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The reduced length of a polynomial with complex coefficients

by

A. Schinzel (Warszawa)

Dedicated to Wolfgang M. Schmidt
on the occasion of his 75th birthday

For $P \in \mathbb{C}[x]$, $P(x) = \sum_{i=0}^{d} a_i x^{d-i} = a_0 \prod_{i=1}^{d}(x - \alpha_i)$, let

$$P^*(x) = \sum_{i=0}^{d} a_i x^i, \qquad H(P) = \max_{0 \le i \le d} |a_i|,$$

$$L(P) = \sum_{i=0}^{d} |a_i|, \qquad M(P) = |a_0| \prod_{i=1}^{d} \max\{1, |\alpha_i|\},$$

$$l(P) = \inf L(PG), \qquad \widehat{l}(P) = \min\{l(P), l(P^*)\},$$

where G runs through all monic polynomials in $\mathbb{C}[x]$. This notation is consistent with that of [3] and [4], since if $P \in \mathbb{R}[x]$ the above infimum coincides with $\inf L(PG)$, where G runs through all monic polynomials in $\mathbb{R}[x]$. Some of the results about $l(P)$ stated in [1] and [3] for $P \in \mathbb{R}[x]$ carry over with essentially the same proof to $P \in \mathbb{C}[x]$. Thus we have

Proposition 1. *Suppose that $\omega, \eta, \psi \in \mathbb{C}$, $|\omega| \ge 1$, $|\eta| < 1$. Then for every $Q \in \mathbb{C}[x]$,*

(i) $l(\psi Q) = |\psi| l(Q)$,
(ii) $l(x + \omega) = 1 + |\omega|$,
(iii) *if* $T(x) = Q(x)(x - \eta)$, *then* $l(T) = l(Q)$,
(iv) $l(\overline{Q}) = l(Q)$, *where* $\overline{Q}$ *denotes the complex conjugate of* Q.

Proposition 2. *For all monic polynomials P, Q in $\mathbb{C}[x]$, all $\eta \in \mathbb{C}$ with $|\eta| = 1$ and all positive integers k,*

(i) $\max\{l(P), l(Q)\} \le l(PQ) \le l(P)l(Q)$,
(ii) $M(P) \le l(P)$,

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(iii) $l(P(\eta x)) = l(P(x))$,
(iv) $l(P(x^k)) = l(P(x))$.

THEOREM 1. *Let $P, Q \in \mathbb{C}[x]$, Q be monic and have all zeros on the unit circle. Then for all $m \in \mathbb{N}$,*

$$l(PQ^m) = l(PQ).$$

THEOREM 2. *If $P \in \mathbb{C}[x] \setminus \mathbb{C}$ is monic and has all zeros on the unit circle, then $\widehat{l}(P) = l(P) = 2$, with $l(P)$ attained if all zeros are roots of unity and simple ($l(P)$ is attained means that $l(P) = L(Q)$, where Q/P is a monic polynomial).*

Theorems 1 and 2 correspond to Theorems 3 and 4 of [3], respectively. Also Theorem 6 of [3] extends to polynomials over $\mathbb{C}$, but the extension requires a different proof. We shall prove the following more general

THEOREM 3. *Let $P = P_0 P_1$, where $P_\nu \in \mathbb{C}[x]$ $(\nu = 1, 2)$, $L(P_0) \le 2|P_0(0)|$ and P_1 is monic. Then*

$$l(P) \ge L(P_0) + (2|P_0(0)| - L(P_0))(l(P_1) - 1).$$

COROLLARY 1. *If $P \in \mathbb{C}[x]$ and $L(P) \le 2|P(0)|$, then*

$$l(P) = L(P).$$

Conversely, if $l(P) = L(P)$ and all coefficients of P are real and positive, then $L(P) \le 2P(0)$.

COROLLARY 2. *If $P(x) = (x - \alpha)(x - \beta)$, where $|\alpha| \ge |\beta| \ge 1$, then*

$$l(P) \ge 1 + |\alpha| - |\beta| + |\alpha\beta|, \tag{1}$$

with equality if $\alpha/\beta \in \mathbb{R}$ and either $\alpha/\beta < 0$ or $|\beta| = 1$.

One can prove that if $\alpha/\beta + \beta/\alpha \in \mathbb{R}$, the two cases given in Corollary 2 are the only ones for which there is equality in (1).

COROLLARY 3. *If $P(x) = (x - \alpha)(x - \beta)$, where $|\alpha| \ge |\beta| \ge 1$, then*

$$l(P) \ge 2|\alpha|,$$

with equality only possible if $|\beta| = 1$. If moreover $\alpha/\beta \in \mathbb{R}$, then the equality really holds.

COROLLARY 4. *Let $P = P_0 P_1$, where $P_\nu \in \mathbb{C}[x]$ $(\nu = 0, 1)$, $\deg P_1 \ge 1$ and all zeros z of P_ν satisfy $|z| > 1$ for $\nu = 0$, $|z| = 1$ for $\nu = 1$. If*

$$l(P_0) = L(P_0), \tag{2}$$

then

$$l(P) \ge 2M(P). \tag{3}$$

It remains a problem whether (3) holds without the assumption (2). The following results point towards an affirmative answer.

THEOREM 4. *If $P \in \mathbb{C}[x] \setminus \{0\}$ has a zero z with $|z| = 1$, then*

$$L(P) > \sqrt{2}\, M(P), \quad l(P) \geq \sqrt{2}\, M(P).$$

THEOREM 5. *If $P(x) = (x - \alpha)(x - \beta)(x - 1)$, where α, β are real and at least one of them is positive, then* (3) *holds.*

The validity of (3) for all polynomials P over $\mathbb{C}$ or over $\mathbb{R}$ with a zero on the unit circle is equivalent to the validity of a simpler inequality $L(P) \geq 2M(P)$ for all polynomials P over $\mathbb{C}$ or $\mathbb{R}$, respectively, with a zero on the unit circle. E. Dobrowolski has verified that the latter inequality is true for all such polynomials $P \in \mathbb{C}[x]$ of degree at most 4.

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Proof of Theorem 3. Let G be any monic polynomial in $\mathbb{C}[x]$ and let

$$T(x) = P_1(x)G(x) = x^n + \sum_{i=1}^{n} b_i x^{n-i}, \qquad b_0 = 1.$$

We have

$$\sum_{i=1}^{n} |b_i| = L(T) - 1 \geq l(P_1) - 1. \tag{4}$$

Now, let

$$P_0(x) = \sum_{i=0}^{d} a_i x^{d-i}.$$

We have

$$\begin{aligned} Q(x) = P(x)G(x) = P_0(x)T(x) &= \Big(\sum_{i=0}^{d} a_i x^{d-i}\Big)\Big(\sum_{i=0}^{n} b_i x^{n-i}\Big) \\ &= \sum_{j=0}^{d+n} \Big(\sum_{i=0}^{\min\{j,d\}} a_i b_{j-i}\Big) x^{d+n-j}. \end{aligned}$$

Now

$$\Big|\sum_{i=0}^{\min\{j,d\}} a_i b_{j-i}\Big| \geq \begin{cases} |a_j| - \sum_{i=0}^{j-1} |a_i|\,|b_{j-i}| & \text{for } j \leq d, \\ |a_d|\,|b_{j-d}| - \sum_{i=0}^{d-1} |a_i|\,|b_{j-i}| & \text{for } j > d, \end{cases}$$

hence

$$\begin{aligned}L(Q) &= \sum_{j=0}^{d+n}\Big|\sum_{i=0}^{\min\{j,d\}} a_i b_{j-i}\Big| \\ &\ge \sum_{j=0}^{d}|a_j| - \sum_{j=1}^{d}\sum_{i=0}^{j-1}|a_i|\,|b_{j-i}| + \sum_{j=d+1}^{d+n}|a_d|\,|b_{j-d}| - \sum_{j=d+1}^{d+n}\sum_{i=0}^{d-1}|a_i|\,|b_{j-i}| \\ &= L(P_0) + |a_d|(L(T)-1) - \sum_{j=1}^{d+n}\sum_{i=0}^{\min\{j-1,\,d-1\}}|a_i|\,|b_{j-i}| \\ &= L(P_0) + |P_0(0)|(L(T)-1) - (L(P_0)-|P_0(0)|)(L(T)-1)\end{aligned}$$

and since, by the assumption, $|2P_0(0)| - L(P_0) \ge 0$ it follows from (4) that

$$L(Q) \ge L(P_0) + (2|P_0(0)| - L(P_0))(l(P_1)-1).$$

Proof of Corollary 1. In order to obtain the first statement we take $P_1 = 1$ in Theorem 3. In order to obtain the second statement, let

$$P(x) = \sum_{i=0}^{d} a_i x^{d-i}, \qquad \eta = \min_{0<i\le d}\frac{a_i}{a_{i-1}}$$

and assume that $L(P) > 2a_d$. Then

$$L(P(x)(x-\eta)) = a_0 + \sum_{i=1}^{d}(a_i - a_{i-1}\eta) + a_d\eta = L(P) - (L(P)-2a_d)\eta < L(P).$$

Proof of Corollary 2. Taking $P_0 = x-\alpha$, $P_1 = x-\beta$ in Theorem 3 and using Corollary 1 to evaluate $l(P_1)$ we obtain (1). If $\alpha/\beta \in \mathbb{R}$ and $\alpha/\beta < 0$ we have $L(P) = 1 + |\alpha+\beta| + |\alpha\beta| = 1 + |\alpha| - |\beta| + |\alpha\beta|$, hence $l(P) = L(P)$. If $\alpha/\beta \in \mathbb{R}$, $\alpha/\beta > 0$ and $|\beta| = 1$, then for $|\alpha| = 1$ we have $l(P) = 2 = 1 + |\alpha| - |\beta| + |\alpha\beta|$ by Theorem 2. For $|\alpha| > 1$ we infer from the divisibility

$$P \,\Big|\, x^{n+1} - \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha^n-\beta^n}\,x^n + (\alpha\beta)^n\,\frac{\alpha-\beta}{\alpha^n-\beta^n}$$

that

$$\begin{aligned}l(P) &\le 1 + \lim_{n\to\infty}\Big|\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha^n-\beta^n}\Big| + \lim_{n\to\infty}\Big|\alpha^n\,\frac{\alpha-\beta}{\alpha^n-\beta^n}\Big| = 1 + |\alpha| + |\alpha-\beta| \\ &= 1 + |\alpha\beta| + |\alpha| - |\beta|,\end{aligned}$$

hence again $l(P) = 1 + |\alpha| - |\beta| + |\alpha\beta|$.

Proof of Corollary 3. The first part of the corollary follows from the first part of Corollary 2 and the identity

$$1 + |\alpha| - |\beta| + |\alpha\beta| - 2|\alpha| = (|\alpha|-1)(|\beta|-1).$$

The second part follows from the second part of Corollary 2.

Proof of Corollary 4. Multiplying P_0 by a constant we may assume that P_1 is monic. If $l(P_0) \geq 2M(P_0)$ we have

$$l(P) \geq l(P_0) \geq 2M(P_0) = 2M(P).$$

If $l(P_0) < 2M(P_0) = 2|P_0(0)|$ we have $L(P_0) < 2|P_0(0)|$ by (2), and since, by Theorem 2, $l(P_1) = 2$, Theorem 3 gives

$$l(P) \geq L(P_0) + 2|P_0(0)| - L(P_0) = 2M(P_0) = 2M(P).$$

For the proof of Theorem 4 we need

Lemma 1. *If $P \in \mathbb{C}[x]$ has at least one zero ε with $|\varepsilon| = 1$, then*

$$L(P) \geq 2H(P).$$

Proof. Let $P(x) = (x - \varepsilon)\sum_{i=0}^{d-1} b_i x^{d-i-1}$, where $|\varepsilon| = 1$. We have

$$P(x) = \sum_{i=0}^{d} (b_i - \varepsilon b_{i-1}) x^{d-i}, \quad \text{where} \quad b_{-1} = b_d = 0.$$

Assuming that

$$H(P) = |b_j - \varepsilon b_{j-1}|$$

we have

$$\begin{aligned} L(P) &= \sum_{i=0}^{j-1} |b_i - \varepsilon b_{i-1}| + H(P) + \sum_{i=j+1}^{d} |b_i - \varepsilon b_{i-1}| \\ &\geq \sum_{i=0}^{j-1} (|b_i| - |b_{i-1}|) + H(P) + \sum_{i=j+1}^{d} (|b_{i-1}| - |b_i|) \\ &\geq |b_{j-1}| + H(P) + |b_j| \geq 2H(P). \end{aligned}$$

Proof of Theorem 4. Let $P(x) = \sum_{i=0}^{d} a_i x^{d-i}$. Then

$$\|P\|^2 := \sum_{i=0}^{d} |a_i|^2 \leq H(P)L(P)$$

and, by Lemma 1,

$$L(P)^2 \geq 2\|P\|^2. \tag{5}$$

However, P has at least two non-zero coefficients, hence by Theorem 40 of [2],

$$\|P\| > M(P). \tag{6}$$

The first inequality of Theorem 4 follows from (5) and (6). Hence for every monic G in $\mathbb{C}[x]$, $L(PG) > \sqrt{2}\, M(PG) \geq \sqrt{2}\, M(P)$, which implies $l(P) \geq \sqrt{2}\, M(P)$.

For the proof of Theorem 5 we need five lemmas.

Lemma 2. *If $k > l \geq 1$, the function*

$$g(x) = \frac{x^k - 1}{x^l - 1}$$

is strictly increasing for $x > 1$.

Proof. We have

$$g'(x) = \frac{x^{l-1} f(x)}{(x^l - 1)^2}, \quad \text{where} \quad f(x) = (k-l)x^k - kx^{k-l} + l.$$

Now $f(1) = 0$, $f'(x) = (k-l)k(x^{k-1} - x^{l-1}) > 0$ for $x > 1$, hence $f(x) > 0$ and $g(x)$ is strictly increasing.

Lemma 3. *Let $\alpha, \beta \in \mathbb{R}$ and $\alpha \geq \beta > 1$, $k > l \geq 1$,*

$$D(k, l; \alpha, \beta) = \begin{cases} (\alpha^k - 1)(\beta^l - 1) - (\alpha^l - 1)(\beta^k - 1) & \text{if } \alpha \neq \beta, \\ k\alpha^{k-1}(\alpha^l - 1) - l\alpha^{l-1}(\alpha^k - 1) & \text{if } \alpha = \beta. \end{cases}$$

Then

$$D(k, l; \alpha, \beta) > 0. \tag{7}$$

Proof. For $\alpha > \beta$ we have, in the notation of Lemma 2,

$$D(k, l; \alpha, \beta) = (\beta^k - 1)(\beta^l - 1)(g(\alpha) - g(\beta)),$$

and (7) follows from Lemma 2. For $\alpha = \beta$ we have

$$D(k, l; \alpha, \beta) = \alpha^{l-1} f(\alpha),$$

and (7) follows from the inequality $f(x) > 0$ for $x > 1$ established in the proof of Lemma 2.

Lemma 4. *If $P(x) = (x-\alpha)(x-\beta)(x-1)$, $\alpha \geq \beta > 1$, then every monic polynomial divisible by P with at most four non-zero coefficients is of the form*

$$x^m + ax^n + bx^p + c,$$

where $m > n > p > 0$ and

$$a = -\frac{D(m, p; \alpha, \beta)}{D(n, p; \alpha, \beta)}, \ b = \frac{D(m, n; \alpha, \beta)}{D(n, p; \alpha, \beta)}, \ c = -(\alpha\beta)^p \frac{D(m-p, n-p; \alpha, \beta)}{D(n, p; \alpha, \beta)}.$$

Proof. The above values of a, b, c are obtained by solving the systems of linear equations

$$\alpha^m + a\alpha^n + b\alpha^p + c = 0, \tag{8}$$

$$\beta^m + a\beta^n + b\beta^p + c = 0, \tag{9}$$

$$1 + a + b + c = 0$$

if $\alpha > \beta$, and

$$m\alpha^{m-1} + na\alpha^{n-1} + pb\alpha^{p-1} = 0,$$
$$\alpha^m + a\alpha^n + b\alpha^p + c = 0,$$
$$1 + a + b + c = 0$$

otherwise, with the determinant $D(n, p; \alpha, \beta)$, which is non-zero by virtue of Lemma 3.

LEMMA 5. *If r is a positive integer, and $t, x \geq 1$, then*

$$\frac{t^r x^{r+1}}{1 + t + \cdots + t^{r-1}} \geq tx^2 - 1.$$

Proof. For $r = 1$ the inequality is clear. For $r \geq 2$ let t_0 be the unique positive root of the equation

$$h(t) = 2 + 2t + \cdots + 2t^{r-1} - (r+1)t^{r-1} = 0.$$

We have

$$(10) \qquad 1 < t_0 < \frac{r+1}{r-1},$$

since $h(1) = r - 1 > 0$ and $h\left(\frac{r+1}{r-1}\right) = 1 - r < 0$.

Put

$$x_0(t) = \left(\frac{2 + 2t + \cdots + 2t^{r-1}}{(r+1)t^{r-1}}\right)^{1/(r-1)}.$$

The function

$$F(t, x) = t^r \frac{x^{r+1}}{1 + \cdots + t^{r-1}} - tx^2 + 1$$

is decreasing for $x < x_0(t)$ and increasing for $x > x_0(t)$. If $t < t_0$ we have $x_0(t) > 1$, if $t \geq t_0$ we have $x_0(t) \leq 1 \leq x$. Therefore, for $t \geq t_0$,

$$F(t, x) \geq F(t, 1) = \frac{1}{1 + \cdots + t^{r-1}} > 0.$$

For $t < t_0$ we have

$$\begin{aligned} F(t, x) \geq F(t, x_0(t)) &= t^r \frac{x_0(t)^{r+1}}{1 + \cdots + t^{r-1}} - tx_0(t)^2 + 1 \\ &= tx_0(t)^2 \left(\frac{t^{r-1}}{1 + \cdots + t^{r-1}} x_0(t)^{r-1} - 1\right) + 1 = tx_0(t)^2 \left(\frac{2}{r+1} - 1\right) + 1 \\ &= 1 - \frac{r-1}{r+1} tx_0(t)^2. \end{aligned}$$

Assuming that the right-hand side is negative we obtain

$$tx_0(t)^2 > \frac{r+1}{r-1},$$

thus

$$t^{(r-1)/2}\cdot 2\cdot\frac{1+\cdots+t^{r-1}}{(r+1)t^{r-1}}>\left(\frac{r+1}{r-1}\right)^{(r-1)/2}$$

and

$$2(t^{-(r-1)/2}+\cdots+t^{(r-1)/2})>\left(\frac{r+1}{r-1}\right)^{(r-1)/2}(r+1).$$

The function $t^{-1}+t$ is increasing for $t\ge 1$ and so is $t^{-(r-1)/2}+\cdots+t^{(r-1)/2}$. Hence $t<t_0$ implies

$$2\,\frac{t_0^r-1}{t_0-1}\,t_0^{-(r-1)/2}=2(t_0^{-(r-1)/2}+\cdots+t_0^{(r-1)/2})>\left(\frac{r+1}{r-1}\right)^{(r-1)/2}(r+1),$$

thus, by the definition of t_0,

$$(r+1)t_0^{(r-1)/2}>\left(\frac{r+1}{r-1}\right)^{(r-1)/2}(r+1)$$

and

$$t_0>\frac{r+1}{r-1}$$

contrary to (10).

Lemma 6. *If $\alpha\ge\beta>1$, then in the notation of Lemma* 4,

$$a<0,\qquad b\ge\alpha\beta-1,\qquad c<0.$$

Proof. By Lemma 3 we have in this case $a<0$, $b>0$, $c<0$, hence

$$\alpha^m+|b|\alpha^p=|a|\alpha^n+|c|<(|a|+|c|)\alpha^n$$

and

$$b+1=|a|+|c|>\alpha^{n-m}\ge\alpha\beta,$$

unless $m=n+1$.

Consider first the case $\alpha>\beta$. Assuming $m=n+1$ we infer from (7) and (8) that

$$\alpha-\beta+b(\alpha^{p-n}-\beta^{p-n})+c(\alpha^{-n}-\beta^{-n})=0,$$

and since $\alpha^{p-n}-\beta^{p-n}<0$ and $c(\alpha^{-n}-\beta^{-n})>0$,

$$b\ge\frac{\alpha-\beta}{\beta^{p-n}-\alpha^{p-n}}.$$

Putting $n-p=r$, $\alpha=t\beta$ we obtain $t\ge 1$, and by Lemma 5,

$$b\ge\frac{t^r\beta^{r-1}}{1+\cdots+t^{r-1}}\ge t\beta^2-1=\alpha\beta-1.$$

Consider now the case $\alpha=\beta$. Then by the case already proved,

$$b=\lim_{\beta\to\alpha-0}\frac{D(m,n;\alpha,\beta)}{D(n,p;\alpha,\beta)}\ge\lim_{\beta\to\alpha-0}(\alpha\beta-1)=\alpha^2-1.$$

Proof of Theorem 5. Let $|\alpha| \geq |\beta|$. If $|\alpha| < 1$, then by Proposition 1(ii), (iii),

$$l(P) = l(x-1) = 2 = 2M(P).$$

If $|\beta| < 1 \leq |\alpha|$, then by Proposition 1(iii) and by Corollary 3,

$$l(P) = l((x-\alpha)(x-1)) = 2|\alpha| = 2M(P).$$

If $\beta = 1$ the same is true by Theorem 1 and Corollary 3. If $|\beta| \geq 1$ and $\alpha/\beta < 0$, then by Corollary 2,

$$l(P) = l((x-\alpha)(x-\beta)) = L((x-\alpha)(x-\beta)),$$

hence, by Corollary 4,

$$l(P) \geq 2M(P).$$

Finally, if $|\beta| \geq 1$, $\beta \neq 1$ and $\alpha/\beta > 0$, then by the assumption that at least one of α, β is positive we have $\alpha \geq \beta > 1$ and by Theorem 1 of [3],

$$l(P) = \inf_{Q \in S_3(P)} L(Q),$$

while, by Lemma 4, each element Q of $S_3(P)$ is of the form $Q = x^m + ax^n + bx^p + c$, where a, b, c are given by the formulae of Lemma 4. Now, by Lemma 6,

$$a < 0, \quad b \geq \alpha\beta - 1, \quad c < 0,$$

hence $L(Q) = L(Q) + Q(1) = 2(1+b) \geq 2\alpha\beta = 2M(P)$.

Note added in proof. Concerning Proposition 2(ii) E. Dobrowolski has observed that if $M(P) > |a_0|$, then $l(P) > M(P)$. Indeed, then for every monic Q we have $L(PQ) \geq |a_0| + \sqrt{\|PQ\|^2 - |a_0|^2} > |a_0| + \sqrt{M(PQ)^2 - |a_0|^2} \geq |a_0| + \sqrt{M(P)^2 - |a_0|^2}$, hence $l(P) \geq |a_0| + \sqrt{M(P)^2 - |a_0|^2} > M(P)$.

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Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8, P.O. Box 21
00-956 Warszawa, Poland
E-mail: schinzel@impan.gov.pl

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