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On the Linnik–Sprindžuk theorem about the zeros of L-functions

by

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Dedicated to Professor W. M. Schmidt on the occasion of his 75th birthday

1. Introduction. One of the many mysteries of the zeros of L-functions is embodied by the following theorem of Sprindžuk [11], [12], obtained by a development of Linnik's [8] ideas. Assume the Riemann Hypothesis (RH) for the Riemann zeta function $\zeta(s)$; then the generalized RH for the Dirichlet L-functions is equivalent to the asymptotic formulae

$$\sum_{\gamma} |\gamma|^{i\gamma} e^{-i\gamma-\pi|\gamma|/2}\left(x+2\pi i\,\frac{a}{q}\right)^{-1/2-i\gamma} = -\frac{\mu(q)}{x\sqrt{2\pi}\,\varphi(q)} + O(x^{-1/2-\varepsilon})$$

as $x \to 0^+$, where γ runs over the imaginary parts of the non-trivial zeros of $\zeta(s)$, and $q \geq 2$ and a are integers with $(a,q)=1$, $0<|a|\leq q/2$. Roughly speaking, the Linnik–Sprindžuk theorem says that the generalized RH is equivalent to RH plus a suitable property of the vertical distribution of the zeros of $\zeta(s)$. Another way of looking at this theorem is to say that the generalized RH is equivalent to RH plus a suitable behaviour of certain "twists" of the zeta-zeros. In other words, the zeros of $\zeta(s)$ contain information on the zeros of $L(s,\chi)$, and conversely. Such a result has been extended in various ways by Fujii [2]–[4] and by Suzuki [13]. In particular, Suzuki [13] extended the Linnik–Sprindžuk theorem to the Selberg class $\mathcal{S}$ of L-functions, thus obtaining a similar relation between the zeros of a function $F(s)$ and those of the twists $F(s,\chi)$ by primitive Dirichlet characters, provided both $F(s)$ and $F(s,\chi)$ belong to $\mathcal{S}$.

Our aim in this paper is to obtain a different form of the above Linnik–Sprindžuk phenomenon. We formulate our results in the framework of the

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Selberg class $\mathcal{S}$, defined as follows. Every function $F \in \mathcal{S}$ is a Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

absolutely convergent for $\sigma > 1$, and there exists an integer $m \geq 0$ such that $(s-1)^m F(s)$ is an entire function of finite order; the minimum of such integers is denoted by m_F. Moreover, $F(s)$ satisfies a functional equation of type

$$(1) \qquad Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) F(s) = \omega Q^{1-s} \prod_{j=1}^{r} \Gamma(\lambda_j(1-s) + \overline{\mu}_j) \overline{F}(1-s),$$

where $\overline{F}(s) = \overline{F(\overline{s})}$, $|\omega| = 1$, $Q > 0$, $\lambda_j > 0$ and $\Re\mu_j \geq 0$. In addition, $a(n) \ll n^{\varepsilon}$ for every $\varepsilon > 0$, and $F(s)$ has an Euler product satisfying

$$\log F(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s},$$

where $b(n) = 0$ unless $n = p^k$ with $k \geq 1$, and $b(n) \ll n^{\vartheta}$ for some $\vartheta < 1/2$. We refer to our surveys [5], [6], [9] and [10] for the basic theory of the Selberg class. We also use the notation

$$\psi(s) = \frac{\Gamma'}{\Gamma}(s)$$

for the logarithmic derivative of $\Gamma(s)$.

For $F \in \mathcal{S}$ and $\alpha \in \mathbb{R} \setminus \{0\}$ we write

$$H_F(s,\alpha) = \sum_{\varrho} \Gamma(\varrho - s)(2\pi i \alpha)^{s-\varrho},$$

$$G_F(s,\alpha) = H_F(s,\alpha) + \sum_{j=1}^{r} \lambda_j \psi(\lambda_j s + \mu_j),$$

where ϱ runs over the non-trivial zeros of $F(s)$. By the Riemann–von Mangoldt and Stirling's formulae, the series for $H_F(s,\alpha)$ converges absolutely and uniformly on compact sets for $\sigma > 3/2$ and $s \neq \varrho + l$ with $l = 1, 2, \ldots$ (see beginning of the next section). Moreover, assuming the General Riemann Hypothesis (GRH) for $F(s)$, the above condition $\sigma > 3/2$ can be replaced by $\sigma > 1$. The analytic properties of $G_F(s,\alpha)$ are given by the following theorem.

THEOREM 1. *Let $F \in \mathcal{S}$ and $m \in \mathbb{Z}\setminus\{0\}$. Then $G_F(s,m)$ is meromorphic on $\mathbb{C}$. Moreover, $G_F(s,m)$ is holomorphic for $\sigma < 1$, while for $\sigma \geq 1$ it has simple poles at $s = \varrho + k$, where ϱ runs over the non-trivial zeros of $F(s)$ and $k = 1, 2, \ldots,$ and at $s = 1$ if $m_F \neq 0$.*

Theorem 1 immediately yields the analytic properties of $H_F(s,m)$.

Corollary 1. *Let $F \in \mathcal{S}$ and $m \in \mathbb{Z} \setminus \{0\}$. Then $H_F(s,m)$ is meromorphic on $\mathbb{C}$. Moreover, $H_F(s,m)$ has simple poles at the points $s = -(\mu_j + k)/\lambda_j$, $j = 1,\dots,r$ and $k = 0,1,\dots,$ for $\sigma < 1$, while for $\sigma \ge 1$ it has simple poles at $s = \varrho + k$, where ϱ runs over the non-trivial zeros of $F(s)$ and $k = 1,2,\dots,$ and at $s = 1$ if $m_F \neq 0$.*

Note that the polar structure of $H_F(s,m)$ does not depend on m; this is already clear for $\sigma > 3/2$ from the convergence properties of the series for $H_F(s,\alpha)$. Note also that the poles of $H_F(s,m)$ in the half-plane $\sigma < 1$ almost coincide with the trivial zeros of $F(s)$, the only difference occurring at $s = 0$ if $m_F \neq 0$; moreover, such poles lie in the half-plane $\sigma \le 0$.

Let now $\chi \pmod q$, $q \ge 2$, be a primitive Dirichlet character and write

$$l^*(s,\chi) = 2\chi(-1)\omega_\chi q^{s-1/2} l(s,\overline{\chi}) \cos\left(\frac{\pi(s + a(\chi))}{2}\right),$$

where

$$l(s,\chi) = \sum_{0<a<q/2} \frac{\chi(a)}{a^s}, \qquad a(\chi) = \begin{cases} 0 & \text{if } \chi(-1) = 1, \\ 1 & \text{if } \chi(-1) = -1, \end{cases} \qquad \omega_\chi = \frac{\tau(\chi)}{i^{a(\chi)}\sqrt{q}}$$

and $\tau(\chi)$ is the Gauss sum. Moreover, let

$$F(s,\chi) = \sum_{n=1}^{\infty} \frac{a(n)\chi(n)}{n^s}$$

be the twist of $F(s)$ by χ, and write

$$H_F(s,\chi) = \sum_{\varrho} \Gamma(\varrho - s)(2\pi)^{s-\varrho}\, l^*(\varrho - s,\chi),$$

$$G_F(s,\chi) = H_F(s,\chi) - \frac{F'}{F}(s,\chi),$$

where again the summation is over the non-trivial zeros of $F(s)$. The function $H_F(s,\chi)$ is a kind of twist of $H_F(s,\alpha)$ (see Lemma 4 below), and its convergence properties are similar to those of $H_F(s,\alpha)$ (i.e. convergence for $\sigma > 3/2$ with $s \neq \varrho + l$, and for $\sigma > 1$ under GRH). We have

Theorem 2. *Let $F \in \mathcal{S}$ and $\chi \pmod q$, $q \ge 2$, be a primitive Dirichlet character. Then $G_F(s,\chi)$ is meromorphic on $\mathbb{C}$. Moreover, $G_F(s,\chi)$ is holomorphic for $\sigma < 1$, while for $\sigma \ge 1$ it has simple poles at $s = \varrho + k$, where ϱ runs over the non-trivial zeros of $F(s)$ and $k = 1,2,\dots,$ provided $l^*(-k,\overline{\chi}) \neq 0$, and at $s = 1$ if $m_F \neq 0$ and $l^*(-1,\overline{\chi}) \neq 0$.*

We briefly discuss the meaning of Theorem 2 after Corollary 2 below. Note that the zeros of $l^*(-k,\overline{\chi})$ come from those of $l(-k,\overline{\chi})$ and of $\cos(\pi(-k + a(\chi))/2)$, and the zeros of the latter are easily described; in particular, the cosine factor cancels the poles of $G_F(s,\chi)$ in infinitely many strips of type $k \le \sigma \le k+1$.

Given $F \in \mathcal{S}$, it is expected that the conductor $q_F = (2\pi)^{d_F} Q^2 \prod_{j=1}^r \lambda_j^{2\lambda_j}$ is an integer (where $d_F = 2\sum_{j=1}^r \lambda_j$ is the degree and Q, λ_j are given by (1)), and that the twists $F(s,\chi)$ belong to the class $\mathcal{S}$ for every primitive character $\chi \pmod q$ with $(q,q_F)=1$; see [7]. However, at present nothing is known in general about the analytic properties of $F(s,\chi)$ outside the half-plane $\sigma > 1$ of absolute convergence, although the above twist conjecture is known to hold for most classical L-functions. Theorem 2 shows that the continuation properties of $F(s,\chi)$ and of $H_F(s,\chi)$ are closely related; in particular, the meromorphic continuation to the whole complex plane of $F(s,\chi)$ is equivalent to that of $H_F(s,\chi)$, and more precise information can be obtained assuming the twist conjecture. Therefore, in this case we cannot switch from $G_F(s,\chi)$ to the more interesting function $H_F(s,\chi)$ for $\sigma \le 1$. We can, however, immediately deduce the meromorphic continuation of $H_F(s,\chi)$ to $\sigma > 1$ without assuming GRH.

Corollary 2. *Let $F \in \mathcal{S}$ and $\chi \pmod q$, $q \ge 2$, be a primitive Dirichlet character. Then $H_F(s,\chi)$ is meromorphic for $\sigma > 1$ with simple poles at $s = \varrho + k$, where ϱ runs over the non-trivial zeros of $F(s)$ and $k = 1, 2, \ldots$, provided $l^*(-k,\overline{\chi}) \neq 0$, and at $s = 1$ if $m_F \neq 0$ and $l^*(-1,\overline{\chi}) \neq 0$.*

In the spirit of the Linnik–Sprindžuk theorem, we now assume the twist conjecture and observe the behaviour of the poles when switching from $H_F(s,\alpha)$ to $H_F(s,\chi)$. We first note from Corollary 1 that $H_F(s,m)$ has poles at (essentially) the trivial zeros of $F(s)$, is holomorphic for $0 < \sigma < 1$ and has poles at the shifted non-trivial zeros of $F(s)$ in each strip $k \le \sigma \le k+1$ with integer $k \ge 1$. Then, if we "twist" $H(s,\alpha)$ to get $H_F(s,\chi)$, by Theorem 2 the poles in the strips $k \le \sigma \le k+1$ remain unchanged (if $l^*(-k,\overline{\chi}) \neq 0$) or disappear (if $l^*(-k,\overline{\chi}) = 0$), but for $\sigma < 1$ simple poles at the zeros of $F(s,\chi)$ pop up. In particular, $H_F(s,\chi)$ is defined by means of the non-trivial zeros of $F(s)$, and its poles keep track of the non-trivial zeros of both $F(s)$ and $F(s,\chi)$.

We finally remark that suitable variants of Theorem 2 can be obtained by the arguments in this paper. For example, in the prototypical case of $\zeta(s)$ we may consider functions of type

$$K(s,\chi) = \sum_{\gamma>0} \frac{g^*(\varrho - s, \overline{\chi})}{(\varrho/i)^{s+1/2-\varrho}},$$

where $\varrho = \beta + i\gamma$ runs over the non-trivial zeros of $\zeta(s)$ and

$$g^*(s,\chi) = \left(\frac{q}{2\pi}\right)^s g(s,\chi), \qquad g(s,\chi) = \sum_{a=1}^q \frac{\chi(a)}{a^s}.$$

Then $K(s,\chi)$ is convergent for $\sigma > 3/2$ (for $\sigma > 1$ under RH), has meromorphic continuation to the whole complex plane, and its poles are located at the points $s = \varrho_\chi - k$, with integer $k \geq 0$ and ϱ_χ running over the non-trivial zeros of $L(s,\chi)$, and at $s = k$ with integer $k \leq 1$.

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2. Proofs. Let $\alpha \in \mathbb{R} \setminus \{0\}$, $X > 0$, $z_X(\alpha) = 1/X + 2\pi i\alpha$, $e(x) = e^{2\pi i x}$ and for $\sigma > 1$

$$-\frac{F'}{F}(s) = \sum_{n=1}^{\infty} \frac{b(n)\log n}{n^s} = \sum_{n=1}^{\infty} \frac{\varLambda_F(n)}{n^s},$$

say. By Mellin's transform and then shifting the line of integration to $-\infty$ we have

$$\begin{aligned}(2)\qquad &\sum_{n=1}^{\infty} \frac{\varLambda_F(n)}{n^s} e(-n\alpha)e^{-n/X} \\ &= \frac{1}{2\pi i} \int_{(2)} \left\{-\frac{F'}{F}(s+w)\right\} \varGamma(w) z_X(\alpha)^{-w}\, dw \\ &= m_F \varGamma(1-s) z_X(\alpha)^{s-1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left\{-\frac{F'}{F}(s-k)\right\} z_X(\alpha)^k \\ &\quad - \sum_{\varrho} \varGamma(\varrho - s) z_X(\alpha)^{s-\varrho} - \sum_{j=1}^{r}\sum_{l=0}^{\infty} \varGamma\left(-s - \frac{l+\mu_j}{\lambda_j}\right) z_X(\alpha)^{s+(l+\mu_j)/\lambda_j}\end{aligned}$$

for s different from the poles of the $\varGamma$-functions involved and of $-\frac{F'}{F}(s-k)$. In fact, the series on the right hand side of (2) are convergent and

$$\int_{(-K)} \left\{-\frac{F'}{F}(s+w)\right\} \varGamma(w) z_X(\alpha)^{-w}\, dw \to 0$$

as $K \to \infty$ over a suitable sequence such that the lines $\sigma = K$ are free from the poles of the integrand.

As we already remarked in the Introduction, the series

$$H_F(s,\alpha) = \sum_{\varrho} \varGamma(\varrho - s) z_\infty(\alpha)^{s-\varrho}$$

converges absolutely and uniformly on compact sets for $\sigma > 3/2$ and $s \neq \varrho + l$ with $l = 1, 2, \ldots$ (for $\sigma > 1$ under GRH). Indeed,

$$\varGamma(\varrho - s) \ll e^{-\pi|\gamma|/2} |\gamma|^{1/2-\sigma} \quad \text{and} \quad z_\infty^{s-\varrho} \ll e^{\pi|\gamma|/2},$$

hence for $\varrho - s \notin \mathbb{Z}$,

$$H_F(s,\alpha) \ll \sum_{\varrho} (1+|\gamma|)^{1/2-\sigma},$$

which is convergent for $\sigma > 3/2$ thanks to the Riemann–von Mangoldt formula for the number of zeros in the critical strip (similarly under GRH). Therefore, letting $X \to \infty$ in (2), for $\sigma > 3/2$ and s different from the poles of the Γ-functions involved and of $-\frac{F'}{F}(s-k)$ we get

$$\begin{aligned}(3)\qquad H_F(s,\alpha) = &-\sum_{n=1}^{\infty} \frac{\Lambda_F(n)}{n^s}\, e(-n\alpha) + m_F \Gamma(1-s) z_\infty(\alpha)^{s-1} \\ &+ \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \Big\{ -\frac{F'}{F}(s-k) \Big\} z_\infty(\alpha)^k \\ &- \sum_{j=1}^{r} \sum_{l=0}^{\infty} \Gamma\Big(-s - \frac{l+\mu_j}{\lambda_j} \Big) z_\infty(\alpha)^{s+(l+\mu_j)/\lambda_j}.\end{aligned}$$

We have $z_\infty(\alpha) = 2\pi i \alpha$ and, by the functional equation,

$$-\frac{F'}{F}(s) = 2\log Q + \sum_{j=1}^{r} \lambda_j \psi(\lambda_j s + \mu_j) + \sum_{j=1}^{r} \lambda_j \psi(\lambda_j(1-s) + \overline{\mu}_j) + \frac{\overline{F}'}{\overline{F}}(1-s),$$

hence (3) becomes, for the same values of s,

$$\begin{aligned} H_F(s,\alpha) = &-\sum_{n=1}^{\infty} \frac{\Lambda_F(n)}{n^s}\, e(-n\alpha) + \sum_{n=1}^{\infty} \frac{\Lambda_F(n)}{n^s} + m_F \Gamma(1-s)(2\pi i \alpha)^{s-1} \\ &+ \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \Big\{ 2\log Q + \sum_{j=1}^{r} \lambda_j \psi(\lambda_j(s-k)+\mu_j) \\ &+ \sum_{j=1}^{r} \lambda_j \psi(\lambda_j(1-s+k) + \overline{\mu}_j) + \frac{\overline{F}'}{\overline{F}}(1-s+k) \Big\} (2\pi i \alpha)^k \\ &- \sum_{j=1}^{r} \sum_{l=0}^{\infty} \Gamma\Big(-s - \frac{l+\mu_j}{\lambda_j} \Big) (2\pi i \alpha)^{s+(l+\mu_j)/\lambda_j} \\ = &\sum_{n=1}^{\infty} \frac{\Lambda_F(n)(1-e(-n\alpha))}{n^s} + m_F \Gamma(1-s)(2\pi i\alpha)^{s-1} \\ &+ 2e(-\alpha)\log Q - \sum_{j=1}^{r} \lambda_j \psi(\lambda_j s + \mu_j) \\ &+ \sum_{j=1}^{r} \Big(\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \lambda_j \psi(\lambda_j(s-k)+\mu_j)(2\pi i \alpha)^k \end{aligned}$$

$$
\begin{aligned}
&\qquad - \sum_{l=0}^{\infty} \Gamma\Big(-s - \frac{l+\mu_j}{\lambda_j}\Big)(2\pi i\alpha)^{s+(l+\mu_j)/\lambda_j}\Big) \\
&\qquad + \sum_{j=1}^{r} \lambda_j \sum_{k=1}^{\infty} \frac{(-1)^k}{k!}\, \psi(\lambda_j(1-s+k)+\overline{\mu}_j)(2\pi i\alpha)^k \\
&\qquad + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \frac{\overline{F}'}{\overline{F}}(1-s+k)(2\pi i\alpha)^k \\
&= \sum_{n=1}^{\infty} \frac{\Lambda_F(n)(1-e(-n\alpha))}{n^s} + m_F\Gamma(1-s)(2\pi i\alpha)^{s-1} + 2e(-\alpha)\log Q \\
&\qquad - \sum_{j=1}^{r} \lambda_j\psi(\lambda_j s+\mu_j) + A(s,\alpha) + B(s,\alpha) + C(s,\alpha),
\end{aligned}
$$

say. With this notation, we have proved the following

LEMMA 1. *Let $F \in \mathcal{S}$ and $\alpha \in \mathbb{R}\setminus\{0\}$. Then $H_F(s,\alpha)$ is meromorphic for $\sigma > 3/2$, and*

$$
\begin{aligned}
H_F(s,\alpha) &+ \sum_{j=1}^{r} \lambda_j\psi(\lambda_j s+\mu_j) \\
&= \sum_{n=1}^{\infty} \frac{\Lambda_F(n)(1-e(-n\alpha))}{n^s} + m_F\Gamma(1-s)(2\pi i\alpha)^{s-1} \\
&\quad + 2e(-\alpha)\log Q + A(s,\alpha) + B(s,\alpha) + C(s,\alpha).
\end{aligned}
$$

The functions $B(s,\alpha)$ and $C(s,\alpha)$ are easy to deal with. Indeed, for $\sigma < 2$ and $k \ge 1$ we have $\Re(\lambda_j(1-s+k)+\overline{\mu}_j) \ge \lambda_j(2-\sigma) > 0$, therefore

$$\psi(\lambda_j(1-s+k)+\overline{\mu}_j) \ll_s \log k$$

and hence the series in $B(s,\alpha)$ is absolutely convergent. Thus

(4) $\qquad B(s,\alpha)$ is holomorphic for $\sigma < 2$.

Moreover, for $\sigma < 2$ and $k \ge 2$ we have $\Re(1-s+k) \ge 3-\sigma > 1$, hence

$$C(s,\alpha) = -2\pi i\alpha \frac{\overline{F}'}{\overline{F}}(2-s) + O_s\Big(\sum_{k=2}^{\infty} \frac{(2\pi|\alpha|)^k}{k!}\Big). \tag{5}$$

Therefore the O-term in (5) is holomorphic for $\sigma < 2$, and in particular

(6) $\qquad C(s,\alpha)$ is meromorphic for $\sigma < 2$ and holomorphic for $\sigma < 1$.

In order to study $A(s,\alpha)$ we write

$$A(s,\alpha) = \sum_{j=1}^{r} A(s;\lambda_j,\mu_j,\alpha), \tag{7}$$

where

$$A(s;\lambda,\mu,\alpha)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\,\lambda\psi(\lambda(s-k)+\mu)(2\pi i\alpha)^k - \sum_{l=0}^{\infty}\Gamma\Big(-s-\frac{l+\mu}{\lambda}\Big)(2\pi i\alpha)^{s+(l+\mu)/\lambda}, \tag{8}$$

and prove the following

LEMMA 2. *For $\lambda>0$, $\mu\in\mathbb{C}$ and $\alpha\in\mathbb{R}$ the function $A(s;\lambda,\mu,\alpha)$ is entire.*

Proof. We write (8) as

$$A(s;\lambda,\mu,\alpha)=A_1(s;\lambda,\mu,\alpha)-A_2(s;\lambda,\mu,\alpha) \tag{9}$$

and investigate first $A_2(s;\lambda,\mu,\alpha)$. For $\sigma>0$ and $\xi>0$ we have

$$\Gamma(s)=\int_0^{\infty}e^{-x}x^{s-1}\,dx=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\int_0^{\xi}x^{s+k-1}\,dx+\int_{\xi}^{\infty}e^{-x}x^{s-1}\,dx$$
$$=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\,\frac{\xi^{s+k}}{s+k}+\int_{\xi}^{\infty}e^{-x}x^{s-1}\,dx,$$

and by analytic continuation this holds for every $s\in\mathbb{C}\setminus\{0,-1,-2,\dots\}$. Hence, assuming

$$\xi>2\pi|\alpha|, \tag{10}$$

we have

$$A_2(s;\lambda,\mu,\alpha)=\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\,\frac{\xi^{-s-(l+\mu)/\lambda+k}}{-s-(l+\mu)/\lambda+k}\,(2\pi i\alpha)^{s+(l+\mu)/\lambda}$$
$$+\sum_{l=0}^{\infty}(2\pi i\alpha)^{s+(l+\mu)/\lambda}\int_{\xi}^{\infty}e^{-x}x^{-s-(l+\mu)/\lambda-1}\,dx$$
$$=-\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-\xi)^k}{k!}\Big(\frac{2\pi i\alpha}{\xi}\Big)^{s+(l+\mu)/\lambda}\frac{1}{s+(l+\mu)/\lambda-k}$$
$$+(2\pi i\alpha)^{s+\mu/\lambda}\int_{\xi}^{\infty}e^{-x}x^{-s-\mu/\lambda-1}\sum_{l=0}^{\infty}\Big(\frac{2\pi i\alpha}{x}\Big)^{l/\lambda}dx$$
$$=-\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-\xi)^k}{k!}\Big(\frac{2\pi i\alpha}{\xi}\Big)^{s+(l+\mu)/\lambda}\frac{1}{s+(l+\mu)/\lambda-k}$$
$$+(2\pi i\alpha)^{s+\mu/\lambda}\int_{\xi}^{\infty}e^{-x}x^{-s-\mu/\lambda-1}\frac{1}{1-(2\pi i\alpha/x)^{1/\lambda}}\,dx.$$

Since the last summand is an entire function of s we get

$$(11)\quad A_2(s;\lambda,\mu,\alpha) = -\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-\xi)^k}{k!}\left(\frac{2\pi i\alpha}{\xi}\right)^{s+(l+\mu)/\lambda}\frac{1}{s+(l+\mu)/\lambda-k}+E_1(s,\xi),$$

where $E_1(s,\xi)$ is an entire function.

In order to deal with $A_1(s;\lambda,\mu,\alpha)$ we recall (see eq. (3) on p. 15 of [1]) that for $s\neq 0,-1,\ldots,$

$$\psi(s)=-\gamma+\sum_{l=0}^{\infty}\frac{s-1}{(l+1)(s+l)},$$

where γ is Euler's constant. Hence

$$(12)\quad A_1(s;\lambda,\mu,\alpha) = -\gamma\lambda e(-\alpha)+\lambda\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^k}{k!}\frac{\lambda(s-k)+\mu-1}{(l+1)(\lambda(s-k)+\mu+l)}(2\pi i\alpha)^k$$
$$= -\gamma\lambda e(-\alpha)+\lambda\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-2\pi i\alpha)^k}{k!}\frac{s+(\mu-1)/\lambda-k}{(l+1)(s+(l+\mu)/\lambda-k)}.$$

Thus from (9), (11) and (12) we obtain

$$(13)\quad A(s;\lambda,\mu,\alpha) = \lambda\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-2\pi i\alpha)^k}{k!}\frac{s+(\mu-1)/\lambda-k}{(l+1)(s+(l+\mu)/\lambda-k)}$$
$$+\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-\xi)^k}{k!}\left(\frac{2\pi i\alpha}{\xi}\right)^{s+(l+\mu)/\lambda}\frac{1}{s+(l+\mu)/\lambda-k}+E_2(s,\xi)$$
$$=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-2\pi i\alpha)^k}{k!}\left(\frac{\lambda s+\mu-1-k\lambda}{l+1}+\left(\frac{2\pi i\alpha}{\xi}\right)^{s+(l+\mu)/\lambda-k}\right)\times\frac{1}{s+(l+\mu)/\lambda-k}+E_2(s,\xi)$$
$$=\sum_{\substack{k,l\geq 0\\ |s+(l+\mu)/\lambda-k|<1}}\frac{(-2\pi i\alpha)^k}{k!}\left(\frac{\lambda s+\mu-1-k\lambda}{l+1}+\left(\frac{2\pi i\alpha}{\xi}\right)^{s+(l+\mu)/\lambda-k}\right)\times\frac{1}{s+(l+\mu)/\lambda-k}$$

$$+ \sum_{\substack{k,l\geq 0\\ |s+(l+\mu)/\lambda-k|\geq 1}} \frac{(-2\pi i\alpha)^k}{k!}\left(\frac{\lambda s+\mu-1-k\lambda}{l+1}+\left(\frac{2\pi i\alpha}{\xi}\right)^{s+(l+\mu)/\lambda-k}\right) \times \frac{1}{s+(l+\mu)/\lambda-k}+E_2(s,\xi)$$

$$= S_1(s,\xi)+S_2(s,\xi)+E_2(s,\xi)$$

say, with an entire function $E_2(s,\xi)$. In $S_1(s,\xi)$ we always have $k \ll l \ll k$ and, recalling (10), also

$$\left|\frac{\frac{\lambda s+\mu-1-k\lambda}{l+1}+(2\pi i\alpha/\xi)^{s+(l+\mu)/\lambda-k}}{s+(l+\mu)/\lambda-k}\right| = \left|\frac{\lambda}{l+1}+\frac{(2\pi i\alpha/\xi)^{s+(l+\mu)/\lambda-k}-1}{s+(l+\mu)/\lambda-k}\right| \ll 1,$$

therefore

$$S_1(s,\xi) \ll \sum_{k=0}^{\infty}\frac{1}{k!}\sum_{k\ll l\ll k} 1.$$

Since the series is convergent, $S_1(s,\xi)$ is an entire function. Moreover, again recalling (10) we obtain

$$S_2(s,\xi) \ll \sum_{\substack{k,l\geq 0\\ |s+(l+\mu)/\lambda-k|\geq 1}} \frac{|2\pi\alpha|^k}{k!}\left(\frac{k+1}{l+1}+\left|\frac{2\pi\alpha}{\xi}\right|^{l/\lambda-k}\right)\frac{1}{|s+(l+\mu)/\lambda-k|}$$

$$\ll \sum_{k=0}^{\infty}\frac{|2\pi\alpha|^k(k+1)}{k!}\sum_{\substack{l\geq 0\\ |s+(l+\mu)/\lambda-k|\geq 1}}\frac{1}{(l+1)|s+(l+\mu)/\lambda-k|} + \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\xi^k}{k!}\left|\frac{2\pi\alpha}{\xi}\right|^{l/\lambda}.$$

The two series are convergent, and hence $S_2(s,\xi)$ is an entire function as well. Lemma 2 then follows from (13). ∎

REMARK. If $\lambda \in \mathbb{Q}^+$ the proof of Lemma 1 can be simplified: one just has to compute the residue at each suspected pole and to show that it vanishes.

From Lemma 1, (4), (6), (7) and Lemma 2 we immediately deduce the following basic formula.

LEMMA 3. *Let $F \in \mathcal{S}$ and $\alpha \in \mathbb{R}\setminus\{0\}$. Then*

$$H_F(s,\alpha) + \sum_{j=1}^{r}\lambda_j\psi(\lambda_j s+\mu_j) = \sum_{n=1}^{\infty}\frac{\Lambda_F(n)(1-e(-n\alpha))}{n^s} + M_F(s,\alpha),$$

where $M_F(s,\alpha)$ is meromorphic on $\mathbb{C}$ and holomorphic for $\sigma < 1$.

The proof of Theorem 1 is now easy. By Lemma 3, for $\alpha = m \in \mathbb{Z}\setminus\{0\}$ we have

$$G_F(s,m) = H_F(s,m) + \sum_{j=1}^{r} \lambda_j \psi(\lambda_j s + \mu_j) = M_F(s,m),$$

and the first two statements of Theorem 1 follow. To prove the last statement we first note that $H_F(s,m)$, $G_F(s,m)$ and $M_F(s,m)$ have the same poles for $\sigma \ge 1$. Thanks to the convergence properties of the series, for $\sigma > 3/2$ the function $H_F(s,m)$ is holomorphic apart from simple poles at $s = \varrho + k$, where ϱ and k run over the non-trivial zeros of $F(s)$ and over the integers ≥ 2, respectively. Concerning the remaining range $1 \le \sigma \le 3/2$, from Lemma 1, (4), (5) and Lemma 2 we have

$$M_F(s,m) = m_F \Gamma(1-s)(2\pi i m)^{s-1} - 2\pi i m \frac{\overline{F}'}{\overline{F}}(2-s) + h(s),$$

where $h(s)$ is holomorphic for $\sigma < 2$. Hence in that range the poles of $M_F(s,m)$ are a simple pole at $s = 1$ (if $m_F \ne 0$) and simple poles at $2 - s = \bar{\varrho}$, since the zeros of $\overline{F}(s)$ are at $\bar{\varrho}$. Therefore $M_F(s,m)$ has simple poles at $s = 2 - \bar{\varrho} = 2 - \beta + i\gamma = 1 + 1 - \beta + i\gamma = 1 + \varrho$ by the functional equation. Theorem 1 is thus proved. ∎

Turning to the proof of Theorem 2, the next lemma supports the assertion that $H_F(s,\chi)$ is a kind of twist of $H_F(s,\alpha)$.

LEMMA 4. *Let* $\chi \pmod q$, $q > 2$, *be a primitive Dirichlet character. Then for* $\sigma > 3/2$ *we have*

$$H_F(s,\chi) = \frac{\tau(\chi)}{q} \sum_{0<|a|<q/2} \overline{\chi}(a) H_F\left(s, \frac{a}{q}\right).$$

Proof. Thanks to the convergence properties of $H_F(s,\alpha)$, for $\sigma > 3/2$ we have

$$\frac{\tau(\chi)}{q} \sum_{0<|a|<q/2} \overline{\chi}(a) H_F\left(s, \frac{a}{q}\right) = \sum_{\varrho} \Gamma(\varrho - s)(2\pi)^{s-\varrho} \frac{\tau(\chi)}{q} \sum_{0<|a|<q/2} \overline{\chi}(a) \left(i\frac{a}{q}\right)^{s-\varrho}. \tag{14}$$

But, writing $w = \varrho - s$,

$$\frac{\tau(\chi)}{q} \sum_{0<|a|<q/2} \overline{\chi}(a) \left(i\frac{a}{q}\right)^{-w} = \frac{\tau(\chi)}{q} \sum_{0<a<q/2} \overline{\chi}(a) \left(\frac{a}{q}\right)^{-w} (i^{-w} + \chi(-1)(-1)^{-w})$$

$$= \chi(-1)\,\frac{\tau(\chi)}{q}\,q^w \sum_{0<a<q/2} \frac{\overline{\chi}(a)}{a^w}(\chi(-1)e^{-i\pi w/2} + e^{i\pi w/2})$$

$$= \chi(-1)\,\frac{\tau(\chi)}{\sqrt{q}}\,q^{w-1/2}\,l(w,\overline{\chi})i^{-a(\chi)}(e^{-i\pi(w+a(\chi))/2} + e^{i\pi(w+a(\chi))/2})$$

$$= 2\chi(-1)\,\frac{\tau(\chi)}{i^{a(\chi)}\sqrt{q}}\,q^{w-1/2}l(w,\overline{\chi})\cos\left(\frac{\pi(w+a(\chi))}{2}\right) = l^*(w,\chi),$$

and the lemma follows from (14). ∎

Theorem 2 now follows from Lemmas 3 and 4. In fact, for a primitive character $\chi \pmod q$ we have, since $q > 2$ and hence $(q/2, q) > 1$ if $q/2 \in \mathbb{N}$,

$$(15)\qquad \frac{\tau(\chi)}{q} \sum_{0<|a|<q/2} \overline{\chi}(a)e\left(-\frac{na}{q}\right) = \overline{\frac{1}{\tau(\chi)} \sum_{0<|a|<q/2} \chi(a)e\left(\frac{na}{q}\right)} = \chi(n)$$

and hence, using the orthogonality of the characters, by Lemmas 4 and 3 for $\sigma > 3/2$ we obtain

$$H_F(s,\chi) = \frac{\tau(\chi)}{q} \sum_{0<|a|<q/2} \overline{\chi}(a)\bigg(\sum_{n=1}^{\infty} \frac{\Lambda_F(n)}{n^s} - \sum_{n=1}^{\infty} \frac{\Lambda_F(n)e(-na/q)}{n^s}$$
$$- \sum_{j=1}^{r} \lambda_j \psi(\lambda_j s + \mu_j) + M_F\left(s, \frac{a}{q}\right)\bigg)$$
$$= \frac{F'}{F}(s,\chi) + \frac{\tau(\chi)}{q} \sum_{0<|a|<q/2} \overline{\chi}(a) M_F\left(s, \frac{a}{q}\right).$$

Hence Lemma 3 implies that $G(s,\chi)$ is meromorphic on $\mathbb{C}$ and holomorphic for $\sigma < 1$. As in the proof of Theorem 1, the poles of $G_F(s,\chi)$ in the half-plane $\sigma > 3/2$ are detected by means of the convergence properties of the series defining $H_F(s,\chi)$.

The remaining range $1 \le \sigma \le 3/2$ is treated as follows. By the properties of $H_F(s,\alpha)$ in Lemma 1, (4), (6) and Lemma 2, for $1 < \sigma < 2$ we have

$$(16)\qquad H_F\left(s, \frac{a}{q}\right) = \frac{F'}{F}(s) - \sum_{n=1}^{\infty} \frac{\Lambda_F(n)e(-an/q)}{n^s}$$
$$+ m_F \Gamma(1-s)\left(2\pi i\,\frac{a}{q}\right)^{s-1} - 2\pi i\,\frac{a}{q}\,\frac{\overline{F}'}{\overline{F}}(2-s) + k_1(s)$$

where $k_1(s)$ is holomorphic for $1 \le \sigma < 2$. Inserting (16) in Lemma 4, by (15) we get

$$(17)\quad H_F(s,\chi)=\frac{F'}{F}(s,\chi)+\frac{m_F\tau(\chi)}{q}\,\Gamma(1-s)\sum_{0<|a|<q/2}\overline{\chi}(a)\left(2\pi i\,\frac{a}{q}\right)^{s-1}$$
$$-\frac{2\pi i\tau(\chi)}{q^2}\,\frac{\overline{F}'}{\overline{F}}(2-s)\sum_{0<|a|<q/2}a\overline{\chi}(a)+k_2(s)$$
$$=\frac{F'}{F}(s,\chi)+\frac{m_F\tau(\chi)}{q}\,\Gamma(1-s)g(1-s,\chi)$$
$$-2\pi\,\frac{\overline{F}'}{\overline{F}}(2-s)l^*(-1,\chi)+k_2(s),$$

say, where $k_2(s)$ is holomorphic for $1\le\sigma<2$. But, by the orthogonality of characters, $g(0,\chi)=0$ and hence the corresponding term in (17) is also holomorphic for $1\le\sigma<2$. Therefore, (17) takes the form

$$G_F(s,\chi)=-2\pi\,\frac{\overline{F}'}{\overline{F}}(2-s)l^*(-1,\chi)+k_3(s)$$

with $k_3(s)$ holomorphic for $1\le\sigma<2$. This means that $G_F(s,\chi)$ has poles at the points $\varrho+1$ if $l^*(-1,\chi)=l^*(-1,\overline{\chi})\neq0$, and also at $s=1$ if $m_F\neq0$ and $l^*(-1,\chi)\neq0$. ■

References

[1] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions*, Vol. I, McGraw-Hill, 1953.

[2] A. Fujii, *Zeta zeros and Dirichlet L-functions*, Proc. Japan Acad. 64 (1988), 215–218.

[3] —, *Zeta zeros and Dirichlet L-functions. II*, ibid. 64 (1988), 296–299.

[4] —, *Some observations concerning the distribution of the zeros of the zeta function. III*, ibid. 68 (1992), 105–110.

[5] J. Kaczorowski, *Axiomatic theory of L-functions: the Selberg class*, in: Analytic Number Theory (Cetraro, 2002), A. Perelli and C. Viola (eds.), Lecture Notes in Math. 1891, Springer, 2006, 133–209.

[6] J. Kaczorowski and A. Perelli, *The Selberg class: a survey*, in: Number Theory in Progress, Proc. Conf. in Honor of A. Schinzel, K. Győry *et al.* (eds.), de Gruyter, 1999, 953–992.

[7] —, —, *On the structure of the Selberg class, II: invariants and conjectures*, J. Reine Angew. Math. 524 (2000), 73–96.

[8] Yu. V. Linnik, *On the expression of L-series by the ζ-function*, Dokl. Akad. Nauk SSSR 57 (1947), 435–437 (in Russian).

[9] A. Perelli, *A survey of the Selberg class of L-functions, part I*, Milan J. Math. 73 (2005), 19–52.

[10] —, *A survey of the Selberg class of L-functions, part II*, Riv. Mat. Univ. Parma (7) 3* (2004), 83–118.

[11] V. G. Sprindžuk, *Vertical distribution of zeros of the zeta-function and the extended Riemann hypothesis*, Acta Arith. 27 (1975), 317–332 (in Russian).

[12] —, *On extended Riemann hypothesis*, in: Topics in Number Theory, P. Turán (ed.), Colloq. Math. Soc. János Bolyai 13, North-Holland, 1976, 369–372.

[13] M. Suzuki, *A relation between the zeros of an L-function belonging to the Selberg class and the zeros of an associated L-function twisted by a Dirichlet character*, Arch. Math. (Basel) 83 (2004), 514–527.

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