theorem 2 is pure analysis of the function $f : [2,+\infty) \times \mathbb{R}^+ \to \mathbb{R}^+$, $f(m,s) = (m/s + 1/m + 1)\sqrt{s}$. Note that for fixed $m = m_0$ the global minimum $2\sqrt{m_0+1}$ of f only appears if $s = s_0 = m_0^2/(m_0+1)$. Blomer [1] has found thin bases A with

$$\overline{\mathrm{d}}_2(\mathrm{A}) = \frac{\sqrt{3}}{(\sqrt{2}-1)\sqrt[4]{8}} = 2.48635\ldots + \varepsilon.$$

Therefore by Theorem 2, PR-bases cannot give an improvement over thin bases.

4. Uniformly thin sets including P**.** Here we exhibit uniformly thin sets Q including P. In the following section we substitute $\mathrm{Q}\setminus\mathrm{P}$ by the set R of all residues.

Theorem 3. *Let* $\mathrm{P}_n = [\,0,(m^n)\,,b_n\,]$, $n \in \mathbb{N}$, $m \in (\mathbb{N}\setminus\{1\})$, $b_n \cong cm^{2n}$, *and* $c < m^2/(m-1)$. *Then for* $\mathrm{P} = \bigcup_{n=1}^{\infty}\mathrm{P}_n = \{p_0 < p_1 < p_2 < \cdots\}$ *there is a uniformly thin set* $\mathrm{Q} = \{q_0 < q_1 < q_2 < \cdots\}$ *satisfying*

(a) $\mathrm{P} \subseteq \mathrm{Q}$,

(b) $\mathrm{d}_2(\mathrm{Q}) = \left(\dfrac{m}{c} + \dfrac{1}{m} + 1\right)\sqrt{c}$,

(c) $q_k = \dfrac{1}{\mathrm{d}_2^2(\mathrm{Q})}\,k^2 + r_k, \quad 0 \le r_k < \dfrac{1}{\mathrm{d}_2^2(\mathrm{Q})}\,(2k+1) \quad \forall k \ge k_0.$

Proof. We can apply Cassels' lemma to P: indeed,

$$\limsup_{k\to\infty}\frac{\sqrt{p_k}}{p_{k+1}-p_k} = \sqrt{c},$$

and since $c < m^2/(m-1)$ we get $\sqrt{c} < \mu/2$ with $\mu = (m/c + 1/m + 1)\sqrt{c}$.

Remark 6. For fixed m_0 we must choose $c = m_0^2/(m_0+1)$ to get the minimal lower estimate of $\overline{\mathrm{d}}_2(\mathrm{A})$ for PR-bases by Theorem 2 and Remark 5. Here we need $c < m_0^2/(m_0-1)$. Then Theorem 3 can be applied to the progressions P_n of the PR-bases with best possible upper asymptotic density.

5. A basis B **with** $d_2(B) = 2\sqrt{3}$**.** We only look at the case $m = 2$, $c = 4/3$. We should mention that for all $c < m^2/(m-1)$ we can prove the existence of a uniformly thin PR-basis B with $d_2(B) = (m/c + 1/m + 1)\sqrt{c}$. Observe that this is best possible for PR-bases by Theorem 2. In particular, we get the best possible asymptotic density for fixed m.

THEOREM 4. *Let* $b_0 = 0$*. For any positive integer let*

$$b_n = \frac{1}{3}\,4^{n+1} + x_n, \qquad 0 \le x_n < 2^{n+1},$$

be such that $2^{n+1} \mid b_n$*, and set*

$$P_n = [\,0, (2^n)\,, b_n\,].$$

Further let $P = \bigcup_{n=1}^{\infty} P_n$*. Then there is a set* R *such that* $B = P \cup R$ *is a uniformly thin PR-basis with asymptotic density*

$$d_2(B) = 2\sqrt{3}\,.$$

If A *is a* (*uniformly*) *thin PR-basis*, *then*

$$d_2(B) = 2\sqrt{3} \le \overline{d}_2(A)$$

by Theorem 2. *So the* (*upper*) *asymptotic density of the PR-basis* B *is best possible for PR-bases.*

Proof. We show the following four items:

(1) There is a uniformly thin set $Q = \{q_0 < q_1 < q_2 < \cdots\}$ such that $P \subseteq Q$, $d_2(Q) = 2\sqrt{3}$, and $q_i = \frac{1}{12}i^2 + r_i$ with $0 \le r_i < \frac{1}{12}(2i+1)$ for all $i \ge i_0$.

(2) If a set T satisfies $T(b_{n+1}) - T(b_n) \le c_0$ for all $n \ge n_1$ and some $c_0 \in \mathbb{N}$ then $d_2(T) = 0$. If $d_2(T) = 0$, then any (uniformly) thin set A satisfies $\overline{d}_2(A) = \overline{d}_2(A \cup T) = \overline{d}_2(A \setminus T)$ (and $d_2(A) = d_2(A \cup T) = d_2(A \setminus T)$).

(3) Let $S^{(n)} = (Q \setminus P) \cap [\,b_{n-2}, b_{n-1}\,]$. Then $2^{n-1} - 3 \le \#S^{(n)} \le 2^{n-1} + 2$ for all $n \ge n_0$.

(4) Let R_0 be a complete set of residues modulo 2^{n_0-1}. Then the sets $S^{(n)}$ can be substituted by sets $R^{(n)}$ for all $n \ge n_0$ such that:

   (a) $R^{(n)} \subseteq [\,b_{n-2}, b_{n-1}\,]$.

   (b) $R_n = R_0 \cup \bigcup_{l=n_0}^{n} R^{(l)}$ is a complete set of residues modulo 2^n.

   (c) Let, in particular, $R_0 = [\,0, 2^{n_0-1} - 1\,]$. Then

$$B = P \,\cup\, R_0 \,\cup\, \bigcup_{l=n_0}^{\infty} R^{(l)}$$

   is a uniformly thin PR-basis with asymptotic density

$$d_2(B) = 2\sqrt{3}.$$

Proof of (1). Use Theorem 3 with $m = 2$ and $c = 4/3$.

Proof of (2). For any $k \in \mathbb{N}$ there is an $n \in \mathbb{N}$ with $b_n < k \le b_{n+1}$. Then $\mathrm{T}(k) \le \mathrm{T}(b_{n+1}) \le c_0(n+1) + c$ with a constant c for sufficiently large k. Notice that $\sqrt{k} \ge \sqrt{b_n} = (1/\sqrt{3}\,)2^{n+1}$, so $\mathrm{d}_2(\mathrm{T}) = 0$. If the (upper) asymptotic density of a set T vanishes, then adding it to or subtracting it from another set A does not change the (upper) asymptotic density of A: for example,

$$\mathrm{d}_2(\mathrm{A}) = \lim_{k\to\infty} \frac{\mathrm{A}(k)}{\sqrt{k}} = \lim_{k\to\infty} \frac{\mathrm{A}(k) + \mathrm{T}(k)}{\sqrt{k}} \ge \limsup_{k\to\infty} \frac{(\mathrm{A} \cup \mathrm{T})(k)}{\sqrt{k}}$$

and

$$\mathrm{d}_2(\mathrm{A}) = \lim_{k\to\infty} \frac{\mathrm{A}(k)}{\sqrt{k}} \le \liminf_{k\to\infty} \frac{(\mathrm{A} \cup \mathrm{T})(k)}{\sqrt{k}},$$

so $\mathrm{d}_2(\mathrm{A} \cup \mathrm{T}) = \mathrm{d}_2(\mathrm{A})$.

Proof of (3). Since $\mathrm{P} \cap [\,b_{n-2}, b_{n-1}\,] = \mathrm{P}_{n-1} \cap [\,b_{n-2}, b_{n-1}\,]$ we obtain

$$\#\mathrm{S}^{(n)} = \mathrm{Q}(b_{n-1}) - \mathrm{Q}(b_{n-2}) - \frac{b_{n-1} - b_{n-2}}{2^{n-1}} \qquad \forall n \ge n_0.$$

Further from the definition of the b_n we get

$$\frac{b_{n-1} - b_{n-2}}{2^{n-1}} = \frac{1}{3}\,(2^{n+1} - 2^{n-1}) + \frac{x_{n-1} - x_{n-2}}{2^{n-1}} \in \{2^{n-1}, 2^{n-1} + 1\}.$$

For sufficiently large n by (1) we have

$$b_n = \frac{1}{3}\,4^{n+1} + x_n \ge \frac{1}{12}\,(2^{n+2})^2 = q_{2^{n+2}} - r_{2^{n+2}},$$

$$q_{2^{n+2}+3} \ge \frac{1}{12}\,(2^{n+2} + 3)^2 > b_n.$$

Note that $\mathrm{Q} \cap [\,q_{2^{n+2}} - r_{2^{n+2}}, q_{2^{n+2}}\,] = \{q_{2^{n+2}}\}$. Thus

$$q_{2^{n+2}} \le b_n < b_{n+1} < q_{2^{n+3}+3}.$$

As $0 \in \mathrm{Q}$ the index of the q_i equals the counting function: $\mathrm{Q}(q_i) = i$. So

$$2^{n+2} \le \mathrm{Q}(b_n) \le 2^{n+2} + 2.$$

Summarizing, for sufficiently large n we obtain

$$2^n - 2 \le \mathrm{Q}(b_{n-1}) - \mathrm{Q}(b_{n-2}) \le 2^n + 2, \qquad 2^{n-1} - 3 \le \#\mathrm{S}^{(n)} \le 2^{n-1} + 2.$$

Proof of (4). We will prove that for all $n \ge n_0$ we can replace $\mathrm{S}^{(n)}$ by a set $\mathrm{R}_0^{(n)}$ of residues of distinct congruence classes modulo 2^n with $\#\mathrm{R}_0^{(n)} = \#\mathrm{S}^{(n)}$, $\mathrm{R}_0^{(n)} \cap \mathrm{P}_{n-1} = \emptyset$, $\mathrm{R}_0^{(n)} \subseteq [\,b_{n-2}, b_{n-1}\,]$, without changing asymptotic density:

$$\mathrm{d}_2\Big(\Big(\mathrm{Q} \setminus \bigcup_{n=n_0}^{\infty} \mathrm{S}^{(n)}\Big) \cup \bigcup_{n=n_0}^{\infty} \mathrm{R}_0^{(n)}\Big) = \mathrm{d}_2(\mathrm{Q}),$$

where the residues of $R_0^{(n)}$ can be chosen in any congruence classes which are not contained in P_{n-1}.

Assume this statement is already proved.

By the definition of R_0 we can assume that R_{n-1} satisfies (4)(a) and (4)(b). To pass from R_{n-1} to R_n we need residues of exactly 2^{n-1} distinct congruence classes modulo 2^n contained in $[\, b_{n-2}, b_{n-1}\,]$.

First case: There are residues in $[\, b_{n-2}, b_{n-1}\,] \setminus P_{n-1}$ for any congruence class modulo 2^n.

Then $R_0^{(n)}$ contains residues of at least $2^{n-1} - 3$ distinct congruence classes modulo 2^n. If $\#R_0^{(n)} < 2^{n-1}$ then we add the residues of the missing congruence classes modulo 2^n (by (3) at most three) using any of their residues contained in $[\, b_{n-2}, b_{n-1}\,]$. If $\#R_0^{(n)} > 2^{n-1}$ then we omit the superfluous residues (by (3) at most two). If $\#R_0^{(n)} = 2^{n-1}$ then there is nothing to do. In all cases we have found $R^{(n)} \subseteq [\, b_{n-2}, b_{n-1}\,]$ which contains exactly the residues of the missing 2^{n-1} congruence classes modulo 2^n. So we set $R_n = R_{n-1} \cup R^{(n)}$. Now we apply (2) to obtain

$$d_2\Big(\Big(Q \setminus \bigcup_{n=n_0}^{\infty} S^{(n)}\Big) \cup \bigcup_{n=n_0}^{\infty} R^{(n)}\Big) = d_2\Big(\Big(Q \setminus \bigcup_{n=n_0}^{\infty} S^{(n)}\Big) \cup \bigcup_{n=n_0}^{\infty} R_0^{(n)}\Big).$$

Other cases: There are only residues of at most two distinct congruence classes modulo 2^n, which are not in $[\, b_{n-2}, b_{n-1}\,] \setminus P_{n-1}$, as P_{n-1} itself contains only residues of exactly two distinct congruence classes modulo 2^n. Then if we first take residues of at most two unnecessary congruence classes modulo 2^n into $R_0^{(n)}$ instead those at most two necessary ones, we can use the above considerations and obtain the same asymptotic density. In the end we replace the unnecessary residues by the necessary ones, which are also contained in the arithmetic progression $[\, b_{n-2}, (m^{n-1})\,, b_{n-1}\,]$. By (2) this does not change the asymptotic density. Again we have found $R^{(n)} \subseteq [\, b_{n-2}, b_{n-1}\,]$ and R_n satisfying (4)(a) and (4)(b).

Now we can turn to (4)(c). As adding finitely many elements to a set does not change its asymptotic density we see that

$$\begin{aligned} d_2(B) &= d_2\Big(P \cup R_0 \cup \bigcup_{n=n_0}^{\infty} R^{(n)}\Big) = d_2\Big(P \cup \bigcup_{n=n_0}^{\infty} R^{(n)}\Big) \\ &= d_2\Big(((Q \setminus P) \cap [\,0, b_{n_0-2}\,]) \cup P \cup \bigcup_{n=n_0}^{\infty} R^{(n)}\Big) \\ &= d_2\Big(\Big(Q \setminus \bigcup_{n=n_0}^{\infty} S^{(n)}\Big) \cup \bigcup_{n=n_0}^{\infty} R^{(n)}\Big) \end{aligned}$$

$$= \mathrm{d}_2\Big(\Big(\mathrm{Q} \setminus \bigcup_{n=n_0}^{\infty} \mathrm{S}^{(n)}\Big) \cup \bigcup_{n=n_0}^{\infty} \mathrm{R}_0^{(n)}\Big) = \mathrm{d}_2(\mathrm{Q}) = 2\sqrt{3}.$$

Probably one should read these equations backwards. Further note that

$$((\mathrm{Q} \setminus \mathrm{P}) \cap [\,0, b_{n_0-2}\,]) \cup \mathrm{P} \cup \bigcup_{n=n_0}^{\infty} \mathrm{R}^{(n)} = \Big(\mathrm{Q} \setminus \bigcup_{n=n_0}^{\infty} \mathrm{S}^{(n)}\Big) \cup \bigcup_{n=n_0}^{\infty} \mathrm{R}^{(n)}.$$

By (4)(a)–(c), B is a PR-basis.

Thus it remains to prove the statement from the beginning of the proof. There are only two congruence classes modulo 2^n in the arithmetic progression P_{n-1}. So there remain $2^n - 2 > \#\mathrm{S}^{(n)}$ distinct congruence classes, residues from which can be chosen to be put in $\mathrm{R}_0^{(n)}$. Let $s_n = \#\mathrm{S}^{(n)}$, $\mathrm{S}^{(n)} = \{s_1^{(n)} < \cdots < s_{s_n}^{(n)}\}$, and let $\varrho^{(n)} = \{\varrho_1^{(n)} < \cdots < \varrho_{s_n}^{(n)}\}$ be a set of s_n distinct congruence classes modulo 2^n for $n \geq n_0$. Note that $b_{n-1} - b_{n-2} > 2^n$. Then

$$\{s_j^{(n)} - (2^n - r),\ s_j^{(n)} + r\} \cap [\,b_{n-2}, b_{n-1}\,] \neq \emptyset$$

for any $s_j^{(n)} \in \mathrm{S}^{(n)}$ and $r < 2^n$. So there exists $t_j^{(n)}$ with $|t_j^{(n)}| < 2^n$ such that

$$r_j^{(n)} = s_j^{(n)} + t_j^{(n)} \in [\,b_{n-2}, b_{n-1}\,], \qquad r_j^{(n)} \in \varrho_j^{(n)}.$$

Let $\mathrm{R}_0^{(n)} = \{r_1^{(n)} < \cdots < r_{s_n}^{(n)}\}$ for $n \geq n_0$ and

$$\mathrm{B}_0 = \Big(\mathrm{Q} \setminus \bigcup_{n=n_0}^{\infty} \mathrm{S}^{(n)}\Big) \cup \bigcup_{n=n_0}^{\infty} \mathrm{R}_0^{(n)}.$$

Finally, we must investigate $\mathrm{B}_0(k)$. By the construction of $r_j^{(n)}$ we know that $\mathrm{B}_0(b_{n-2}) = \mathrm{Q}(b_{n-2})$ for all $n \in \mathbb{N}$. If $q_i \in [\,b_{n-2}, b_{n-1}\,]$ and $j > 6$ with $q_{i\pm j} \in [\,b_{n-2}, b_{n-1}\,]$ then

$$|q_{i\pm j} - q_i| \geq \frac{j-1}{12}\,(2 \cdot 2^n + 1) > 2^n,$$

as we must cross j–1 intervals $\big[\frac{1}{12}l^2, \frac{1}{12}(l+1)^2\big)$ by (1). Using $\mathrm{Q}(b_{n-2}) \geq 2^n - 1$ we find that $l \geq 2^n$ and the minimal length of the intervals is $\frac{1}{12}2^{n+1} + \frac{1}{12}$. So for any substituted $q_i = s_j^{(n)}$ we obtain

$$\max\{q_{i-7}, b_{n-2}\} \leq r_j^{(n)} \leq \min\{q_{i+7}, b_{n-1}\},$$

i.e. there are only two possibilities for any $q_i \in \mathrm{Q}$: either $q_i \in \mathrm{B}_0$ or q_i is substituted by a new element $r_j^{(n)} \in \mathrm{B}_0$ which satisfies the above condition. Then $\mathrm{B}_0(q_{i-7}) \leq \mathrm{Q}(q_i) = i$ and $\mathrm{B}_0(q_{i+7}) \geq \mathrm{Q}(q_i) = i$. So B_0 satisfies

$$\mathrm{Q}(q_i) - 7 = \mathrm{Q}(q_{i-7}) \leq \mathrm{B}_0(q_i) \leq \mathrm{Q}(q_{i+7}) = \mathrm{Q}(q_i) + 7.$$

For any $k \in \mathbb{N}$ there is an i such that $q_i \le k < q_{i+1}$. Then $i = \mathrm{Q}(q_i) = \mathrm{Q}(k) < \mathrm{Q}(q_{i+1}) = i+1$. Since $\mathrm{B}_0(q_i) \le \mathrm{B}_0(k) \le \mathrm{B}_0(q_{i+1})$ we find that

$$\mathrm{B}_0(k) \ge \mathrm{B}_0(q_i) \ge \mathrm{Q}(q_i) - 7 = \mathrm{Q}(k) - 7,$$
$$\mathrm{B}_0(k) \le \mathrm{B}_0(q_{i+1}) \le \mathrm{Q}(q_{i+1}) + 7 = \mathrm{Q}(k) + 8.$$

By the definition of the asymptotic density it follows that

$$\mathrm{d}_2(\mathrm{B}_0) = \mathrm{d}_2(\mathrm{Q}) = 2\sqrt{3}.$$

Remark 7. Note that $\mathrm{d}_2(\mathrm{B}) = 2\sqrt{3}$ is best possible for PR-bases by Theorem 2 for upper asymptotic density. We should mention some results of [4]. Neither the famous thin basis A_0 of Stöhr with $\overline{\mathrm{d}}_2(\mathrm{A}_0) = \frac{3}{2}\sqrt{3}$ nor the thin basis A_1 of Hofmeister [3] with $\overline{\mathrm{d}}_2(\mathrm{A}_1) = 2\sqrt{5/3}$ can be embedded in a uniformly thin set. Possibly Blomer's [1] UR-bases can be transformed to uniformly thin sets, but it would be a surprise. Further the uniformly thin bases of Cassels [2] and Hofmeister [3] contain subsets C_0 and H_0, which are bases themselves. But we can prove that any uniformly thin set C^* or H^* which includes C_0 or H_0 satisfies $\mathrm{d}_2(\mathrm{C}^*) \ge 3\sqrt{3} = 5.19615\ldots$ and $\mathrm{d}_2(\mathrm{H}^*) \ge (10/\sqrt{6})\sqrt[4]{5/3} = 4.63859\ldots$. So Lemma 1 is sharp in these cases. If we further look at the lower asymptotic density $\underline{\mathrm{d}}_2(\mathrm{A}) = \liminf_{n\to\infty} \mathrm{A}(n)/\sqrt{n}$, then there exists no function of $\underline{\mathrm{d}}_2(\mathrm{A}), \overline{\mathrm{d}}_2(\mathrm{A})$ which gives the asymptotic density for a uniformly thin set D containing A, i.e. generally $\mathrm{d}_2(\mathrm{D}) \ne f(\underline{\mathrm{d}}_2(\mathrm{A}), \overline{\mathrm{d}}_2(\mathrm{A}))$. Proofs can be found in [4].

Acknowledgements. The author would like to thank Gerd Hofmeister for his helpful comments.

References

[1] V. Blomer, *Thin bases of order h*, J. Number Theory 98 (2003), 34–46.
[2] J. W. S. Cassels, *Über Basen der natürlichen Zahlenreihe*, Abh. Math. Sem. Univ. Hamburg 21 (1957), 247–257.
[3] G. Hofmeister, *Thin bases of order two*, J. Number Theory 86 (2001), 118–132.
[4] C. Schmitt, *Gleichmäßig dünne Basen zweiter Ordnung*, Dissertation, Johannes-Gutenberg-Univ. Mainz, 2005.

Südring 73, D-65795 Hattersheim, Germany
E-mail: chrisheinschmitt@aol.com

Received on 13.8.2004
and in revised form on 6.3.2006 (4825)