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The arithmetic of the coefficients of half-integral weight Eisenstein series

by

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1. Introduction and statement of results. If $H(-n)$ denotes the Hurwitz–Kronecker class number of positive definite binary quadratic forms with discriminant $-n$, then Zagier's weight $3/2$ Eisenstein series is given by ($q = e^{2\pi iz}$ throughout)

$$(1.1)\qquad H_{3/2}(z) = -\frac{1}{12} + \sum_{n=1}^{\infty} H(1,n)q^n = -\frac{1}{12} + \sum_{0<n\equiv 0,3 \pmod 4} H(-n)q^n.$$

This series fits into Cohen's general theory of weight $k+1/2$ Eisenstein series

$$(1.2)\qquad H_{k+1/2}(z) = \sum_{n=0}^{\infty} H(k,n)q^n,$$

where $H(k,n) = H_1(k,n)$ is defined in (1.4). If $k \geq 2$, then $H_{k+1/2}(z)$ is a weight $k+1/2$ modular form on the congruence subgroup $\Gamma_0(4)$ (see [3, Th. 3.1]).

If $D \equiv 0,1 \pmod 4$, and $\chi_D(\bullet) = \left(\frac{D}{\bullet}\right)$ denotes the usual Kronecker character, then it is a classical fact (see [1, Ch. 5, §4]) that

$$H(1,-D) = H(D) = \sum_{a=0}^{|D|-1} \chi_D(a)a,$$

when $D < -4$ is a fundamental discriminant.

We generalize this classical result in a variety of ways. See, for example, Corollary 1.2. Moreover, we obtain an elegant uniform description of the

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$H(k,n)$ in terms of values of the Bernoulli polynomials $B_k(x)$, weighted against the mod n representation numbers of sums of squares.

To state our result, if χ is a Dirichlet character, we let $L(s,\chi) = \sum \chi(n)n^{-s}$ be its L-series. For positive integers r,n and k with r odd, let $D := (-1)^k n$ and set

$$h_r(k,n) = \begin{cases} \dfrac{(-1)^{[k/2]}\chi_8(r)(k-1)!n^{k-r/2}L(k,\chi_D)}{2^{k-(r+1)/2}\pi^k} & \text{if } (-1)^k n \equiv 0 \pmod 4, \\ \dfrac{(-1)^{[k/2]}(k-1)!n^{k-r/2}L(k,\chi_D)}{2^{k-1}\pi^k} & \text{if } (-1)^k n \equiv 1 \pmod 4, \\ 0 & \text{if } (-1)^k n \equiv 2,3 \pmod 4. \end{cases} \tag{1.3}$$

Define also

$$H_r(k,n) = \begin{cases} \sum_{d^2|n} h_r(k,n/d^2) & \text{if } (-1)^k n \equiv 0,1 \pmod 4, \\ \zeta(1-2k) & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1.4}$$

Note that $h_r(k,n) \ll n^{k-r/2}\log(2n)$ (where the $\log(2n)$ is necessary only for $k=1$). In particular,

$$H_r(k,n) \ll n^{\max(k-r/2,0)+\varepsilon}. \tag{1.5}$$

Next, we define, for positive integers r and n,

$$R_r(a,n) := \#\{(\nu_1,\ldots,\nu_r) \in (\mathbb{Z}/n\mathbb{Z})^r : \nu_1^2+\ldots+\nu_r^2 \equiv a \pmod n\}. \tag{1.6}$$

If k is a positive integer, then let B_k be the kth Bernoulli number and let $B_k(x)$ denote the usual kth Bernoulli function with the convention that

$$B_1(x) = \begin{cases} x-1/2 & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \tag{1.7}$$

Finally, define the two formal Dirichlet series $H_{r,k}(s)$ and $F_{r,k}(s)$ by

$$H_{r,k}(s) = \sum_{n=1}^{\infty} \frac{H_r(k,n)}{n^s}, \tag{1.8}$$

$$F_{r,k}(s) = -\frac{\chi_{(-1)^k}(r)}{k} \sum_{n=1}^{\infty} \frac{\sum_{a=0}^{n-1} R_r(a,n)B_k(a/n)}{n^s}. \tag{1.9}$$

By (1.5), $H_{r,k}(s)$ converges for $\mathrm{Re}(s) > \max(1, k+1-r/2)$. For $F_{r,k}(s)$, we note that $B_k(x)$ is bounded in the interval $[0,1]$ and $\sum R_r(a,n) = n^r$. This implies that $F_{r,k}(s)$ converges for $\mathrm{Re}(s) > r+1$. As a consequence of the following theorem, we can extend the region of convergence for $F_{r,k}(s)$ to $\mathrm{Re}(s) > \max(r/2+1, r-k+1)$.

By modifying an argument of J. Bruinier (see [2, Th. 11]) in the case where $k=1$, for positive odd integers r, we obtain the following description of $H_{r,k}(s)$ as a formal product of $F_{r,k}(s)$ and a simple quotient of the Riemann zeta function $\zeta(s)$. In particular, if $r = 1$, we obtain a convenient description of the Mellin transform of Cohen's half-integral weight Eisenstein series.

THEOREM 1.1. *If r is a positive odd integer and $k \geq 1$, then*

$$H_{r,k}(s) = \frac{\zeta(2s)}{\zeta(s)}\, F_{r,k}(s - k + r).$$

As is well known, $\zeta(2s)/\zeta(s) = \sum_{n=1}^{\infty} \lambda(n) n^{-s}$, where λ is the Liouville function, the totally multiplicative function on positive integers defined on primes p by $\lambda(p) = -1$. Then Theorem 1.1 gives an immediate formula for $H_r(k, n)$.

COROLLARY 1.2. *For $r, n, k \in \mathbb{Z}$ with r odd,*

$$H_r(k, n) = -\frac{\chi_{(-1)^k}(r)}{k} \sum_{d|n} \lambda(n/d) d^{k-r} \sum_{a=0}^{d-1} R_r(a, d) B_k(a/d).$$

We cannot resist interpreting the $r = 1$ case of Theorem 1.1 in terms of the *logarithmic derivatives* of the infinite products

$$F_k(q) = \prod_{n=1}^{\infty} (1 - q^n)^{H(k,n)/n}. \tag{1.10}$$

COROLLARY 1.3. *If k is a positive integer, then*

$$\frac{q\frac{d}{dq}(F_k(q))}{F_k(q)} + \sum_{\substack{a,b \geq 1\\ c \geq 2\\ b \text{ square-free}}} H(k, a) q^{abc^2} = \frac{1}{k} \sum_{n=1}^{\infty} \sum_{a=0}^{n-1} R_1(a, n) B_k(a/n) n^{k-1} q^n.$$

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2. Proofs. We begin with the following elementary lemmas. The first lemma is standard and can be found in [4, p. 522].

LEMMA 2.1. *The Fourier expansion of $B_k(x)$ is given by*

$$B_k(x) = \frac{-k!}{(2\pi i)^k} {\sum_{n\in\mathbb{Z}}}' \frac{e(nx)}{n^k}, \tag{2.1}$$

where $\sum'$ indicates that we are summing over all non-zero integers, and $e(x) = e^{2\pi i x}$.

The proof of the following lemma is a straightforward analogue of the proof of Möbius inversion and will be left to the reader.

LEMMA 2.2. *Let $f:\mathbb{Z}^+\to\mathbb{C}$. Define $F,G:\mathbb{Z}^+\to\mathbb{C}$ by*

$$F(N)=\sum_{d\mid N} f(N/d),\qquad G(N)=\sum_{d^2\mid N} f(N/d^2). \tag{2.2}$$

Then

$$\begin{aligned} &\text{(i)}\quad F(N)=\sum_{\substack{a\mid N\\ N/a\ \text{square-free}}} G(a),\\ &\text{(ii)}\quad G(N)=\sum_{a\mid N}\lambda(N/a)F(a). \end{aligned} \tag{2.3}$$

Fixing notation, we define, for integers m and a,

$$G(m,a)=\sum_{\nu\in\mathbb{Z}/a\mathbb{Z}} e(m\nu^2). \tag{2.4}$$

In the following lemma, we recall some basic facts on the Gauss sums in (2.4). For their proofs, we refer the reader to [5, IV.3].

LEMMA 2.3. *Suppose that a, b and c are integers with $b,c\ge 1$.*

(i) *If $c\mid(a,b)$, then $G(a,b)=c\cdot G(a/c,b/c)$.*
(ii) *If $(b,c)=1$ and $(a,bc)=1$, then $G(a,bc)=G(ab,c)G(ac,b)$.*
(iii) *If b is odd and $(a,b)=1$, then $G(a,b)=\left(\frac{a}{b}\right)G(1,b)$.*
(iv) *If a is odd, then $G(a,2^s)=\left(\frac{-2^s}{a}\right)\varepsilon((a-1)/2)G(1,2^s)$.*
(v) $G(1,b)=\begin{cases}(1+i)\sqrt{b} & \text{if } b\equiv 0 \pmod 4,\\ \sqrt{b} & \text{if } b\equiv 1 \pmod 4,\\ 0 & \text{if } b\equiv 2 \pmod 4,\\ i\sqrt{b} & \text{if } b\equiv 3 \pmod 4.\end{cases}$

Here, $\varepsilon(b)=1$ if b is even and $\varepsilon(b)=i$ if b is odd.

For positive integers k and r, let

$$\mathfrak{g}_{k,r}(m,a)=\begin{cases}\mathrm{Re}(G(m,a)^r) & \text{if } k\in 2\mathbb{Z},\\ \mathrm{Im}(G(m,a)^r) & \text{if } k\in 2\mathbb{Z}+1.\end{cases} \tag{2.5}$$

Then we see immediately from the definition and (i) above that, for $b\mid(m,a)$,

$$\mathfrak{g}_{k,r}(m,a)=b^r\mathfrak{g}_{k,r}(m/b,a/b). \tag{2.6}$$

Then we have the following proposition.

PROPOSITION 2.4. *For integers m and a such that $(m,a)=1$,*

$$\begin{aligned}\mathfrak{g}_{k,r}(m,a)&\\ &=a^{r/2}\left(\frac{(-1)^k a}{m}\right)\cdot\begin{cases}2^{(r-1)/2}\chi_{(-1)^k 8}(r) & \text{if } (-1)^k a\equiv 0 \pmod 4,\\ \chi_{(-1)^k}(r) & \text{if } (-1)^k a\equiv 1 \pmod 4,\\ 0 & \text{if } (-1)^k a\equiv 2,3 \pmod 4.\end{cases}\end{aligned} \tag{2.7}$$

Proof. If $(-1)^k a \equiv 1 \pmod 4$, then $G(m,a)^r = \left(\frac{m}{a}\right)^r \varepsilon(k)^r a^{r/2}$. Therefore,

$$\mathfrak{g}_{k,r}(m,a) = a^{r/2}\left(\frac{m}{a}\right)\varepsilon(k)^{r-1} = a^{r/2}\left(\frac{m}{a}\right)\chi_{(-1)^k}(r). \tag{2.8}$$

Clearly, by Lemma 2.3, if $(-1)^k a \equiv 2 \pmod 4$, then $\mathfrak{g}_{k,r}(m,a) = 0$. If $(-1)^k a \equiv 3 \pmod 4$, then $G(m,a)^r = \left(\frac{m}{a}\right)^r \varepsilon(k-1)^r a^{r/2}$ and $\mathfrak{g}_{k,r}(m,a) = 0$.

Therefore, the only case left is $(-1)^k a \equiv 0 \pmod 4$. Suppose $a = 2^s a_0$ with a_0 odd; then, by Lemma 2.3, we see that

$$G(m,a) = \left(\frac{-m}{a_0}\right)\left(\frac{-2^s}{m}\right)\varepsilon\left(\frac{ma_0-1}{2}\right)G(1,2^s)G(1,a_0). \tag{2.9}$$

Therefore, if $a_0 \equiv 1 \pmod 4$, then

$$\begin{aligned} G(m,a) &= \left(\frac{-2^s a_0}{m}\right)\varepsilon\left(\frac{m-1}{2}\right)(1+i)2^{s/2}\sqrt{a_0} \\ &= \left(\frac{-a}{m}\right)\varepsilon\left(\frac{m-1}{2}\right)\sqrt{a}\,(1+i), \end{aligned} \tag{2.10}$$

and if $a_0 \equiv 3 \pmod 4$, then

$$\begin{aligned} G(m,a) &= -\left(\frac{-a_0}{m}\right)\left(\frac{-2^s}{m}\right)\varepsilon\left(\frac{m+1}{2}\right)i\sqrt{a_0}\,(1+i)2^{s/2} \\ &= -\left(\frac{a}{m}\right)\varepsilon\left(\frac{m+1}{2}\right)i\sqrt{a}\,(1+i) \\ &= \left(\frac{-a}{m}\right)\varepsilon\left(\frac{m-1}{2}\right)\sqrt{a}\,(1+i). \end{aligned} \tag{2.11}$$

Then we see that

$$G(m,a)^r = 2^{(r-1)/2}a^{r/2}\left(\frac{-a}{m}\right)\varepsilon\left(\frac{m-1}{2}\right)^r i^{(r-1)/2}. \tag{2.12}$$

After a lengthy verification, we obtain

$$\mathfrak{g}_{k,r}(m,a) = \left(\frac{(-1)^k a}{m}\right)2^{(r-1)/2}a^{r/2}\chi_{(-1)^k 8}(r), \tag{2.13}$$

as desired. ∎

The following proposition provides the foundation for Theorem 1.1.

PROPOSITION 2.5. *For $r,k,n \in \mathbb{Z}^+$ with r odd,*

$$\sum_{a=0}^{n-1} R_r(a,n)B_k(a/n) = -\frac{\chi_{(-1)^k}(r)k}{n^{k-r}}\sum_{d\mid n} h_r(k,d).$$

Proof. Let N be a positive integer; then by Lemma 2.1,

$$\sum_{a=0}^{N-1} R_r(a,N)B_k(a/N) \tag{2.14}$$
$$= \sum_{\nu_1,\dots,\nu_r \,(\mathrm{mod}\, N)} B_k\left(\frac{\nu_1^2}{N}+\dots+\frac{\nu_r^2}{N}\right)$$
$$= \frac{-k!}{(2\pi i)^k}\sideset{}{'}\sum_{n\in\mathbb{Z}} n^{-k} \sum_{\nu_1,\dots,\nu_r \,(\mathrm{mod}\, N)} e\left(n\left(\frac{\nu_1^2}{N}+\dots+\frac{\nu_r^2}{N}\right)\right)$$
$$= \frac{-k!}{(2\pi i)^k}\sideset{}{'}\sum_{n\in\mathbb{Z}} n^{-k} G(n,N)^r$$
$$= \frac{-k!}{(2\pi i)^k}\sideset{}{'}\sum_{n\in\mathbb{Z}} n^{-k}\left(\mathrm{Re}(G(n,N)^r)+i\,\mathrm{Im}(G(n,N)^r)\right)$$
$$= \frac{-2\varepsilon(k)k!}{(2\pi i)^k}\sum_{n\ge 1} n^{-k}\mathfrak{g}_{k,r}(n,N).$$

Reindexing this sum on the divisors of N, we find

$$\sum_{a=0}^{N-1} R_r(a,N)B_k(a/N) \tag{2.15}$$
$$= \frac{-2\varepsilon(k)k!}{(2\pi i)^k}\sum_{a\mid N}\sum_{\substack{n\ge 1\\ (n,N)=N/a}} n^{-k}\mathfrak{g}_{k,r}(n,N)$$
$$= \frac{-2\varepsilon(k)k!}{(2\pi i)^k} N^{r-k}\sum_{a\mid N} a^{k-r}\sum_{\substack{m\ge 1\\ (m,a)=1}} m^{-k}\mathfrak{g}_{k,r}(m,a).$$

Since $\chi_a(m)=0$ if $(m,a)\neq 1$, Proposition 2.4 yields

$$a^{k-r}\sum_{\substack{m\ge 1\\ (m,a)=1}} m^{-k}\mathfrak{g}_{k,r}(m,a) \tag{2.16}$$
$$= 2^{(r-1)/2}\chi_{(-1)^k 8}(r)a^{k-r/2}\sum_{m\ge 1} m^{-k}\chi_{(-1)^k a}(m)$$
$$= 2^{(r-1)/2}\chi_{(-1)^k 8}(r)a^{k-r/2}L(k,\chi_{(-1)^k a})$$

if $(-1)^k a \equiv 0 \pmod 4$, and

$$a^{k-r}\sum_{\substack{m\ge 1\\ (m,a)=1}} m^{-k}\mathfrak{g}_{k,r}(m,a) = \chi_{(-1)^k}(r)a^{k-r/2}\sum_{m\ge 1} m^{-k}\chi_{(-1)^k a}(m) \tag{2.17}$$
$$= \chi_{(-1)^k}(r)a^{k-r/2}L(k,\chi_{(-1)^k a})$$

if $(-1)^k a \equiv 1 \pmod 4$. If $(-1)^k a \equiv 2, 3 \pmod 4$, the sum is zero. (1.3) yields

$$\sum_{a=0}^{N-1} R_r(a,N) B_k(a/N) = -\frac{\chi_{(-1)^k}(r) k}{N^{k-r}} \sum_{d|N} h_r(k,d) \tag{2.18}$$

as desired. ■

We are now prepared to prove the main theorem.

Proof of Theorem 1.1. Combining Proposition 2.5 with Lemma 2.2(ii) yields

$$H_r(k,n) = -\frac{\chi_{(-1)^k}(r)}{k} \sum_{d|n} \lambda(n/d) d^{k-r} \sum_{a=0}^{d-1} R_r(a,d) B_k(a/d). \tag{2.19}$$

A straightforward calculation involving formal Dirichlet series shows that, for any series $\sum_{n\ge 1} b(n) n^{-s}$,

$$\frac{\zeta(2s)}{\zeta(s)} \sum_{n\ge 1} b(n) n^{-s} = \sum_{n\ge 1} \Big(\sum_{d|n} \lambda(n/d) b(d) \Big) n^{-s}.$$

Applying this to (2.19), we obtain the theorem. ■

Proof of Corollary 1.2. This is (2.19) in the proof of Theorem 1.1. ■

Proof of Corollary 1.3. If we define a formal product

$$F(q) = \prod_{n=1}^{\infty} (1-q^n)^{c(n)}, \tag{2.20}$$

then one obtains formally

$$\frac{q\frac{d}{dq}(F(q))}{F(q)} = -\sum_{n=1}^{\infty} \Big(\sum_{d|n} c(d) d \Big) q^n. \tag{2.21}$$

Combining Proposition 2.5 with Lemma 2.2(i), we get

$$-\frac{\chi_{(-1)^k}(r)}{n^{r-k} k} \sum_{a=0}^{n-1} R_r(a,n) B_k(a/n) = \sum_{\substack{d|n \\ n/d \text{ square-free}}} H_r(k,d). \tag{2.22}$$

The result follows from this and (2.21) when $r = 1$. ■

3. Remarks. We remark that these calculations can be carried out in similar settings with similar results. We state some interesting formulas that are obtained when r is even in the above calculations.

THEOREM 3.1. *With notation as above*:

(i) *If* $r \equiv 2 \pmod 4$ *and* k *is even*,

$$\frac{1}{B_k}\sum_{a=0}^{n-1} R_r(a,n)B_k(a/n) = n^{r-k}\sum_{\substack{d|n\\ 2\nmid d}}\left(\frac{-1}{d}\right)d^{k-r/2}\prod_{p|d}\left(1-\frac{1}{p^k}\right).$$

(ii) *If* $r \equiv 2 \pmod 4$ *and* k *is odd*,

$$\frac{(-1)^{(r-2)/4}}{2^{r/2}B_k(1/4)}\sum_{a=0}^{n-1} R_r(a,n)B_k(a/n) = n^{r-k}\sum_{\substack{d|n\\ 4|d}} d^{k-r/2}\prod_{\substack{p|d\\ p\neq 2}}\left(1-\frac{\left(\frac{-1}{p}\right)}{p^k}\right).$$

(iii) *If* $r \equiv 0 \pmod 4$ *and* k *is even*,

$$\frac{1}{B_k}\sum_{a=0}^{n-1} R_r(a,n)B_k(a/n) = n^{r-k}\sum_{d|n} d^{k-r/2}\prod_{p|d}\left(1-\frac{1}{p^k}\right)\eta(d), \qquad \textit{where}$$

$$\eta(d) = \begin{cases} 1 & \textit{if } 2\nmid d,\\ 0 & \textit{if } d\equiv 2 \pmod 4,\\ (-1)^{r/4}2^{r/2} & \textit{if } 4\,|\,d. \end{cases}$$

(iv) *If* $r \equiv 0 \pmod 4$ *and* k *is odd*,

$$\sum_{a=0}^{n-1} R_r(a,n)B_k(a/n) = 0.$$

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