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# Asymptotics for a class of arithmetic functions

by

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**1. Introduction.** At the 1991 Czechoslovak number theory conference, Professor A. Schinzel asked the following question: *For a positive integer $n$, let as usual $r(n)$ denote the number of ways $n$ can be written as a sum of two squares. What is the sharpest state-of-the-art error bound in the asymptotics for the quadratic moment*

$$\sum_{n\le x} r^2(n) = 4x\log x + Cx + O(???) \ ?$$

In [4], M. Z. Garaev, M. Kühleitner, F. Luca and W. G. Nowak gave an interesting and fairly general theorem which includes applications to sums like

$$\sum_n r^2(n),\ \sum_n d^2(n),\ \sum_n r(n^3),\ \sum_n d(n^3),\ \sum_n r(n)d(n),$$

where $d(n)$ is the Dirichlet divisor function. For papers in this direction, see also [14, 5, 6, 21, 28, 32, 34].

The main theorem of [4] can be stated as follows.

THEOREM GKLN. *Let $0 \le f(n) \ll n^{\varepsilon}$ for every $\varepsilon > 0$, with a Dirichlet series*

$$F(s) = \sum_{n\ge 1} \frac{f(n)}{n^s} = \frac{\prod_{m=1}^{M} \zeta_{\mathbb{K}_m^*}(s)}{\prod_{j=1}^{J} (\zeta_{\mathbb{K}_j}(2s))^{\tau_j}} G(s) \qquad (\Re s > 1).$$

*Assume that:*

- *$\zeta_{\mathbb{K}_m^*}$ are Dedekind zeta-functions of number fields $\mathbb{K}_m^*$ of degrees $d_m = [\mathbb{K}_m^* : \mathbb{Q}]$ equal to $1$ or $2$, with*
$$d_1 + d_2 + d_3 + d_4 = 4.$$
- *$\mathbb{K}_1, \dots, \mathbb{K}_J$ are arbitrary algebraic number fields, $J \ge 0$.*

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- *$\tau_1, \ldots, \tau_J$ are fixed real numbers.*
- *The "harmless" factor $G(s)$ is holomorphic, bounded from above and away from 0 uniformly in a half-plane $\Re s > \sigma_0$, where $\sigma_0 < 1/2$.*

*Under the above assumptions,*

$$\sum_{n\le x} f(n) = K(x) + O(x^{1/2}(\log x)^{M+1}(\log\log x)^{|\tau_1|+\cdots+|\tau_J|}) \tag{1.1}$$

*with*

$$K(x) = \mathrm{Res}_{s=1}\biggl(F(s)\frac{x^s}{s}\biggr) = xP_{M-1}(\log x),$$

*where $P_{M-1}(\cdot)$ is a polynomial of degree $M-1$.*

*Furthermore, we have the short interval result*

$$\sum_{x<n\le x+y} f(n) \sim B_0 y(\log x)^{M-1}, \tag{1.2}$$

*where $B_0$ is the leading coefficient of $P_{M-1}(\cdot)$, as long as $y = y(x)$ satisfies*

$$y = o(x), \qquad \frac{y}{x^{1/2}\log x(\log\log x)^{|\tau_1|+\cdots+|\tau_J|}} \to \infty \qquad (x\to\infty). \tag{1.3}$$

For Schinzel's question, Theorem GKLN implies that

$$\sum_{n\le x} r^2(n) = 4x\log x + Cx + O(x^{1/2}(\log x)^3\log\log x). \tag{1.4}$$

When $f(n) = d^2(n)$, the error term in the formula (1.1) has the estimate $O(x^{1/2}(\log x)^5\log\log x)$, which was proved independently by Ramachandra and Sankaranarayanan [28]. Recently, Jia and Sankaranarayanan [14] proved that the $\log\log x$ factor in this case can be removed. This is the best result one can obtain by the present methods of analytic number theory.

In this paper we shall show that under the assumptions of Theorem GKLN, the $(\log\log x)^{|\tau_1|+\cdots+|\tau_J|}$ in both (1.1) and (1.3) can be removed. Hence we get the best result one can obtain by the present approaches of analytic number theory. Furthermore, our result holds for a more general class of arithmetic functions.

Remark 1.1. The most important ingredient of this paper is the application of the twisted mean square of the Dedekind zeta-function (see [9]) over the critical line. Another important point is that we do not use the large and small values of $\zeta(1+it)$ (or other similar functions) as previous papers did; instead, we use the zero-free region result and the mean value of Dirichlet polynomials. The main tools used in this paper are (for $\zeta(s)$ or other relevant functions):

- the zero-free region,
- the mean value of Dirichlet polynomials,

- higher power moments over the critical line,
- the twisted mean square of the Dedekind zeta-function over the critical line.

**Notation.** Throughout this paper, $r(n)$ denotes the number of ways $n$ can be written as a sum of two squares; $d(n)$ is the Dirichlet divisor function, i.e., the number of ways $n$ can be written as a product of two natural numbers; $d_l(n)$ denotes the number of ways $n$ can be written as a product of $l$ natural numbers; $\omega(n)$ is the number of distinct prime divisors of $n$. $\mathbb{Q}$ denotes the rational number field, and $\mathbb{K}$ an algebraic number field. $\zeta(s)$ denotes the Riemann zeta-function, $L(s;\chi,q)$ (or $L(s;\chi)$) the Dirichlet $L$-function for a Dirichlet character $\chi$ modulo some $q \geq 1$, and $\zeta_{\mathbb{K}}(s)$ the Dedekind zeta-function for an algebraic number field $\mathbb{K}$.

**2. Statement of the main result.** Before stating our result, we introduce a convenient class of functions. For any fixed $\beta \geq 0$, let $\mathbb{D}_\beta$ denote the set of functions $H(s)$ defined by the Dirichlet series

$$H(s) := \sum_{n \geq 1} \frac{h(n)}{n^s} \qquad (\Re s > 1)$$

for some arithmetic function $h(n)$, and satisfying the following conditions:

- $H(s)$ can be analytically continued to the region

$$\sigma \geq 1 - \frac{c_H}{(\log(|t|+2))^\beta} \tag{2.1}$$

  for some constant $c_H > 0$, and has a possible pole at $s = 1$ of order $\delta_H \geq 0$,
- $H(s)$ satisfies the estimate

$$H(\sigma + it) \ll (\log(|t|+2))^{\theta_H} \qquad (|t| \geq 1) \tag{2.2}$$

  for some constant $\theta_H > 0$ in the region (2.1),
- the arithmetic function $h(n)$ satisfies

$$h(n) \ll d_l(n) \tag{2.3}$$

  for some positive integer $l = l(H) \geq 1$.

Remark 2.1. The assumptions for $\mathbb{D}_\beta$ are natural, being closely related to the zero-free regions of important functions in analytic number theory, such as the Riemann zeta-function $\zeta(s)$, etc.

Let $L(s)$ be the Riemann zeta-function $\zeta(s)$, or the Dirichlet $L$-function $L(s;\chi,q)$ with respect to the Dirichlet character $\chi$ modulo a fixed $q \geq 1$, or the Dedekind zeta-function $\zeta_{\mathbb{K}}(s)$ for an algebraic number field $\mathbb{K}$. According to Lemma 3.1, we have $L(\cdot) \in \mathbb{D}_\beta$ and $1/L(\cdot) \in \mathbb{D}_\beta$ for any $\beta > 2/3$.

Suppose $f : \mathbb{N} \to \mathbb{C}$ is an arithmetic function such that the Dirichlet series

$$F(s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

is absolutely convergent for $\sigma > 1$ and

$$F(s) = \zeta(s)L(s;\chi,q)M(s)H(2s)G(s) \quad (\sigma > 1), \tag{2.4}$$

where

- $L(s;\chi,q)$ is a Dirichlet $L$-function with respect to a Dirichlet character $\chi$ modulo some fixed $q \geq 1$,
- $M(s)$ is a meromorphic function in the extended Selberg class (see Section 7, or [16, 30, 31]) such that $s = 1$ is a possible pole of order $\delta_M \geq 0$ and

$$\int_1^T |M(1/2+it)|^2\,dt \ll T(\log T)^{\theta_M} \quad \text{for some } \theta_M \geq 0, \tag{2.5}$$

- $H(\cdot) \in \mathbb{D}_\beta$ for some $\beta > 0$,
- The "harmless" factor $G(s)$ is holomorphic, bounded from above and away from 0 uniformly in a half-plane $\Re s > \sigma_0$, where $\sigma_0 < 1/2$.

The expected form of the asymptotic formula for the summatory function of $f(n)$ is

$$\sum_{n\leq x} f(n) = xP_1(\log x) + x^{1/2}P_2(\log x) + E_f(x), \tag{2.6}$$

where $P_1(u)$ is a polynomial in $u$ of degree $\delta_\chi + \delta_M$ with $\delta_\chi = 1$ when $\chi$ is the principal character modulo $q$, and $\delta_\chi = 0$ otherwise; $P_2(u)$ is a polynomial in $u$ of degree $\delta_H - 1$ when $\delta_H \geq 1$, and $P_2(u) = 0$ when $\delta_H = 0$; and $E_f(x)$ is the error term. Actually, we have

$$xP_1(\log x) = \operatorname{Res}_{s=1} F(s)\frac{x^s}{s}, \qquad x^{1/2}P_2(\log x) = \operatorname{Res}_{s=1/2} F(s)\frac{x^s}{s}. \tag{2.7}$$

Our main result is the following

Main Theorem. *Suppose the above conditions hold.*

(1) *We have the estimate*

$$E_f(x) = O(x^{1/2}(\log x)^{2+\delta_\chi+\theta_M/2}). \tag{2.8}$$

(2) *If $f(n) \geq 0$ for all $n$, then*

$$\sum_{x<n\leq x+y} f(n) \sim C_0 y(\log x)^{\delta_\chi+\delta_M} \tag{2.9}$$

*for $y = y(x)$ with*

$$y = o(x), \qquad \frac{y}{x^{1/2}(\log x)^{1+\theta_M/2-\delta_M}} \to \infty \qquad (x \to \infty),$$

*where $C_0$ is the leading coefficient of $P_1(\cdot)$.*

REMARK 2.2. When $1 \le \delta_H \le 3 + \delta_\chi + \theta_M/2$, the term $x^{1/2}P_2(\log x)$ does not appear since it is absorbed into $O(x^{1/2}(\log x)^{2+\delta_\chi+\theta_M/2})$. However, if $\delta_H > 3 + \delta_\chi + \theta_M/2$, then $x^{1/2}P_2(\log x)$ is a true main term.

REMARK 2.3. The condition (2.3) on $\mathbb{D}_\beta$ is important but not essential, and it can be relaxed in some sense. For example, it is easy to see that the function $H_1(s) = 1/\zeta(2s-1)$ is not in $\mathbb{D}_\beta$ for any $\beta > 0$ since (2.3) is not true. However, our theorem still holds if we replace $H(2s)$ by $H_1(2s)$.

Now we define an extension of $\mathbb{D}_\beta$, denoted by $\mathbb{D}_\beta^\diamond$, to be the set of functions $H^\diamond(s)$ defined by the Dirichlet series

$$H^\diamond(s) := \sum_{n \ge 1} \frac{h^\diamond(n)}{n^s} \qquad (\Re s > 1)$$

for some arithmetic function $h^\diamond(n)$, and satisfying the following conditions:

- $H^\diamond(s)$ can be analytically continued to the region

$$\sigma \ge 1 - \frac{c_{H^\diamond}}{(\log(|t|+2))^\beta} \tag{2.10}$$

  for some constant $c_{H^\diamond} > 0$, and has a possible pole at $s = 1$ of order $\delta_{H^\diamond} \ge 0$,
- $H^\diamond(s)$ satisfies the estimate

$$H^\diamond(\sigma + it) \ll (\log(|t|+2))^{\theta_{H^\diamond}} \qquad (|t| \ge 1) \tag{2.11}$$

  for some constant $\theta_{H^\diamond} > 0$ in the region (2.10),
- There exists $H(\cdot) \in \mathbb{D}_\beta$ such that

$$H^\diamond(1+it) \ll |H(1+ait)| \quad (|t| \ge 1)$$

  for some positive integer $a$.

Obviously $\mathbb{D}_\beta \subset \mathbb{D}_\beta^\diamond$, and our Main Theorem still holds if $H(s)$ is replaced by some $H^\diamond(s) \in \mathbb{D}_\beta^\diamond$.

## 3. Preliminary lemmas

LEMMA 3.1. *Let $L(s)$ be the Riemann zeta-function $\zeta(s)$, or the Dirichlet $L$-function $L(s;\chi,q)$ with respect to the Dirichlet character $\chi$ modulo a fixed $q \ge 1$, or the Dedekind zeta-function $\zeta_{\mathbb{K}}(s)$ for an algebraic number field $\mathbb{K}$. Then:*

(1) *There exists a positive constant* $c_L$ *such that* $L(s)\neq 0$ *in the region*

$$\sigma\geq 1-\frac{c_L}{(\log(|t|+2))^{2/3}(\log\log(|t|+2))^{1/3}}.$$

(2) *There exists a positive constant* $C_L$ *such that in the above region*

$$L(s)\ll(\log(|t|+2))^{C_L},\qquad 1/L(s)\ll(\log(|t|+2))^{C_L}.$$

*Proof.* For $\zeta(s)$ and $L(s;\chi,q)$, see Ivić [11, Chapter 6] and C. D. Pan and C. B. Pan [27, Chapters 10 and 17], respectively. Note that we do not consider the effects of the possible Siegel zero of $L(s;\chi,q)$ when $\chi$ is a real character, since we always suppose that $q$ is fixed. For the case of the Dedekind zeta-function, see Mitsui [23]. ∎

Lemma 3.2 (Ivić [11, Theorem 5.2]). *Suppose* $a(1),\ldots,a(N)$ *are arbitrary complex numbers. Then*

$$\int_1^T\Big|\sum_{n\leq N}a(n)n^{it}\Big|^2\,dt=T\sum_{n\leq N}|a(n)|^2+O\Big(\sum_{n\leq N}n|a(n)|^2\Big).$$

Lemma 3.3. *Let* $L(s)$ *be* $\zeta(s)$ *or the Dirichlet L-function* $L(s;\chi,q)$ *with respect to a Dirichlet character* $\chi$ *modulo a fixed* $q\geq 1$. *Then for any* $4<A<12$,

$$\int_1^T|L(1/2+it)|^A\,dt\ll T^{1+(A-4)/8+\varepsilon}.$$

*Proof.* When $L(s)=\zeta(s)$, the lemma follows from the fourth moment estimate (see Ivić [11])

$$\int_1^T|\zeta(1/2+it)|^4\,dt\ll T(\log T)^4,$$

the twelfth moment estimate (see Heath-Brown [8])

$$\int_1^T|\zeta(1/2+it)|^{12}\,dt\ll T^2(\log T)^{17}$$

and Hölder's inequality.

When $L(s)=L(s;\chi,q)$ the proof is the same, using the fourth moment of $L(s;\chi,q)$, the twelfth moment of $L(s;\chi,q)$ over the critical line (see [22]) and Hölder's inequality. ∎

Lemma 3.4. *Let* $L(s)=L(s;\chi,q)$ *be the Dirichlet L-function with respect to a Dirichlet character* $\chi$ *modulo a fixed* $q\geq 1$. *Then*

$$\int_1^T|\zeta(1/2+it)L(1/2+it;\chi,q)|^2\,dt\ll T(\log T)^{2+2\delta_\chi}.\tag{3.1}$$

*Proof.* When $\chi = \chi_0$ is the principal character, Lemma 3.4 follows from the fourth moment estimate for $\zeta(s)$ over the critical line. When $\chi \neq \chi_0$, it is a consequence of Müller's result [25]. ∎

LEMMA 3.5. *Let $l \geq 1$ be a fixed integer. Then*

$$\sum_{n\leq x} d_l(n) \ll x(\log x)^{l-1}, \qquad \sum_{n\leq x} d_l^2(n) \ll x(\log x)^{l^2-1}.$$

*Proof.* Well-known. ∎

**4. A mean value estimate.** In this section we shall prove the following mean value estimate, which plays the most important role in the proof of our Main Theorem.

PROPOSITION. *Let $L(s;\chi,q)$, $M(s)$ and $H(s)$ satisfy the conditions of the Main Theorem. Then*

$$\int_1^T |\zeta(1/2+it)L(1/2+it;\chi,q)M(1/2+it)H(1+2it)|\,dt \ll T(\log T)^{1+\delta_\chi+\theta_M/2}. \tag{4.1}$$

REMARK 4.1. The Proposition is still true if we replace $H(1+2it)$ by $H(1+ait)$ for any positive integer $a \geq 1$.

It suffices to prove that

$$\int_1^T |\zeta^2(1/2+it)L^2(1/2+it;\chi,q)H^2(1+2it)|\,dt \ll T(\log T)^{2+2\delta_\chi}, \tag{4.2}$$

or equivalently

$$\int_T^{2T} |\zeta^2(1/2+it)L^2(1/2+it;\chi,q)H^2(1+2it)|\,dt \ll T(\log T)^{2+2\delta_\chi}. \tag{4.3}$$

Actually, (4.1) follows immediatly from (4.2), (2.5) and Cauchy's inequality.

We shall prove the estimate (4.3).

**4.1. An approximation of $H(1+2it)$.** Define $\mathbf{H}(x) := \sum_{n\leq x} h(n)$. Recall that $H(\cdot) \in \mathbb{D}_\beta$ for some fixed $\beta > 0$. By the contour integration method and the conditions on $\mathbb{D}_\beta$, we can prove that

$$\mathbf{H}(x) = \eta x P^*(\log x) + O(xe^{-c(\log x)^{1/(1+\beta)}}), \tag{4.4}$$

where $c > 0$ is a positive constant, $\eta = 1$ if $s = 1$ is a pole of $H(s)$ of order $\delta_H \geq 1$ and $P^*(u)$ is a polynomial in $u$ of degree $\delta_H - 1$, and $\eta = 0$ if $H(s)$ is analytic at $s = 1$. We omit the proof of (4.4), since it is routine and the same as the proof of the prime number theorem. Let

$$E_{\mathbf{H}}(x) := \mathbf{H}(x) - \eta x P^*(\log x).$$

Then

$$E_{\mathbf{H}}(x) \ll xe^{-c(\log x)^{1/(1+\beta)}}. \tag{4.5}$$

Suppose $\xi = u + iv$ with $1 < u < 2$ and $v \asymp T$. Let $B > 1$ be a large parameter to be determined later. Then

$$H(\xi) = \sum_{n=1}^{\infty} \frac{h(n)}{n^{\xi}} = \sum_{n \le B} \frac{h(n)}{n^{\xi}} + \sum_{n > B} \frac{h(n)}{n^{\xi}}. \tag{4.6}$$

By partial summation,

$$\begin{aligned} \sum_{n>B} \frac{h(n)}{n^{\xi}} &= \int_B^{\infty} \frac{1}{x^{\xi}}\, d\mathbf{H}(x) \\ &= \int_B^{\infty} \frac{1}{x^{\xi}}\, d\eta\, xP^*(\log x) + \int_B^{\infty} \frac{1}{x^{\xi}}\, dE_{\mathbf{H}}(x) =: \int_1 + \int_2 . \end{aligned} \tag{4.7}$$

If $\delta_H = 0$ then $\int_1 = 0$, and if $\delta_H \ge 1$ then it is easy to see that

$$\int_1 = \int_B^{\infty} \frac{P^*(\log x) + (P^*(\log x))'}{x^{\xi}}\, dx \ll \frac{(\log B)^{\delta_H - 1}}{|\xi - 1|}. \tag{4.8}$$

By partial integration we get

$$\int_2 = -\frac{E_{\mathbf{H}}(B)}{B^{\xi}} + \xi \int_B^{\infty} \frac{E_{\mathbf{H}}(x)}{x^{1+\xi}}\, dx. \tag{4.9}$$

From (4.5) we see that the integral on the right-hand side of (4.9) is absolutely convergent on the line $\xi = 1$. So by taking $\xi = 1 + 2it$ with $t \asymp T$ we deduce from (4.6)–(4.9) that

$$\begin{aligned} H(1+2it) &= \sum_{n \le B} \frac{h(n)}{n^{1+2it}} + O(\eta (\log B)^{\delta_H - 1}/T + |E_{\mathbf{H}}(B)|/B) \\ &\quad + O\Big(T \int_B^{\infty} \frac{|E_{\mathbf{H}}(x)|}{x^2}\, dx\Big) \\ &= \sum_{n \le B} \frac{h(n)}{n^{1+2it}} + O(\eta (\log B)^{\delta_H - 1}/T + |E_{\mathbf{H}}(B)|/B|) \\ &\quad + O(Te^{-c_1(\log B)^{1/(1+\beta)}}) \end{aligned}$$

for some $0 < c_1 < c$. Now taking

$$B := e^{(2c_1^{-1}\log T)^{1+\beta}}, \qquad y := T^{1/24}, \tag{4.10}$$

we get

$$H(1+2it) = \sum_{n\le B} \frac{h(n)}{n^{1+2it}} + O((\log B)^{\delta_H-1}/T) \tag{4.11}$$
$$= H_1(t) + H_2(t) + O((\log B)^{\delta_H-1}/T),$$

where

$$H_1(t) := \sum_{n\le y} \frac{h(n)}{n^{1+2it}}, \qquad H_2(t) := \sum_{y<n\le B} \frac{h(n)}{n^{1+2it}}.$$

Inserting (4.11) into the integral on the left-hand side of (4.3) and then using Lemma 3.4 we get

$$\int_T^{2T} |\zeta^2(1/2+it)L^2(1/2+it;\chi,q)H^2(1+2it)|\,dt \tag{4.12}$$
$$\ll S_1 + S_2 + T^{-1}(\log T)^{2\delta_H+2\delta_\chi},$$

where

$$S_1 := \int_T^{2T} |\zeta^2(1/2+it)L^2(1/2+it;\chi,q)H_1^2(t)|\,dt,$$
$$S_2 := \int_T^{2T} |\zeta^2(1/2+it)L^2(1/2+it;\chi,q)H_2^2(t)|\,dt.$$

**4.2. Estimation of $S_2$.** By Cauchy's inequality we get

$$S_2 \ll (S_{21}S_{22})^{1/2}, \tag{4.13}$$

where

$$S_{21} := \int_T^{2T} |\zeta^4(1/2+it)H_2^2(t)|\,dt, \qquad S_{22} := \int_T^{2T} |L^4(1/2+it;\chi,q)H_2^2(t)|\,dt.$$

Suppose $1<\mathbf{p}<5/4$ is a real number, and $\mathbf{q}>1$ is a natural number such that $1/\mathbf{p}+1/\mathbf{q}=1$. By Hölder's inequality and Lemma 3.3,

$$S_{21} \ll \Big(\int_T^{2T} |\zeta(1/2+it)|^{4\mathbf{p}}\,dt\Big)^{1/\mathbf{p}} \Big(\int_T^{2T} |H_2(t)|^{2\mathbf{q}}\,dt\Big)^{1/\mathbf{q}} \tag{4.14}$$
$$\ll T^{\frac{4\mathbf{p}+4}{8\mathbf{p}}+\varepsilon}\Big(\int_T^{2T} |H_2^{\mathbf{q}}(t)|^2\,dt\Big)^{1/\mathbf{q}}.$$

We write

$$H_2^{\mathbf{q}}(t) = \sum_{y^{\mathbf{q}}<n\le B^{\mathbf{q}}} \frac{h^*(n)}{n^{1+2it}}, \tag{4.15}$$

where (recalling (2.3))

$$h^*(n) = \sum_{\substack{n=n_1\cdots n_{\mathbf{q}} \\ y<n_1,\ldots,n_{\mathbf{q}}\le B}} h(n_1)\cdots h(n_{\mathbf{q}}) \ll \sum_{n=n_1\cdots n_{\mathbf{q}}} d_l(n_1)\cdots d_l(n_{\mathbf{q}}) = d_{l\mathbf{q}}(n). \tag{4.16}$$

So by Lemma 3.5 we get

$$\sum_{n\le x} h^{*2}(n) \ll \sum_{n\le x} d_{l\mathbf{q}}^2(n) \ll x(\log x)^{l^2\mathbf{q}^2-1}. \tag{4.17}$$

By Lemma 3.2 and (4.17), partial summation yields

$$\begin{aligned}\int_T^{2T} |H_2^{\mathbf{q}}(t)|^2\,dt &\ll T\sum_{y^{\mathbf{q}}<n\le B^{\mathbf{q}}} \frac{h^{*2}(n)}{n^2} + \sum_{y^{\mathbf{q}}<n\le B^{\mathbf{q}}} \frac{h^{*2}(n)}{n} \\ &\ll \frac{T}{y^{\mathbf{q}}}(\log y)^{l^2\mathbf{q}^2-1} + (\log B)^{l^2\mathbf{q}^2} \\ &\ll \frac{T}{y^{\mathbf{q}}}(\log T)^{l^2\mathbf{q}^2-1} + (\log T)^{l^2\mathbf{q}^2(1+\beta)}.\end{aligned} \tag{4.18}$$

From (4.14) and (4.18) we get

$$\begin{aligned} S_{21} &\ll T^{1+\frac{4\mathbf{p}-4}{8\mathbf{p}}+\varepsilon} y^{-1}(\log T)^{\frac{l^2\mathbf{q}^2-1}{\mathbf{q}}} + T^{\frac{4\mathbf{p}+4}{8\mathbf{p}}+\varepsilon}(\log T)^{l^2\mathbf{q}(1+\beta)} \\ &\ll T^{1+\frac{4\mathbf{p}-4}{8\mathbf{p}}-\frac{1}{24}+\varepsilon}(\log T)^{\frac{l^2\mathbf{q}^2-1}{\mathbf{q}}} + T^{\frac{4\mathbf{p}+4}{8\mathbf{p}}+\varepsilon}(\log T)^{l^2\mathbf{q}(1+\beta)} \\ &\ll T^{47/48+\varepsilon},\end{aligned} \tag{4.19}$$

where in the last step we took $\mathbf{q} = 24$, $\mathbf{p} = 24/23$. Similarly we have

$$S_{22} \ll T^{47/48+\varepsilon}. \tag{4.20}$$

Thus from (4.13), (4.19) and (4.20) we get

$$S_2 \ll T^{47/48+\varepsilon}. \tag{4.21}$$

**4.3. Estimation of $S_1$.** The estimate of $S_1$ is closely related to the integral

$$\int_T^{2T} |\zeta(1/2+it)|^4 |M(1/2+it)|^2\,dt, \tag{4.22}$$

where

$$M(s) := \sum_{h\le T^\theta} a(h)h^{-s}$$

is a Dirichlet polynomial of length $T^\theta$ $(0 < \theta < 1)$ with complex coefficients $a(h)$. The evaluation of the integral in (4.22) is an important problem in analytic number theory. It was studied by J.-M. Deshouillers and H. Iwaniec [3], N. Watt [33] and most recently by Y. Motohashi [24], all of whom used powerful methods from the spectral theory of the non-Euclidean Laplacian. Hughes and Young [10] obtained an asymptotic formula for (4.22) when $\theta = 1/11 - \varepsilon$. With the help of that result Ivić and the present author [12, 13] studied the integral $\int_1^T |\zeta(1/2+it)|^4 |\zeta(\sigma+it)|^{2j}\,dt$ for $\sigma \le 1$.

Let $q \ge 1$ be a fixed integer, and $\chi$ be an arbitrary Dirichlet character modulo $q$. Heap [9] generalized the result of Hughes and Young [10] to the integral

$$\int_T^{2T} |\zeta(1/2+it)|^2 |L(1/2+it;\chi,q)|^2 |M(1/2+it)|^2\,dt. \tag{4.23}$$

Expanding $|M(\cdot)|^2$, we can write the integral (4.23) as

$$\sum_{h,k\le T^\theta} \frac{a(h)\overline{a(k)}}{(hk)^{1/2}} I^*(h,k), \tag{4.24}$$

where

$$I^*(h,k) := \int_T^{2T} |\zeta(1/2+it)|^2 |L(1/2+it;\chi,q)|^2 (h/k)^{-it}\,dt.$$

Instead of evaluating $I^*(h,k)$ directly, Heap evaluated the integral

$$\begin{aligned} I(h,k) := \int_{-\infty}^{\infty} & w(t)\zeta(1/2+\alpha+it)L(1/2+\beta+it;\chi,q) \\ & \times \zeta(1/2+\gamma-it)L(1/2+\delta-it;\overline{\chi},q)(h/k)^{-it}\,dt, \end{aligned} \tag{4.25}$$

where $\alpha,\beta,\gamma,\delta \ll (\log T)^{-1}$ are small complex numbers and $w(t)$ is a smooth, non-negative function with support contained in $[T/2,4T]$. When $(h,k)=1$ and $h,k \le T^{1/11-\varepsilon}$, Heap obtained an asymptotic formula for $I(h,k)$, which is of size $\approx \frac{1}{\sqrt{hk}} T(\log T)^{2+2\delta_\chi}$. Correspondingly, if $(h,k)=1$ and $h,k \le T^{1/11-\varepsilon}$, one has

$$I^*(h,k) \approx \frac{1}{\sqrt{hk}} T(\log T)^{2+2\delta_\chi}. \tag{4.26}$$

Now we estimate $S_1$. Opening the square $|H_1(t)|^2$ we get

$$\begin{aligned} |H_1(t)|^2 &= \sum_{n_1\le y}\sum_{n_2\le y} \frac{h(n_1)\overline{h(n_2)}}{n_1 n_2}\left(\frac{n_2^2}{n_1^2}\right)^{it} \\ &= \sum_{r\le y}\frac{1}{r^2} \sum_{\substack{m_1,m_2\le y/r \\ (m_1,m_2)=1}} \frac{h(rm_1)\overline{h(rm_2)}}{m_1 m_2}\left(\frac{m_2^2}{m_1^2}\right)^{it}. \end{aligned} \tag{4.27}$$

Thus

$$S_1=\sum_{r\le y}\frac{1}{r^2}\sum_{\substack{m_1,m_2\le y/r\\(m_1,m_2)=1}}\frac{h(rm_1)\overline{h(rm_2)}}{m_1m_2}I^*(m_1^2,m_2^2). \tag{4.28}$$

By our choice of $y$, we have $m_1^2,m_2^2\le y^2=T^{1/12}<T^{1/11-\varepsilon}$. So from (4.26) and (4.28) we get

$$\begin{aligned}S_1&\ll\sum_{r\le y}\frac{1}{r^2}\sum_{\substack{m_1,m_2\le y/r\\(m_1,m_2)=1}}\frac{h(rm_1)\overline{h(rm_2)}}{m_1m_2}\,\frac{1}{m_1m_2}T(\log T)^{2+2\delta_\chi}\\&\ll\sum_{r\le y}\frac{1}{r^2}\sum_{\substack{m_1,m_2\le y/r\\(m_1,m_2)=1}}\frac{d_k(rm_1)d_k(rm_2)}{m_1^2m_2^2}T(\log T)^{2+2\delta_\chi}\\&\ll T(\log T)^{2+2\delta_\chi}.\end{aligned}\tag{4.29}$$

Finally, (4.3) immediately follows from (4.12), (4.21) and (4.29).

## 5. Proof of Main Theorem

**5.1. Proof of (2.8).** By Perron's formula (see for example [11, (A10)]) we have

$$\sum_{n\le x}f(n)=\frac{1}{2\pi i}\int_{1+\varepsilon-ix}^{1+\varepsilon+ix}F(s)\frac{x^s}{s}\,ds+O(x^\varepsilon), \tag{5.1}$$

where $\varepsilon>0$ is fixed. Consider the contour $C$ consisting of $I_0,I_1,I_2,I_3,I_4$ and $I_c$, where

$$\begin{aligned}I_0&:=\{s=1+\varepsilon+it:-x\le t\le x\},\\I_1&:=\{s=\sigma+ix:\sigma\text{ is from }1+\varepsilon\text{ to }1/2\},\\I_2&:=\{s=1/2+it:t\text{ is from }x\text{ to }\varepsilon\},\\I_c&:=\{s=1/2+\varepsilon e^{i\theta}:\pi/2\le\theta\le3\pi/2\},\\I_3&:=\{s=1/2+it:t\text{ is from }-\varepsilon\text{ to }-x\},\\I_4&:=\{s=\sigma-ix:1/2\le\sigma\le1+\varepsilon\}.\end{aligned}$$

Inside the contour $C$ the integrand $F(s)x^s/s$ has two poles, $s=1$ of order $1+\delta_\chi+\delta_M$ and $s=1/2$ of order $\delta_H$. So by the residue theorem,

$$\begin{aligned}\frac{1}{2\pi i}\int_C F(s)\frac{x^s}{s}\,ds&=\operatorname{Res}_{s=1}F(s)\frac{x^s}{s}+\operatorname{Res}_{s=1/2}F(s)\frac{x^s}{s}\\&=xP_1(\log x)+x^{1/2}P_2(\log x).\end{aligned}\tag{5.2}$$

From (5.1) and (5.2) we get

$$E_f(x) = -\sum_{j=1}^{4} \frac{1}{2\pi i} \int_{I_j} F(s)\frac{x^s}{s}\,ds - \frac{1}{2\pi i}\int_{I_c} F(s)\frac{x^s}{s}\,ds + O(x^{\varepsilon}). \tag{5.3}$$

It is easy to see that

$$\frac{1}{2\pi i}\int_{I_c} F(s)\frac{x^s}{s}\,ds \ll x^{1/2}. \tag{5.4}$$

By the Proposition and partial integration,

$$\begin{aligned}
&\int_{I_2} F(s)\frac{x^s}{s}\,ds + \int_{I_3} F(s)\frac{x^s}{s}\,ds \\
&\ll x^{1/2} + x^{1/2}\int_1^T |F(1/2+it)|t^{-1}\,dt \\
&\ll x^{1/2} + x^{1/2}\int_1^T |\zeta(1/2+it)L(1/2+it;\chi,q)M(1/2+it)H(1+2it)|t^{-1}\,dt \\
&\ll x^{1/2}(\log x)^{2+\delta_\chi+\theta_M/2}.
\end{aligned} \tag{5.5}$$

The estimate (2.5) implies that

$$M(1/2+it) \ll |t|^{1/2+\varepsilon}, \qquad |t| \ge 1,$$

which combined with the well-known bounds

$$\zeta(1/2+it) \ll |t|^{1/6}, \qquad L(1/2+it;\chi,q) \ll |t|^{1/6}$$

gives

$$F(s) \ll |t|^{5|1-\sigma|/3+\varepsilon} \qquad (1/2 \le \sigma \le 1+\varepsilon)$$

via the Phragmén–Lindelöf principle. With the help of this estimate we immediately get

$$\int_{I_1} F(s)\frac{x^s}{s}\,ds + \int_{I_4} F(s)\frac{x^s}{s}\,ds \ll x^{-1}\int_{1/2}^{1+\varepsilon} |F(\sigma+ix)|x^{\sigma}\,d\sigma \ll x^{1/3+\varepsilon}. \tag{5.6}$$

Now (2.8) follows from (5.3)–(5.6).

**5.2. Proof of (2.9).** Suppose that $y \ll x^{1/2+\varepsilon}$, otherwise (2.9) follows from (2.8) directly.

We follow the approach of [6], which is based on a device due to Karatsuba [19]. Suppose $x$ is a large parameter. For any $x/2 < u < 2x$, by the argument in Section 5.1 we have

$$A(u) := \sum_{n\le x} f(n) = L(u) - \sum_{j=2,3}\frac{1}{2\pi i}\int_{I_j} F(s)\frac{u^s}{s}\,ds + O(x^{1/2}), \tag{5.7}$$

where

$$L(u) := \mathrm{Res}_{s=1} F(s)\frac{u^s}{s} + \mathrm{Res}_{s=1/2} F(s)\frac{u^s}{s}.$$

Since $A(u)$ is increasing, we can use a formula of [6],

$$A(x+y) - A(x) \le \frac{1}{h}\int_{x+y}^{x+y+h} A(u)\,du - \frac{1}{h}\int_{x-h}^{x} A(u)\,du, \tag{5.8}$$

where $x^{1/2} \ll h < y$ is a parameter to be determined later. From (5.7) we have

$$\begin{aligned} &\int_{x+y}^{x+y+h} A(u)\,du - \int_{x-h}^{x} A(u)\,du \\ &= \int_{x+y}^{x+y+h} L(u)\,du - \int_{x-h}^{x} L(u)\,du + O(hx^{1/2}) \\ &\quad - \sum_{j=2,3}\frac{1}{2\pi i}\int_{I_j} F(s)\frac{(x+y+h)^{s+1} - (x+y)^{s+1} - x^{s+1} + (x-h)^{s+1}}{s(s+1)}\,ds. \end{aligned} \tag{5.9}$$

By the same argument as in [6],

$$\begin{aligned} &\int_{x+y}^{x+y+h} L(u)\,du - \int_{x-h}^{x} L(u)\,du \\ &\qquad = C_0 hy(\log x)^{\delta_\chi+\delta_M} + O(h^2(\log x)^{\delta_\chi+\delta_M} + hy(\log x)^{\delta_\chi+\delta_M-1}). \end{aligned} \tag{5.10}$$

We have

$$\begin{aligned} &\sum_{j=2,3}\frac{1}{2\pi i}\int_{I_j} F(s)\frac{(x+y+h)^{s+1} - (x+y)^{s+1} - x^{s+1} + (x-h)^{s+1}}{s(s+1)}\,ds \\ &\qquad\qquad \ll \int_{\varepsilon}^{x}\left|F(1/2+it)\frac{H_{x,y,h}(t)}{t^2+1/4}\right|dt, \end{aligned} \tag{5.11}$$

where

$$H_{x,y,h}(t) := (x+y+h)^{3/2+it} - (x+y)^{3/2+it} - x^{3/2+it} + (x-h)^{3/2+it}.$$

Let $z := x/\sqrt{hy}$. When $\varepsilon \le t \le z$, by using the bound (see [6])

$$H_{x,y,h}(t) \ll (1+t^2)hyx^{-1/2}$$

and our Proposition we get

$$\int_{\varepsilon}^{z} \left| F(1/2+it) \frac{H_{x,y,h}(t)}{t^2+1/4} \right| dt \ll hyx^{-1/2} \int_{\varepsilon}^{z} |F(1/2+it)|\, dt$$
$$\ll hyx^{-1/2} z (\log x)^{1+\delta_\chi+\theta_M/2} \ll (hxy)^{1/2} (\log x)^{1+\delta_\chi+\theta_M/2}.$$

When $t > z$, by using the trivial bound $H_{x,y,h}(t) \ll x^{3/2}$, the Proposition and partial integration we get

$$\int_{z}^{x} \left| F(1/2+it) \frac{H_{x,y,h}(t)}{t^2+1/4} \right| dt \ll x^{3/2} \int_{\varepsilon}^{x} |F(1/2+it)| t^{-2}\, dt$$
$$\ll x^{3/2} z^{-1} (\log x)^{1+\delta_\chi+\theta_M/2} \ll (hxy)^{1/2} (\log x)^{1+\delta_\chi+\theta_M/2}.$$

Inserting the above two estimates into (5.11) we get

$$\sum_{j=2,3} \frac{1}{2\pi i} \int_{I_j} F(s) \frac{(x+y+h)^{s+1} - (x+y)^{s+1} - x^{s+1} + (x-h)^{s+1}}{s(s+1)}\, ds \ll (hxy)^{1/2} (\log x)^{1+\delta_\chi+\theta_M/2}. \tag{5.12}$$

Combining (5.8)–(5.10) and (5.12) we obtain

$$\begin{aligned} A(x+y) - A(x) &\le C_0 y (\log x)^{\delta_\chi+\delta_M} + O(h(\log x)^{\delta_\chi+\delta_M} + y(\log x)^{\delta_\chi+\delta_M-1}) \\ &\quad + O(x^{1/2}) + O(h^{-1/2}(xy)^{1/2}(\log x)^{1+\delta_\chi+\theta_M/2}) \\ &= C_0 y (\log x)^{\delta_\chi+\delta_M} + O(y(\log x)^{\delta_\chi+\delta_M-1}) \\ &\quad + O(x^{1/2} + (xy)^{1/3}(\log x)^{2/3+\delta_\chi+(\theta_M+\delta_M)/3}) \end{aligned} \tag{5.13}$$

by taking

$$h := (xy)^{1/3} (\log x)^{2/3+\theta_M/3-2\delta_M/3}.$$

By a similar approach we get

$$\begin{aligned} A(x+y) - A(x) \ge C_0 y (\log x)^{\delta_\chi+\delta_M} &+ O(y(\log x)^{\delta_\chi+\delta_M-1}) \\ &+ O(x^{1/2} + (xy)^{1/3}(\log x)^{2/3+\delta_\chi+(\theta_M+\delta_M)/3}). \end{aligned} \tag{5.14}$$

From (5.13) and (5.14) we have

$$\begin{aligned} A(x+y) - A(x) &= C_0 y (\log x)^{\delta_\chi+\delta_M} + O(y(\log x)^{\delta_\chi+\delta_M-1}) \\ &\qquad + O(x^{1/2} + (xy)^{1/3}(\log x)^{2/3+\delta_\chi+(\theta_M+\delta_M)/3}) \\ &\sim C_0 y (\log x)^{\delta_\chi+\delta_M} \end{aligned}$$

if

$$\frac{y}{x^{1/2}(\log x)^{1+\theta_M/2-\delta_M}} \to \infty \qquad (x \to \infty).$$

## 6. Applications

**6.1. A special divisor problem.** Let $1 \le a_1 \le \cdots \le a_k$ be fixed integers. The general divisor function is defined by

$$d(n; a_1, \ldots, a_k) = \sum_{n = n_1^{a_1} \cdots n_k^{a_k}} 1.$$

Suppose $J \ge 0$ is a fixed integer, $k = J + 4$, $a_j = 1$ $(1 \le j \le 4)$ and $a_j = 2$ $(4 < j \le J + 4)$. Then

$$d(n; 1, 1, 1, 1, 2, 2, \ldots, 2) = \sum_{n = n_1 n_2^2} d_4(n_1) d_J(n_2).$$

When $J = 0$, this function is just $d_4(n)$. So later we suppose $J \ge 1$. It is easy to see that the corresponding Dirichlet series is $\zeta^4(s)\zeta^J(2s)$. Hence our Main Theorem has the following corollary by taking $L(s; \chi, q) = \zeta(s)$, $M(s) = \zeta^2(s)$ and $h(s) = \zeta^J(s)$.

Corollary 1. *We have the asymptotic formula*

$$\begin{aligned}(6.1)\qquad \sum_{n \le x} & d(n; 1, 1, 1, 1, 2, 2, \ldots, 2) \\ &= x Q_1(\log x) + x^{1/2} Q_2(\log x) + O(x^{1/2} (\log x)^5),\end{aligned}$$

*where $Q_1(u)$ is a polynomial in $u$ of degree* 3, *and $Q_2(u)$ is a polynomial in $u$ of degree $J - 1$.*

*The short interval estimate*

$$(6.2)\qquad \sum_{x < n \le x + y} d(n; 1, 1, 1, 1, 2, 2, \ldots, 2) \sim C_1 y (\log x)^3$$

*holds for $y = y(x)$ with*

$$y = o(x), \qquad \frac{y}{x^{1/2} \log x} \to \infty \qquad (x \to \infty),$$

*where $C_1$ is the leading coefficient of $Q_1(u)$.*

Remark 6.1. Obviously the term $x^{1/2} Q_2(\log x)$ in Corollary 1 is a true main term for $J \ge 7$, but it is absorbed in the error term when $1 \le J \le 6$. However, the expected upper bound for the error term in (6.1) is $O(x^{3/8+\varepsilon})$. So we believe that $x^{1/2} Q_2(\log x)$ should be the true main term also for any $1 \le J \le 6$.

**6.2. On direct and unitary factors of finite abelian groups.** Let $\tau_1(n)$ denote the number of direct factors of finite abelian groups of order $n$, and $t_1(n)$ the number of unitary factors of finite abelian groups of order $n$. Both these functions are multiplicative since (see [1])

$$\sum_{n\geq 1}\tau_1(n)n^{-s}=\prod_{k\geq 1}\zeta^2(ks) \qquad (\Re s>1),$$
$$\sum_{n\geq 1}t_1(n)n^{-s}=\prod_{k\geq 1}\zeta^2((2k-1)s)\zeta(2ks) \qquad (\Re s>1).$$

It is a classical problem to study the asymptotic behaviour of the summatory functions of $\tau_1(n)$ and $t_1(n)$. For details, see [1].

In [1], Calderón proved that

$$\sum_{n\leq x}t_1^2(n)=xR_1(\log x)+O(x^{1/2}(\log x)^9), \tag{6.3}$$

where $R_1(u)$ is a polynomial in $u$ of degree 3. Calderón used Perron's formula and the expression

$$\sum_{n\geq 1}\frac{t_1^2(n)}{n^s}=\zeta^4(s)\zeta^6(2s)\zeta^{20}(3s)H_2(s), \tag{6.4}$$

where $H_2(s)$ is a Dirichlet series absolutely convergent for $\Re s>1/4$. From (6.4) and our Main Theorem we immediately get

Corollary 2. *We have the asymptotic formula*

$$\sum_{n\leq x}t_1^2(n)=xR_1(\log x)+O(x^{1/2}(\log x)^5). \tag{6.5}$$

Now we study $\sum_{n\leq x}\tau_1^2(n)$. For any prime $p$, the values of $\tau_1(p^\alpha)$ $(1\leq\alpha\leq 3)$ are (see [1, (3.1)])

$$\tau_1(1)=1,\qquad \tau_1(p)=2,\qquad \tau_1(p^2)=5,\qquad \tau_1(p^3)=10. \tag{6.6}$$

From (6.6) it is easy to see that when $\Re s>1$, we have

$$\begin{aligned}\sum_{n\geq 1}\frac{\tau_1^2(n)}{n^s}&=\prod_p\Big(1+\sum_{\alpha=1}^{\infty}\frac{\tau_1^2(p^\alpha)}{p^{\alpha s}}\Big)\\
&=\prod_p\Big(1+\frac{4}{p^s}+\frac{25}{p^{2s}}+\sum_{\alpha=3}^{\infty}\frac{\tau_1^2(p^\alpha)}{p^{\alpha s}}\Big)\\
&=\prod_p(1-p^{-s})^{-4}\prod_p(1-p^{-s})^4\Big(1+\frac{4}{p^s}+\frac{25}{p^{2s}}+\sum_{\alpha=3}^{\infty}\frac{\tau_1^2(p^\alpha)}{p^{\alpha s}}\Big)\\
&=\zeta^4(s)\prod_p\Big(1-\frac{4}{p^s}+\frac{6}{p^{2s}}+\frac{4}{p^{3s}}+\frac{1}{p^{4s}}\Big)\Big(1+\frac{4}{p^s}+\frac{25}{p^{2s}}+\sum_{\alpha=3}^{\infty}\frac{\tau_1^2(p^\alpha)}{p^{\alpha s}}\Big)\\
&=\zeta^4(s)\prod_p\Big(1+\frac{15}{p^{2s}}+\frac{20}{p^{3s}}+\cdots\Big)=\zeta^4(s)\zeta^{15}(2s)\prod_p\Big(1+\frac{20}{p^{3s}}+\cdots\Big)\\
&=\zeta^4(s)\zeta^{15}(2s)H_3(s),\end{aligned} \tag{6.7}$$

where $H_3(s)$ is a Dirichlet series absolutely convergent for $\Re s>1/3$.

From (6.7) and our Main Theorem we immediately get

Corollary 3. *We have the asymptotic formula*

$$\sum_{n\le x}\tau_1^2(n) = xR_2(\log x) + x^{1/2}R_3(\log x) + O(x^{1/2}(\log x)^5), \tag{6.8}$$

*where $R_2(u), R_3(u)$ are polynomials in $u$ of degrees 3 and 14, respectively.*

**6.3. The average order of $d^2(n)$ and $d(n^3)$.** It was Ramanujan [29] who first studied the mean value of $d^2(n)$. He stated (without proof) the asymptotic formula

$$\sum_{n\le x} d^2(n) = xQ_3(\log x) + O(x^{3/5+\varepsilon}), \tag{6.9}$$

where $Q_3(u)$ is a polynomial in $u$ of degree 3. The estimate $O(x^{3/5+\varepsilon})$ was improved to $O(x^{1/2+\varepsilon})$ by Wilson [34], and to $O(x^{1/2}(\log x)^5\log\log x)$ by K. Ramachandra and A. Sankaranarayanan [28], and by M. Z. Garaev, M. Kühleitner, F. Luca and W. G. Nowak [4]. Recently, Jia and Sankaranarayanan [14] showed that the error term in (6.3) can be improved to $O(x^{1/2}(\log x)^5)$. For $d(n^3)$, they obtained a similar result.

For $\Re s > 1$ we have (see for example [6])

$$\sum_{n\ge 1}\frac{d^2(n)}{n^s} = \frac{\zeta^4(s)}{\zeta(2s)}, \qquad \sum_{n\ge 1}\frac{d(n^3)}{n^s} = \frac{\zeta^4(s)}{\zeta^3(2s)}G_1(s),$$

where $G_1(s)$ has an absolutely convergent Euler product in the half-plane $\Re s > 1/3$. So our Main Theorem implies

Corollary 4. *We have the asymptotic formulas*

$$\sum_{n\le x} d^2(n) = xQ_3(\log x) + O(x^{1/2}(\log x)^5), \tag{6.10}$$

$$\sum_{n\le x} d(n^3) = xQ_4(\log x) + O(x^{1/2}(\log x)^5), \tag{6.11}$$

*where both $Q_3(u)$ and $Q_4(u)$ are polynomials in $u$ of degree 3.*

*Moreover, the short interval estimates*

$$\sum_{x<n\le x+y} d^2(n) \sim C_2 y(\log x)^3, \tag{6.12}$$

$$\sum_{x<n\le x+y} d(n^3) \sim C_3 y(\log x)^3 \tag{6.13}$$

*hold for $y = y(x)$ with*

$$y = o(x), \qquad \frac{y}{x^{1/2}\log x}\to\infty \quad (x\to\infty),$$

*where $C_2$ is the leading coefficient of $Q_3(u)$, and $C_3$ is the leading coefficient of $Q_4(u)$.*

Remark 6.2. Certainly, our Corollary 4 is not new compared to the results of Jia and Sankaranarayanan [14]. However, we give a slightly different and simpler proof.

**6.4. The second moment of quadratic Dedekind-zeta coefficients.** For an arbitrary quadratic number field $\mathbb{K}$ with discriminant $D$, let $O_{\mathbb{K}}$ denote the ring of algebraic integers in $\mathbb{K}$, and $r_{\mathbb{K}}(n)$ the number of integral ideals $I$ in $O_{\mathbb{K}}$ of norm $N(I) = n$. We have (see [26])

$$\sum_{n\geq 1}\frac{r_{\mathbb{K}}^2(n)}{n^s} = \frac{\zeta_{\mathbb{K}}^2(s)}{\zeta(2s)}\prod_{p|D}(1+p^{-s})^{-1} \quad (\Re s > 1).$$

It is well-known that $\zeta_{\mathbb{K}}(s) = \zeta(s)L(s,\chi_D)$, where $L(s,\chi_D)$ is a Dirichlet $L$-function with respect to a certain non-principal real character modulo $|D|$ (cf. Zagier [17, p. 100]). So from our Main Theorem by taking $L(s;\chi,q) = L(s,\chi_D)$, $M(s) = \zeta_{\mathbb{K}}(s)$, $H(s) = 1/\zeta(s)$ and $G(s) = \prod_{p|D}(1+p^{-s})^{-1}$ we get

Corollary 5. *We have the asymptotic formula*

$$\sum_{n\leq x} r_{\mathbb{K}}^2(n) = B_0 x\log x + B_1 x + O(x^{1/2}(\log x)^3), \tag{6.14}$$

*where*

$$B_0 = \frac{6}{\pi^2}L^2(1,\chi_D)\prod_{p|D}\frac{p}{p+1}.$$

*Moreover, the short interval estimate*

$$\sum_{x<n\leq x+y} r_{\mathbb{K}}^2(n) \sim B_0 y\log x \tag{6.15}$$

*holds if $y = o(x)$ and $y/(x^{1/2}\log x) \to \infty$ as $x \to \infty$.*

In particular, when $\mathbb{K} = \mathbb{Q}(i)$, Corollary 5 implies that

$$\sum_{n\leq x} r^2(n) = 4x\log x + Cx + O(x^{1/2}(\log x)^3), \tag{6.16}$$

which is an improvement of (1.4).

Remark 6.3. Corollary 5 improves Corollary 2 of [6].

**6.5. Some Diophantine equations.** Let $\mathbb{K}$ denote a quadratic number field with discriminant $D$. The generating series of $r_{\mathbb{K}}(n^3)$ is (see for example [6])

$$\sum_{n\geq 1}\frac{r_{\mathbb{K}}(n^3)}{n^s} = \frac{\zeta_{\mathbb{K}}^2(s)}{\zeta(2s)\zeta_{\mathbb{K}}(2s)}G_2(s),$$

where $G_2(s)$ can be written as a Dirichlet series absolutely convergent in the half-plane $\Re(s) > 1/3$. So our Main Theorem implies the following improvement of the corresponding result in Theorem GKLN and of [6, Corollary 4].

COROLLARY 6. *We have the asymptotic formula*

$$\sum_{n\le x} r_{\mathbb{K}}(n^3) = B_2 x\log x + B_3 x + O(x^{1/2}(\log x)^3), \tag{6.17}$$

*where*

$$B_2 = \frac{36L^2(1,\chi_D)}{\pi^4 L(2,\chi_D)} G_2(1).$$

*Moreover, the short interval estimate*

$$\sum_{x<n\le x+y} r_{\mathbb{K}}(n^3) \sim B_2 y\log x \tag{6.18}$$

*holds if $y = o(x)$ and $y/(x^{1/2}\log x) \to \infty$ as $x \to \infty$.*

Let $Q = Q(u;v) = au^2 + buv + cv^2$ be an integral, primitive, positive definite binary quadratic form of class number 1 and discriminant $D = b^2 - 4ac$. It is well-known that

$$r_Q(n) := \#\{(u,v)\in\mathbb{Z}^2 : Q(u,v) = n\} = \omega_D r_{\mathbb{K}}(n),$$

where $\mathbb{K} = \mathbb{Q}(\sqrt{D})$ and $\omega_D$ is the number of units in $Q_{\mathbb{Q}(\sqrt{D})}$. We can now apply Corollary 6 to the Diophantine equation

$$Q(u,v) = w^3 \tag{6.19}$$

and get the following improvement of Corollary 5 of [6].

COROLLARY 7. *For every integral, primitive, positive definite binary quadratic form $Q$ of class number $1$ and with discriminant $D$, we have the asymptotic formula*

$$\begin{aligned}\#\{(u,v,w)\in\mathbb{Z}^3 &: Q(u,v) = w^3,\ 1\le w\le x\}\\ &= \omega_D B_2 x\log x + \omega_D B_3 x + O(x^{1/2}(\log x)^3).\end{aligned}$$

*Moreover, the short interval estimate*

$$\#\{(u,v,w)\in\mathbb{Z}^3 : Q(u,v) = w^3,\ x < w\le x+y\} \sim \omega_D B_2 y\log x$$

*holds if $y = o(x)$ and $y/(x^{1/2}\log x) \to \infty$ as $x\to\infty$.*

**6.6. The average order of $d(n)r(n)$.** When $\Re s > 1$, we have (see for example [6, (4.2)])

$$\sum_{n\ge 1} \frac{d(n)r(n)}{n^s} = \frac{4\zeta^2_{\mathbb{Q}(i)}(s)\zeta(2s)}{\zeta_{\mathbb{Q}(i)}(2s)}. \tag{6.20}$$

From our Main Theorem we get the following improvement of Corollary 7 in [6].

Corollary 8. *We have the asymptotic formula*

$$\sum_{n\le x} d(n)r(n) = B_4 x\log x + B_5 x + O(x^{1/2}(\log x)^3), \tag{6.21}$$

*where $B_4 = \pi^2/(4L(2;\chi,4))$ and $\chi$ is the non-principal character modulo* 4.

*Moreover, the short interval estimate*

$$\sum_{x<n\le x+y} d(n)r(n) \sim B_4 y\log x \tag{6.22}$$

*holds if $y = o(x)$ and $y/(x^{1/2}\log x)\to\infty$ as $x\to\infty$.*

Remark 6.4. Corollary 8 also holds for sums like $\sum d(n)r_{\mathbb{K}}(n)$ for any quadratic number field $\mathbb{K}$.

**6.7. The average order of $4^{\omega(n)}$.** The additive function $\omega(n)$ is the number of distinct prime divisors of $n$. For each complex $z\ne 0$, the function $z^{\omega(n)}$ is multiplicative and it is an interesting problem to study the asymptotic behaviour of the summatory function of $z^{\omega(n)}$. In [11, Chapter 14], Ivić proved the following theorem: *for every fixed $R>0$ and every fixed integer $N\ge 0$, there exist functions $A_0(z), A_1(z),\dots,A_N(z)$ regular in $|z|\le R$ such that $A_0(0)=A_1(0)=\cdots=A_N(0)=0$ and*

$$\sum_{n\le x} z^{\omega(n)} = x(\log x)^{z-1}\Big(\sum_{j=0}^{N} A_j(z)(\log x)^{-j} + O((\log x)^{-N-1})\Big), \tag{6.23}$$

*where the $O$-constant is uniform in $|z|\le R$.*

When $z=k\ge 2$ is an integer, the estimate (6.23) is very weak and can be improved substantially. Actually, the Dirichlet series of $k^{\omega(n)}$ is

$$\sum_{n\ge 1}\frac{k^{\omega(n)}}{n^s} = \frac{\zeta^k(s)}{\zeta^{k^2-k-\binom{k}{2}}(2s)}G_k(s) \qquad (\sigma>1), \tag{6.24}$$

where $G_k(s)$ is regular in the range $\sigma\ge 1/3+\varepsilon$. From (6.24), by the contour integration method we immediately get

$$\sum_{n\le x} k^{\omega(n)} = x\mathcal{P}_k(\log x) + O(x^{\eta_k}), \tag{6.25}$$

where $\mathcal{P}_k(u)$ is a polynomial in $u$ of degree $k-1$, and $1/2\le\eta_k<1$ is a constant.

When $k=4$, our Main Theorem implies

Corollary 9. *We have the asymptotic formula*

$$\sum_{n\le x} 4^{\omega(n)} = x\mathcal{P}_4(\log x) + O(x^{1/2}(\log x)^5), \tag{6.26}$$

*where $\mathcal{P}_4(u)$ is a polynomial in $u$ of degree* 3.

*Moreover, the short interval estimate*

$$\sum_{x<n\le x+y} 4^{\omega(n)} \sim B_6 y(\log x)^3 \tag{6.27}$$

*holds if* $y = o(x)$ *and* $y/(x^{1/2}\log x) \to \infty$ *as* $x \to \infty$, *where* $B_6$ *is the leading coefficient of* $\mathcal{P}_4(u)$.

**6.8. An example connected with cusp forms.** Let $g$ be a primitive holomorphic cusp form of weight $k \ge 1$ for the full modular group $\mathrm{SL}_2(\mathbb{Z})$. Let

$$g(z) = \sum_{n=1}^{\infty} \lambda_g(n) n^{(k-1)/2} e(nz), \tag{6.28}$$

with $e(z) = e^{2\pi i z}$, be its normalized Fourier expansion at the cusp $\infty$. Then the automorphic $L$-function

$$\mathcal{L}_g(s) = \sum_{n=1}^{\infty} \lambda_g(n) n^{-s}$$

is an $L$-function of degree 2 satisfying the functional equation

$$(2\pi)^{-s}\Delta(s)\mathcal{L}_g(s) = (-1)^{k/2}(2\pi)^{-(1-s)}\Delta(1-s)\mathcal{L}_g(1-s)$$

with the gamma factor $\Delta(s) = \Gamma(s + (k-1)/2)$. Deligne [2] proved that $|\lambda_g(n)| \le d(n)$. So $\mathcal{L}_g(s)$ is a function in the Selberg class. It is well-known (see for example [15] and [7]) that

$$\mathcal{L}_g(1/2+it) \ll |t|^{1/3+\varepsilon} \qquad (|t| \ge 1) \tag{6.29}$$

and

$$\int_1^T |\mathcal{L}_g(1/2+it)|^2\, dt = A_{g,0} T\log T + A_{g,1}T + O(T^{2/3}(\log T)^C), \tag{6.30}$$

where $A_{g,0}$ and $A_{g,1}$ are constants.

Suppose $H(\cdot) \in \mathbb{D}_\beta$ for some $\beta > 0$, corresponding to an arithmetic function $h(n)$. Define the arithmetic function

$$f(n) = \sum_{n=n_1n_2n_3^2} d(n_1)\lambda_g(n_2)h(n_3).$$

When $\Re s > 1$, the Dirichlet series of $f$ is

$$\sum_{n\ge 1} \frac{f(n)}{n^s} = \zeta^2(s)\mathcal{L}_g(s)H(2s).$$

So from our Main Theorem we get

Corollary 10. *We have the asymptotic formula*

$$\sum_{n\le x} f(n) = B_7 x\log x + B_8 x + x^{1/2}Q_5(\log x) + O(x^{1/2}(\log x)^{7/2}), \tag{6.31}$$

*where $B_7$ and $B_8$ are constants, $Q_5(u)$ is a polynomial in $u$ of degree $\delta_H - 1$ when $\delta_H \ge 1$, and $Q_5(u) = 0$ if $\delta_H = 0$.*

Similarly, define

$$f_1(n) = \sum_{n=n_1n_2n_3^2} r(n_1)\lambda_g(n_2)h(n_3).$$

When $\Re s > 1$, the Dirichlet series of $f_1$ is

$$\sum_{n\ge1}\frac{f_1(n)}{n^s} = 4\zeta(s)L(s;\chi_4)\mathcal{L}_g(s)H(2s),$$

where $\chi_4$ is the non-principal character modulo 4.

So from our Main Theorem we get

Corollary 11. *We have the asymptotic formula*

$$\sum_{n\le x} f_1(n) = B_9 x + x^{1/2}Q_6(\log x) + O(x^{1/2}(\log x)^{5/2}), \tag{6.32}$$

*where $B_9$ is a constant, $Q_6(u)$ is a polynomial in $u$ of degree $\delta_H - 1$ when $\delta_H \ge 1$, and $Q_6(u) = 0$ if $\delta_H = 0$.*

**7. Appendix: the Selberg class.** The well-known Selberg class $\mathcal{S}$ (see for example [16, 30, 31]) consists of all non-vanishing Dirichlet series

$$\mathcal{L}(s) := \sum_{n=1}^{\infty}\frac{a(n)}{n^s}$$

which satisfy the following hypotheses:

I. *Ramanujan's conjecture*: $a(n) \ll n^{\varepsilon}$ for any $\varepsilon > 0$.

II. *Analytic continuation*: There exists a non-negative integer $m_{\mathcal{L}}$ such that $(s-1)^{m_{\mathcal{L}}}\mathcal{L}(s)$ is an entire function of finite order.

III. *Functional equation*: $\mathcal{L}(s)$ satisfies a functional equation of the type

$$\Lambda_{\mathcal{L}}(s) = \omega\,\overline{\Lambda_{\mathcal{L}}(1-\bar{s})}, \tag{7.1}$$

where

$$\Lambda_{\mathcal{L}}(s) := \mathcal{L}(s)Q^s\prod_{j=1}^{L}\Gamma(\alpha_j s + \beta_j), \tag{7.2}$$

and $Q > 0$, $|\omega| = 1$ and $\alpha_j > 0$, $\beta_j \in \mathbb{C}$ with $\Re\beta_j \ge 0$ for all $1 \le j \le L$. The number $d = 2\sum_j \alpha_j$ is called the *degree* of $\mathcal{L}(s)$.

IV. *Euler product*: $\mathcal{L}(s)$ satisfies

$$\mathcal{L}(s) = \prod_p \exp\Big(\sum_{n\ge 1} \frac{b(p^n)}{p^{ns}}\Big)$$

with suitable coefficients $b(p^n)$ satisfying $b(p^n) \ll p^{nc}$ for some $c < 1/2$.

Many well-known functions are contained in the Selberg class $\mathcal{S}$. We recall some examples. The well-known Riemann zeta-function $\zeta(s)$ and Dirichlet $L$-functions are functions in $\mathcal{S}$ of degree 1. The Dedekind zeta-function over an algebraic number field of degree $\kappa \ge 2$ is a function in $\mathcal{S}$ of degree $\kappa$. The automorphic $L$-function $\mathcal{L}_g(s)$ defined in the last section is a function in $\mathcal{S}$ of degree 2.

The extended Selberg class $\mathcal{S}^{\#}$ (see [16, 17, 18] for an introduction) consists of all Dirichlet series $\sum_{n\ge1} a(n)n^{-s}$ which satisfy conditions I*, II and III, where

I*. $\sum_{n\ge1} a(n)n^{-s}$ is absolutely convergent for $\sigma > 1$.

Finally, consider one example. Let $\varphi$ be a primitive Maass form for $\mathrm{SL}_2(\mathbb{Z})$, which is an eigenfunction of the Laplace operator with eigenvalue $\lambda = 1/4 + r^2$, where $r \in \mathbb{R}$. Write its Fourier expansion at infinity in the form

$$\varphi(z) = \sqrt{v} \sum_{n\in\mathbb{Z}\setminus\{0\}} \rho(n)K_{ir}(2\pi|n|v)e(nu) \qquad (z = u + iv,\ u \in \mathbb{R},\ v > 0),$$

where $K_{ir}$ is the modified Bessel function of the third kind. The corresponding automorphic $L$-function is defined by

$$\mathcal{L}(\varphi, s) = \sum_{n=1}^{\infty} \rho(n)n^{-s} \qquad (\sigma > 1),$$

which is a function of degree 2, entire on $\mathbb{C}$, with the functional equation

$$\pi^{-s}\Delta(s)\mathcal{L}(\varphi, s) = (-1)^{\delta}\pi^{-(1-s)}\Delta(1-s)\mathcal{L}(\varphi, 1-s),$$

where $\Delta(s) = \Gamma((s+\delta+ir)/2)\Gamma((s+\delta-ir)/2)$, and $\delta$ is the parity of $\varphi$ defined by $\delta = 0$ if $\varphi$ is even and $\delta = 1$ if $\varphi$ is odd.

We do not know if $\mathcal{L}(\varphi, \cdot) \in \mathcal{S}$ since the best-known upper bound of $\rho(n)$ is

$$|\rho(n)| \le d(n)n^{7/64}, \tag{7.3}$$

due to Kim and Sarnak [20]. However, we have $\mathcal{L}(\varphi, \cdot) \in \mathcal{S}^{\#}$.

The conclusions of Corollaries 10 and 11 still hold if we replace $\mathcal{L}_g(s)$ by $\mathcal{L}(\varphi, s)$.

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